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ON SOME MIXTURE DISTRIBUTIONS

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*Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday*

Abstract

The aim of this paper is to establish some mixture distributions that arise in stochastic processes. Some basic functions associated with the probability mass function of the mixture distributions, such as k -th moments, characteristic function and factorial moments are computed. Further we obtain a three-term recurrence relation for each established mixture distribution.

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1. Introduction

A particular mixture distribution stems when some or all parameters of a distribution vary according to some given probability distribution, called the mixing distribution. A well known example is the Poisson distribution mixture with gamma mixing distribution leading to negative binomial distribution. Such distributions have been used in a number of applications including accident proneness [2] and entomological field data [2].

In a recent paper Ghitany et al. [6] have shown that the hypergeometric generalized negative binomial distribution has moments of all positive

orders, is overdispersed, skewed to the right and obtained a three term recurrence relation. Special functions have been used to define a variety of probability distributions [12, 13]; generalized gamma-type [8, 9], inverse gaussian [7] using a generalized form of Kobayashi's [10] gamma function. We consider here several mixture distributions which can be obtained by mixing discrete distributions with continuous ones. In Section 2, we define some special functions and give some basic results that will be used in latter sections. Our first mixture distribution is obtained in Section 3 by mixing a hyper Poisson distribution $f_1(x/\lambda)$, defined in [2], with the new generalized gamma distribution $g_1(\lambda)$ defined in (18). In Sections 4, 5 and 6 we consider our second, third and fourth mixture distributions which are obtained by mixing the usual Poisson distribution together with the new generalized gamma distributions $g_2(\lambda)$, $g_3(\lambda)$ and $g_4(\lambda)$ defined in (33), (45) and (53) respectively. In each section of the last three sections we derive k -th moments, characteristic function, factorial moments and three-term recurrence relation for the mixture distribution obtained.

It is interesting to observe that the results obtained by Ghitany et al. [6] follow as special case of our general distributions considered in this work. Moreover, our first mixture distribution generalizes the mixture distribution obtained in [2].

2. Definitions and preliminaries

Throughout the sequel, we shall use the following definitions, Laplace integrals and recurrence relations:

Special functions

(1) The *Kummer confluent hypergeometric function*, [14]:

$${}_1F_1(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!}, \quad |z| < \infty, \alpha, \beta > 0, \quad (1)$$

where $(c)_n$ denotes the *Pochhammer's symbol*, $(c)_n = c(c+1) \dots (c+n-1)$.

(2) The ω - *confluent hypergeometric function*, [16, 17]:

$${}_1\Phi_1^{\omega}(\alpha; \beta; z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \omega k)}{\Gamma(\beta + \omega k)} \frac{z^k}{k!}, \quad (2)$$

$$|z| < \infty, \omega > 0, (\beta + \omega k) \neq 0, -1, -2, \dots$$

For $\omega = 1$, (2) reduces to (1).

(3) The *hypergeometric function* [14, 15]:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \quad |z| < 1, \gamma \neq 0, -1, -2, \dots \quad (3)$$

By means of analytic continuation [3] the function can be defined for all values of z , with $|\arg(1 - z)| < \pi$.

(4) The *generalized hypergeometric function*, [3, 15]:

$${}_nF_m(z) = {}_nF_m \left(\begin{matrix} a, (a_i) \\ p, (p_i) \end{matrix}; z \right) = \sum_{k=0}^{\infty} \frac{\prod_{i=2}^n (a_i)_k}{\prod_{i=2}^m (p_i)_k} \frac{(a)_k z^k}{(p)_k k!}, \quad (4)$$

n and m are non-negative integers, and no p_i [$i = 1, \dots, m$] is zero or non-negative integer. If (i) $n < m + 1$, the series converges for all (finite) z , If (ii) $n = m + 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$, If (iii) $n > m + 1$, the series diverges for all z except $z = 0$.

(5) The ω -*hypergeometric function*, [16]:

$${}_2\tilde{R}_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k \Gamma(\beta + \omega k)}{\Gamma(\gamma + \omega k)} \frac{z^k}{k!}, \quad \omega > 0, |z| < 1. \quad (5)$$

For $\omega = 1$, (5) reduces to (3).

(6) We shall use the following *Appell hypergeometric function of two variables*, [14, 4]:

$$F_1(\alpha, b, \beta; c; z, w) = \sum_{n,k=0}^{\infty} \frac{(\alpha)_{n+k} (b)_k z^k (\beta)_n w^n}{(c)_{n+k} k! n!}, \quad |z|, |w| < 1, \quad (6)$$

$$F_2(\alpha, b, \beta; c, \gamma; z, w) = \sum_{n,k=0}^{\infty} (\alpha)_{n+k} \frac{(b)_n z^n}{n! (c)_n} \frac{(\beta)_k w^k}{k! (\gamma)_k}, \quad |z|, |w| < 1. \quad (7)$$

(7) The *Horn's confluent hypergeometric function of two variables*, [3, 4, 14, 15]:

$$\Phi_1(\alpha, \beta; c; z, w) = \sum_{n,k=0}^{\infty} \frac{(\alpha)_{n+k} (\beta)_k z^k w^n}{(c)_{n+k} k! n!}, \quad |z| < 1. \quad (8)$$

The corresponding integral representations for these functions can be found, for example, in [3, 14].

The following **recurrence relations** for hypergeometric functions are used in our analysis [3], [16] and [5]:

$$b(1-z) {}_2F_1(b+1) = (p-b) {}_2F_1(b-1) + (2b-p+(a-b)z) {}_2F_1(b), \quad (9)$$

where

$${}_2F_1(b) = {}_2F_1(a, b; p; z);$$

$$a\omega(1-z) {}_2\check{R}_1(a+1) = (p-\omega a) {}_2\check{R}_1(a-1) + (2\omega a - p + (b-\omega a)z) {}_2\check{R}_1(a), \quad (10)$$

where

$${}_2\check{R}_1(a) = {}_2\check{R}_1(a, b; p; z),$$

$$(2\alpha - p) {}_{n+1}F_n(\alpha) = \alpha {}_{n+1}F_n(\alpha + 1) + (\alpha - p) {}_{n+1}F_n(\alpha - 1) + z \prod_{i=0}^{n-2} \frac{a_{n-i+1}}{p_{n-i}} \\ \times \left\{ (\alpha - a) {}_{n+1}F_n \left(\begin{matrix} \alpha, a, (1+a_i) \\ p, (1+p_i) \end{matrix}; z \right) - \alpha {}_{n+1}F_n \left(\begin{matrix} \alpha + 1, a, (1+a_i) \\ p, (1+p_i) \end{matrix}; z \right) \right\}, \quad (11)$$

and

$${}_{n+1}F_n(\alpha) = {}_{n+1}F_n \left(\begin{matrix} \alpha, a, (a_i) \\ p, (p_i) \end{matrix}; z \right).$$

We also make use of the following **Laplace integrals**, [1], [11] and [14]:

$$\int_0^{\infty} \lambda^{\gamma-1} e^{-s\lambda} {}_1\check{\Phi}_1(a; p; \lambda) d\lambda = \frac{\Gamma(\gamma)}{s^\gamma} {}_2\check{R}_1 \left(\gamma, a; p; \frac{1}{s} \right), \quad (12)$$

$$\Re\gamma, \Re(s-1), \omega > 0,$$

$$\int_0^{\infty} \lambda^{\gamma-1} e^{-s\lambda} {}_nF_m \left(\begin{matrix} a, (a_i) \\ p, (p_i) \end{matrix}; \lambda \right) d\lambda = \frac{\Gamma(\gamma)}{s^\gamma} {}_{n+1}F_m \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{s} \right) \quad (13)$$

$$n < m \quad \Re\gamma, \Re s > 0,$$

$$n = m \quad \Re\gamma > 0, \Re s > 1,$$

$$\int_0^{\infty} \lambda^{\gamma-1} e^{-s\lambda} \Phi_1(a, p; c; \delta, \lambda) d\lambda = \frac{\Gamma(\gamma)}{s^\gamma} F_1 \left(a, p, \gamma; c; \delta, \frac{1}{s} \right) \quad (14)$$

$$\Re\gamma, \Re(s-1) > 0$$

$$\int_0^\infty \lambda^{\gamma-1} e^{-s\lambda} {}_1F_1(\alpha; \beta; \lambda) {}_1F_1(a; b; \lambda) d\lambda = \frac{\Gamma(\gamma)}{s^\gamma} F_2\left(\gamma, \alpha, a; \beta, b; \frac{1}{s}, \frac{1}{s}\right),$$

$$\Re\gamma, \Re(s-2) > 0. \tag{15}$$

The **hyper Poisson distribution**, defined in [2], with parameters b, q, λ , has a probability mass function (pmf):

$$h(x) = \frac{\lambda^x}{x!} \frac{(b)_x}{(q)_x} \frac{1}{{}_1F_1(b; q; \lambda)}, \quad b, q, \lambda > 0, \quad x = 0, 1, 2, \dots \tag{16}$$

which yields for $b = q$, the usual Poisson distribution whose (pmf) is given by

$$k(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad \lambda > 0, \quad x = 0, 1, 2, \dots \tag{17}$$

3. The first mixture distribution $f_1(x)$

In this section we consider a more general family of continuous mixture distribution. This distribution is obtained by mixing a Hyper Poisson distribution $f_1(x/\lambda)$ defined in [2], with a new generalized gamma distribution defined here by relation (18). The recent results of Ghitany et al. [6] follow as special cases of our general results derived in this section. We derive some basic functions associated with this density function, namely the k -th Moment, characteristic function and factorial moments. Furthermore, we establish a three-term recurrence relation for the new mixture distribution. We begin this section by defining the function $g_1(\lambda)$ as

$$g_1(\lambda) = \frac{\lambda^{\gamma-1}}{\Gamma(\gamma)} (\alpha + 2)^\gamma e^{-(\alpha+2)\lambda} \frac{{}_1F_1(a; p; \lambda) {}_1F_1(b; q; \lambda)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}, \quad \lambda > 0. \tag{18}$$

This function is non-negative and satisfies the condition $\int_0^\infty g_1(\lambda) d\lambda = 1$ by virtue of the result (15). Therefore (18) represents a continuous distribution of the product of two confluent hypergeometric functions, involving Appell's function, which yields for $p = \gamma = b = q$, the probability density function defined in [6]. Let X has a conditional hyper Poisson distribution (16) with parameters b, q, λ , that is, X has a conditional probability mass function (pmf)

$$f_1(x/\lambda) = \frac{\lambda^x}{x!} \frac{(b)_x}{(q)_x} \frac{1}{{}_1F_1(b; q; \lambda)}, \quad b, q, \lambda > 0, \quad x = 0, 1, 2, \dots \tag{19}$$

whose characteristic function, for any real t , is given by

$$\Phi(t) = E [e^{itX} / \Lambda = \lambda] = \frac{1}{{}_1F_1(b; q; \lambda)} \sum_{k=0}^n \frac{(b)_x}{(q)_x} \frac{(\lambda e^{it})^x}{x!} = \frac{{}_1F_1(b; q; \lambda e^{it})}{{}_1F_1(b; q; \lambda)}, \quad (20)$$

from which moments of any order could be evaluated by

$$E [X^r / \Lambda = \lambda] = i^{-r} \Phi^{(r)}(t) |_{t=0} = \frac{\frac{d^r}{dt^r} {}_1F_1(b; q; \lambda e^{it}) |_{t=0}}{{}_1F_1(b; q; \lambda)}, \quad r = 1, 2, \dots \quad (21)$$

In particular, the mean is

$$E [X / \Lambda = \lambda] = i^{-1} \Phi^{(1)}(t) |_{t=0} = \frac{\lambda b} {q} \frac{{}_1F_1(b+1; q+1; \lambda)}{{}_1F_1(b; q; \lambda)}. \quad (22)$$

From (21), we obtain the factorial moments of the hyper Poisson distribution

$$E [X(X-1) \dots (X-r+1) / \Lambda = \lambda] = \frac{\lambda^r (b)_r} {(q)_r} \frac{{}_1F_1(b+r; q+r; \lambda)}{{}_1F_1(b; q; \lambda)}. \quad (23)$$

Now we state and prove our first theorem.

THEOREM 1. *The unconditional pmf of X is given by*

$$f_1(x) = \frac{(\gamma)_x (b)_x}{x! (q)_x (\alpha+2)^x} \frac{{}_2F_1\left(a, x+\gamma; p; \frac{1}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}, \quad x = 0, 1, 2, \dots, \quad (24)$$

whose characteristic function, for any real t , is given by

$$\Psi_X(t) = E [e^{itX}] = \frac{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{e^{it}}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}, \quad (25)$$

its factorial moment is

$$\begin{aligned} & E [X(X-1) \dots (X-r+1)] \\ &= \frac{(\gamma)_r (b)_r}{(q)_r (\alpha+2)^r} \frac{F_2\left(\gamma+r, a, b+r; p, q+r; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}, \end{aligned} \quad (26)$$

and its r th-moment is given by

$$E [X^r] = \sum_{n=0}^r S(r, n) \frac{(\gamma)_r (b)_r}{(q)_r (\alpha + 2)^r} \frac{F_2 \left(\gamma + r, a, b + r ; p, q + r ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}, \tag{27}$$

where

$$S(r, n) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k (n - k)^r}{n!}. \tag{28}$$

P r o o f. From (18) and (19) the unconditional pmf of X is

$$\begin{aligned} f_1(x) &= \int_0^\infty f_1(x/\lambda) g_1(\lambda) d\lambda \\ &= \frac{(b)_x (\alpha + 2)^\gamma}{x! (q)_x \Gamma(\gamma)} \frac{1}{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)} \int_0^\infty \lambda^{x+\gamma-1} e^{-(\alpha+2)\lambda} {}_1F_1(a; p; \lambda) d\lambda \\ &= \frac{(\gamma)_x (b)_x}{x! (q)_x (\alpha + 2)^x} \frac{{}_2F_1 \left(a, x + \gamma ; p ; \frac{1}{\alpha + 2} \right)}{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}, \quad x = 0, 1, 2, \dots \end{aligned}$$

Using (20) the characteristic function of X is

$$\begin{aligned} \Psi_X(t) &= E [e^{itX}] = E [E [e^{itX}/\Lambda]] = E \left[\frac{{}_1F_1(b; q; \Lambda e^{it})}{{}_1F_1(b; q; \Lambda)} \right] \\ &= \frac{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{e^{it}}{\alpha + 2} \right)}{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}, \end{aligned}$$

From (22) the factorial moment of X is

$$\begin{aligned} E [X (X - 1) \dots (X - r + 1)] &= E [E [X (X - 1) \dots (X - r + 1) / \Lambda]] \\ &= E \left[\frac{\Lambda^r (b)_r {}_1F_1(b + r; q + r; \Lambda)}{(q)_r {}_1F_1(b; q; \Lambda)} \right] \\ &= \frac{(\gamma)_r (b)_r}{(q)_r (\alpha + 2)^r} \frac{F_2 \left(\gamma + r, a, b + r ; p, q + r ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}{F_2 \left(\gamma, a, b ; p, q ; \frac{1}{\alpha + 2}, \frac{1}{\alpha + 2} \right)}, \end{aligned}$$

Finally, the r th-moment of X is obtained using the relation

$$E[X^r] = \sum_{n=0}^r S(r, n) E[X(X-1)\dots(X-r+1)],$$

which completes the proof. \blacksquare

THEOREM 2. *The distribution of X satisfies a three-term recurrence relation*

$$\begin{aligned} (x+1)f_1(x+1) &= \frac{(p-\gamma-x)(\gamma+x-1)(b+x-1)_2}{x(q+x-1)_2(\alpha+1)_2} f_1(x-1) \\ &+ \frac{b+x}{(q+x)(\alpha+1)} \left[\frac{a-\gamma-x}{\alpha+2} + 2(\gamma+x) - p \right] f_1(x). \end{aligned} \quad (29)$$

P r o o f. Using the recurrence relation (9), for $b = x + \gamma$, $z = \frac{1}{\alpha+2}$, we get

$$\begin{aligned} (\gamma+x)F(x+\gamma+1) &= \frac{\alpha+2}{\alpha+1} \{ (p-\gamma-x)F(x+\gamma-1) \\ &+ \left[\frac{a-\gamma-x}{\alpha+2} + 2(\gamma+x) - p \right] F(x+\gamma) \}, \end{aligned} \quad (30)$$

where

$$F(x) = {}_2F_1\left(a, x; p; \frac{1}{\alpha+2}\right)$$

Rewrite $f_1(x)$, given by (24), as

$$f_1(x) = v_1(x) {}_2F_1\left(a, x+\gamma; p; \frac{1}{\alpha+2}\right), \quad x = 0, 1, 2, \dots \quad (31)$$

where

$$v_1(x) = \frac{(\gamma)_x (b)_x}{x! (q)_x (\alpha+2)^x} \frac{1}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}, \quad x = 0, 1, 2, \dots$$

from which we get

$$v_1(x+1) = \frac{(b+x)(\gamma+x)}{(x+1)(q+x)(\alpha+2)} v_1(x)$$

$$= \frac{(b+x-1)_2 (\gamma+x-1)_2}{(x)_2 (q+x-1)_2 (\alpha+2)^2} v_1(x-1). \tag{32}$$

Using (30), (31) and (32), we obtain, for $x = 1, 2, \dots$

$$\begin{aligned} (x+1) f_1(x+1) &= (x+1) v_1(x+1) {}_2F_1\left(a, x+\gamma+1; p; \frac{1}{\alpha+2}\right) \\ &= \frac{(x+1)(\alpha+2)}{(\gamma+x)(\alpha+1)} \left\{ \frac{(p-\gamma-x)(b+x-1)_2(\gamma+x-1)_2}{(x)_2(q+x-1)_2(\alpha+2)^2} v_1(x-1)F(x+\gamma-1) \right. \\ &\quad \left. + \frac{(b+x)(\gamma+x)}{(x+1)(q+x)(\alpha+2)} \left[\frac{a-\gamma-x}{\alpha+2} + 2(\gamma+x) - p \right] v_1(x) F(x+\gamma) \right\} \\ &= \frac{(p-\gamma-x)(\gamma+x-1)(b+x-1)_2}{x(q+x-1)_2(\alpha+1)_2} f_1(x-1) \\ &\quad + \frac{b+x}{(q+x)(\alpha+1)} \left[\frac{a-\gamma-x}{\alpha+2} + 2(\gamma+x) - p \right] f_1(x), \end{aligned}$$

and the proof is complete. ■

Figure 1 and **Figure 2** respectively, show the probability mass function $f_1(x)$ and its distribution function $F_1(x)$ for the selected specific values of $\gamma, a, b, p, q, \alpha$.

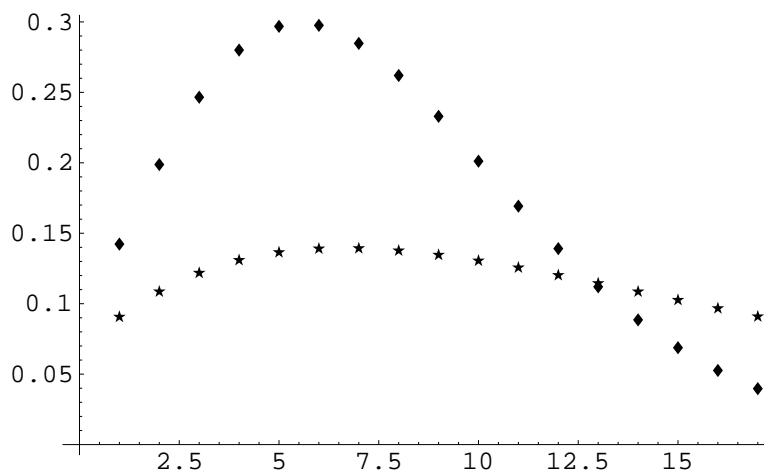


Figure 1: The probability mass function $f_1(x)$. The symbol * represents $f_1(x)$ when $\gamma = 3, a = 4, b = 5, p = 6, q = 7$ and $\alpha = .2$, whereas the symbol \diamond represents $f_1(x)$ when $\gamma = 6, a = 7, b = 5, p = 4, q = 2$ and $\alpha = 7$.

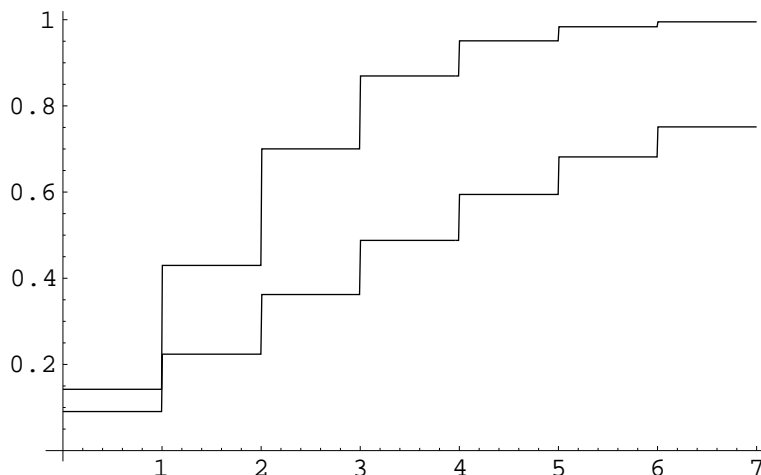


Figure 2: The distribution function $F_1(x)$. The lower graph represents $F_1(x)$ when $\gamma = 3$, $a = 4$, $b = 5$, $p = 6$, $q = 7$ and $\alpha = .2$, whereas the upper graph represents $F_1(x)$ when $\gamma = 6$, $a = 7$, $b = 5$, $p = 4$, $q = 2$ and $\alpha = 7$.

4. The second mixture distribution $f_2(x)$

In this section we consider a more generalized family of continuous mixture distribution. This distribution is obtained by mixing a Poisson distribution $f_2(x/\lambda)$ defined in (17), with a new generalized gamma distribution defined below by the result (33). The work done in [6] follows as a special of this section, when we take $\omega = 1$ and $\gamma = a$. Further, we compute the k -th moments, characteristic function, factorial moments and a three-term recurrence relation. We begin this section by defining the function $g_2(\lambda)$ as,

$$g_2(\lambda) = \frac{\lambda^{\gamma-1}}{\Gamma(\gamma)} (\alpha + 1)^\gamma e^{-(\alpha+1)\lambda} \frac{{}_1\tilde{\Phi}_1(a; p; \lambda)}{{}_2\tilde{R}_1(\gamma, a; p; \frac{1}{\alpha+1})}, \lambda > 0. \quad (33)$$

This function is non-negative and satisfies the condition $\int_0^\infty g_2(\lambda) d\lambda = 1$ by virtue of the result (12). Thus (33) represents a continuous distribution of confluent hypergeometric function type, which yields for $\omega = 1$ and $\gamma = a$, the probability density function considered by Ghitany et al. [6]. Let X

has a conditional Poisson distribution (17) with parameter λ , that is, X has a conditional probability mass function

$$f_2(x/\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad \lambda > 0, \quad x = 0, 1, 2, \dots, \quad (34)$$

whose characteristic function, for any real t , is given by

$$\Phi(t) = E[e^{itX} / \Lambda = \lambda] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{-\lambda(1-e^{it})}. \quad (35)$$

Putting $b = q$ in (23), we obtain the factorial moments of the Poisson distribution

$$E[X(X-1)\dots(X-r+1)/\Lambda] = \Lambda^r. \quad (36)$$

THEOREM 3. *The unconditional pmf of X is given by*

$$f_2(x) = \frac{(\gamma)_x (\alpha + 1)^\gamma}{x! (\alpha + 2)^{x+\gamma}} \frac{{}_2\check{R}_1(x + \gamma, a; p; \frac{1}{\alpha+2})}{{}_2\check{R}_1(\gamma, a; p; \frac{1}{\alpha+1})}, \quad x = 0, 1, 2, \dots, \quad (37)$$

whose characteristic function, for any real t , is given by

$$\Psi_X(t) = E[e^{itX}] = \left(\frac{\alpha + 1}{\alpha + 2 - e^{it}}\right)^\gamma \frac{{}_2\check{R}_1(\gamma, a; p; \frac{1}{\alpha+2-e^{it}})}{{}_2\check{R}_1(\gamma, a; p; \frac{1}{\alpha+1})}, \quad (38)$$

its factorial moment is,

$$E[X(X-1)\dots(X-r+1)] = \frac{(\gamma)_r}{(\alpha + 1)^r} \frac{{}_2\check{R}_1(\gamma + r, a; p; \frac{1}{\alpha+1})}{{}_2\check{R}_1(\gamma, a; p; \frac{1}{\alpha+1})}, \quad (39)$$

and its r th-moment is given by

$$E[X^r] = \sum_{n=0}^r S(r, n) \frac{(\gamma)_n}{(\alpha + 1)^n} \frac{{}_2\check{R}_1(\gamma + n, a; p; \frac{1}{\alpha+1})}{{}_2\check{R}_1(\gamma, a; p; \frac{1}{\alpha+1})}, \quad (40)$$

where $S(r, n)$ are given by (28).

P r o o f. From (33) and (34) the unconditional pmf of X is

$$\begin{aligned} f_2(x) &= \int_0^{\infty} f_2(x/\lambda) g_2(\lambda) d\lambda \\ &= \frac{(\alpha+1)^\gamma}{x! \Gamma(\gamma)} \frac{1}{{}_2\check{R}_1\left(\gamma, a; p; \frac{1}{\alpha+1}\right)} \int_0^{\infty} \lambda^{x+\gamma-1} e^{-(\alpha+2)\lambda} {}_1\check{\Phi}_1(a; p; \lambda) d\lambda \\ &= \frac{(\gamma)_x}{x!} \frac{(\alpha+1)^\gamma}{(\alpha+2)^{x+\gamma}} \frac{{}_2\check{R}_1\left(x+\gamma, a; p; \frac{1}{\alpha+2}\right)}{{}_2\check{R}_1\left(\gamma, a; p; \frac{1}{\alpha+1}\right)}, \quad x = 0, 1, 2, \dots \end{aligned}$$

Using (35) the characteristic function of X is

$$\begin{aligned} \Psi_X(t) &= E[e^{itX}] = E[E[e^{itX}/\Lambda]] = E\left[\frac{{}_1F_1(b; q; \Lambda e^{it})}{{}_1F_1(b; q; \Lambda)}\right] \\ &= \frac{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{e^{it}}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}. \end{aligned}$$

From (36) the factorial moment of X is

$$\begin{aligned} E[X(X-1)\dots(X-r+1)] &= E[E[X(X-1)\dots(X-r+1)/\Lambda]] = E[\Lambda^r] \\ &= \frac{(\gamma)_r (b)_r}{(q)_r (\alpha+2)^r} \frac{F_2\left(\gamma+r, a, b+r; p, q+r; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}. \end{aligned}$$

Finally, the r th-moment is obtained in a similar way as in the previous section. ■

THEOREM 4. *The distribution of X satisfies a three-term recurrence relation*

$$\begin{aligned} (x+1) f_2(x+1) &= \frac{(\gamma+x-1)[p-(\gamma+x)\omega]}{\omega x (\alpha+1)(\alpha+2)} f_2(x-1) \\ &+ \frac{1}{(\alpha+1)\omega} \left[\frac{a-(\gamma+x)\omega}{\alpha+2} + 2(\gamma+x)\omega - p \right] f_2(x). \quad (41) \end{aligned}$$

P r o o f. Using the recurrence relation (10), for $a = x + \gamma$, $z = \frac{1}{\alpha+2}$, we get

$$(\gamma + x) {}_2\check{R}_1(x + \gamma + 1) = \frac{\alpha + 2}{(\alpha + 1) \omega} \left\{ [p - (\gamma + x)\omega] {}_2\check{R}_1(x + \gamma - 1) + \left[\frac{a - (\gamma + x)\omega}{\alpha + 2} + 2(\gamma + x)\omega - p \right] {}_2\check{R}_1(x + \gamma) \right\}, \quad (42)$$

where

$${}_2\check{R}_1(\gamma + x) = {}_2\check{R}_1\left(\gamma + x, a; p; \frac{1}{\alpha + 2}\right).$$

Rewrite $f_2(x)$, given by (37), as

$$f_2(x) = v_2(x) {}_2\check{R}_1\left(x + \gamma, a; p; \frac{1}{\alpha + 2}\right), \quad x = 0, 1, 2, \dots, \quad (43)$$

where

$$v_2(x) = \frac{(\gamma)_x (\alpha + 1)^\gamma}{x! (\alpha + 2)^{x+\gamma}} \frac{1}{{}_2\check{R}_1\left(\gamma, a; p; \frac{1}{\alpha+1}\right)}, \quad x = 0, 1, 2, \dots,$$

from which we get

$$v_2(x + 1) = \frac{(\gamma + x)}{(x + 1)(\alpha + 2)} v_2(x) = \frac{(\gamma + x - 1)_2}{(x)_2 (\alpha + 2)^2} v_2(x - 1) \quad (44)$$

Using (42), (43) and (44), we obtain, for $x = 1, 2, \dots$

$$\begin{aligned} (x + 1) f_2(x + 1) &= (x + 1) v_2(x + 1) {}_2\check{R}_1(x + \gamma + 1) \\ &= \frac{v_2(x + 1) (x + 1) (\alpha + 2)}{\omega (\gamma + x) (\alpha + 1)} \left\{ [p - (\gamma + x)\omega] {}_2\check{R}_1(x + \gamma - 1) + \left[\frac{a - (\gamma + x)\omega}{\alpha + 2} + 2(\gamma + x)\omega - p \right] {}_2\check{R}_1(x + \gamma) \right\} \\ &= \frac{(\gamma + x - 1) [p - (\gamma + x)\omega]}{\omega x (\alpha + 1) (\alpha + 2)} v_2(x - 1) {}_2\check{R}_1(x + \gamma - 1) \\ &+ \frac{1}{(\alpha + 1) \omega} \left[\frac{a - (\gamma + x)\omega}{\alpha + 2} + 2(\gamma + x)\omega - p \right] v_2(x) {}_2\check{R}_1(x + \gamma) \\ &\quad + \frac{(\gamma + x - 1) [p - (\gamma + x)\omega]}{\omega x (\alpha + 1) (\alpha + 2)} f_2(x - 1) \\ &\quad + \frac{1}{(\alpha + 1) \omega} \left[\frac{a - (\gamma + x)\omega}{\alpha + 2} + 2(\gamma + x)\omega - p \right] f_2(x). \end{aligned}$$

and the proof is complete. ■

5. The third mixture distribution $f_3(x)$

In this section we consider another generalized family of continuous mixture distribution. This distribution is obtained by mixing a Poisson distribution $f_3(x/\lambda)$ defined in (17), with a new generalized gamma distribution defined in (45). We derive similar results as obtained for previous mixture distributions for our new mixture distribution $f_3(x)$. We start by defining the function $g_3(\lambda)$ as,

$$g_3(\lambda) = \frac{\lambda^{\gamma-1}}{\Gamma(\gamma)} (\alpha+1)^\gamma e^{-(\alpha+1)\lambda} \frac{{}_nF_m \left(\begin{matrix} a, (a_i) \\ p, (p_i) \end{matrix}; \lambda \right)}{{}_{n+1}F_m \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{\alpha+1} \right)}, \quad n \leq m, \lambda > 0. \quad (45)$$

This function is non-negative and satisfies the condition $\int_0^\infty g_3(\lambda) d\lambda = 1$ by virtue of the result (13). Therefore (45) represents a continuous distribution of generalized hypergeometric function, which yields for $n = m = 1$ and $\gamma = a$, the probability density function defined in [6]. Let X has a conditional Poisson distribution with parameter λ as in the previous section, i.e. its pmf is

$$f_2(x/\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \dots, \lambda > 0.$$

THEOREM 5. *The unconditional pmf of X is given by*

$$f_3(x) = \frac{(\gamma)_x (\alpha+1)^\gamma} {x! (\alpha+2)^{x+\gamma}} \frac{{}_{n+1}F_m \left(\begin{matrix} x+\gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{\alpha+2} \right)}{{}_{n+1}F_m \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{\alpha+1} \right)}, \quad x = 0, 1, 2, \dots, \quad (46)$$

whose characteristic function, for any real t , is given by

$$\Psi_X(t) = E[e^{itX}] = \left(\frac{\alpha+1}{\alpha+2-e^{it}} \right)^\gamma \frac{{}_{n+1}F_m \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{\alpha+2-e^{it}} \right)}{{}_{n+1}F_m \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i) \end{matrix}; \frac{1}{\alpha+1} \right)}, \quad (47)$$

its factorial moment is

$$E[X(X-1)\dots(X-r+1)] = \frac{(\gamma)_r}{(\alpha+1)^r} \frac{{}_{n+1}F_m\left(\begin{matrix} \gamma+r, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}, \tag{48}$$

and its r th-moment is given by

$$E[X^r] = \sum_{n=0}^r S(r, n) \frac{(\gamma)_r}{(\alpha+1)^r} \frac{{}_{n+1}F_m\left(\begin{matrix} \gamma+r, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}, \tag{49}$$

where $S(r, n)$ is defined by (28).

P r o o f. From (45) and (34) the unconditional pmf of X is

$$\begin{aligned} f_3(x) &= \int_0^\infty f_2(x/\lambda) g_3(\lambda) d\lambda \\ &= \frac{(\alpha+1)^\gamma}{x! \Gamma(\gamma)} \frac{1}{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)} \int_0^\infty \lambda^{x+\gamma-1} e^{-(\alpha+2)\lambda} {}_nF_m\left(\begin{matrix} a, (a_i) \\ p, (p_i); \lambda \end{matrix}\right) d\lambda \\ &= \frac{(\gamma)_x (\alpha+1)^\gamma} {x! (\alpha+2)^{x+\gamma}} \frac{{}_{n+1}F_m\left(\begin{matrix} x+\gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+2} \end{matrix}\right)}{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}, \quad x = 0, 1, 2, \dots \end{aligned}$$

Using (35) the characteristic function of X is

$$\begin{aligned} \Psi_X(t) &= E[e^{itX}] = E[E[e^{itX}/\Lambda]] = E[e^{-\Lambda(1-e^{it})}] \\ &= \left(\frac{\alpha+1}{\alpha+2-e^{it}}\right)^\gamma \frac{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+2-e^{it}} \end{matrix}\right)}{{}_{n+1}F_m\left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix}\right)}. \end{aligned}$$

From (36) the factorial moment of X is

$$\begin{aligned} E[X(X-1)\dots(X-r+1)] &= E[E[X(X-1)\dots(X-r+1)/\Lambda]] = E[\Lambda^r] \\ &= \frac{(\gamma)_r (b)_r}{(q)_r (\alpha+2)^r} \frac{F_2\left(\gamma+r, a, b+r; p, q+r; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}. \end{aligned}$$

Finally, the r th-moment of X is obtained as in previous cases. \blacksquare

THEOREM 6. *The distribution of X satisfies a three-term recurrence relation*

$$\begin{aligned} (x+1)f_3(x+1) &= \frac{2(\gamma+x)-p}{\alpha+2} f_3(x) + \frac{(x+\gamma-1)[p-(\gamma+x)]}{x(\alpha+2)^2} f_3(x-1) \\ &+ \prod_{i=0}^{n-2} \frac{a_{n-i+1}}{p_{n-i}} \left\{ v_3(x+1) \frac{x+1}{\alpha+2} {}_{n+1}F_n \left(\begin{matrix} x+\gamma+1, a, (1+a_i) \\ p, (1+p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right. \\ &\left. - \frac{\gamma+x-a}{(\alpha+2)^2} v_3(x) {}_{n+1}F_n \left(\begin{matrix} x+\gamma, a, (1+a_i) \\ p, (1+p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right\}. \end{aligned}$$

P r o o f. Using the recurrence relation (4), for $\alpha = x + \gamma$, $z = \frac{1}{\alpha+2}$, we get

$$\begin{aligned} (\gamma+x)F(x+\gamma+1) &= [2(\gamma+x)-p]F(x+\gamma) + [p-(\gamma+x)]F(x+\gamma-1) \\ &+ \prod_{i=0}^{n-2} \frac{a_{n-i+1}}{p_{n-i}} \left\{ \frac{\gamma+x}{\alpha+2} {}_{n+1}F_n \left(\begin{matrix} x+\gamma+1, a, (1+a_i) \\ p, (1+p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right. \\ &\left. + \frac{\gamma+x-a}{\alpha+2} {}_{n+1}F_n \left(\begin{matrix} x+\gamma, a, (1+a_i) \\ p, (1+p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right\}, \end{aligned} \quad (50)$$

where

$$F(\gamma+x) = {}_{n+1}F_n \left(\begin{matrix} x+\gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+2} \end{matrix} \right)$$

Rewrite $f_3(x)$, given by (46), as

$$f_3(x) = v_3(x) {}_{n+1}F_n \left(\begin{matrix} x+\gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+2} \end{matrix} \right), \quad x = 0, 1, 2, \dots, \quad (51)$$

where

$$v_3(x) = \frac{(\gamma)_x (\alpha + 1)^\gamma}{x! (\alpha + 2)^{x+\gamma}} \frac{1}{{}_{n+1}F_n \left(\begin{matrix} \gamma, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+1} \end{matrix} \right)}, \quad x = 0, 1, 2, \dots,$$

from which we get

$$v_3(x + 1) = \frac{(\gamma + x)}{(x + 1) (\alpha + 2)} v_3(x) = \frac{(\gamma + x - 1)_2}{(x)_2 (\alpha + 2)^2} v_3(x - 1). \quad (52)$$

Using (50), (51) and (52), we obtain, for $x = 1, 2, \dots$

$$\begin{aligned} (x + 1) f_3(x + 1) &= (x + 1) v_3(x + 1) {}_{n+1}F_n \left(\begin{matrix} x + \gamma + 1, a, (a_i) \\ p, (p_i); \frac{1}{\alpha+2} \end{matrix} \right) \\ &= \frac{(x + 1)}{(\gamma + x)} v_3(x + 1) \{ [2(\gamma + x) - p] F(x + \gamma) + [p - (\gamma + x)] F(x + \gamma - 1) \} \\ &\quad + v_3(x + 1) \frac{(x + 1) \prod_{i=0}^{n-2} a_{n-i+1}}{(\gamma + x) \prod_{i=0}^{n-2} p_{n-i}} \left\{ \frac{\gamma + x}{\alpha + 2} {}_{n+1}F_n \left(\begin{matrix} x + \gamma + 1, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right. \\ &\quad \left. - \frac{\gamma + x - a}{\alpha + 2} {}_{n+1}F_n \left(\begin{matrix} x + \gamma, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right\} \\ &\quad = \frac{2(\gamma + x) - p}{\alpha + 2} v_3(x) F(x + \gamma) \\ &\quad + \frac{(x + \gamma - 1) [p - (\gamma + x)]}{x (\alpha + 2)^2} v_3(x - 1) F(x + \gamma - 1) \\ &\quad + \prod_{i=0}^{n-2} \frac{a_{n-i+1}}{p_{n-i}} \left\{ v_3(x + 1) \frac{x + 1}{\alpha + 2} {}_{n+1}F_n \left(\begin{matrix} x + \gamma + 1, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right. \\ &\quad \left. - \frac{\gamma + x - a}{(\alpha + 2)^2} v_3(x) {}_{n+1}F_n \left(\begin{matrix} x + \gamma, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right\} \\ &\quad = \frac{2(\gamma + x) - p}{\alpha + 2} f_3(x) + \frac{(x + \gamma - 1) [p - (\gamma + x)]}{x (\alpha + 2)^2} f_3(x - 1) \\ &\quad + \prod_{i=0}^{n-2} \frac{a_{n-i+1}}{p_{n-i}} \left\{ v_3(x + 1) \frac{x + 1}{\alpha + 2} {}_{n+1}F_n \left(\begin{matrix} x + \gamma + 1, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right. \\ &\quad \left. - \frac{\gamma + x - a}{(\alpha + 2)^2} v_3(x) {}_{n+1}F_n \left(\begin{matrix} x + \gamma, a, (1 + a_i) \\ p, (1 + p_i); \frac{1}{\alpha+2} \end{matrix} \right) \right\} \end{aligned}$$

and the proof is complete. ■

6. The fourth mixture distribution $f_4(x)$

In this section we obtain our last mixture distribution $f_4(x)$. This distribution is obtained by mixing a Poisson distribution $f_2(x/\lambda)$ defined before with a new generalized gamma distribution defined in (53). We begin by defining our last generalized gamma distribution $g_4(\lambda)$ as

$$g_4(\lambda) = \frac{\lambda^{\gamma-1}}{\Gamma(\gamma)} (\alpha+1)^\gamma e^{-(\alpha+1)\lambda} \frac{\Phi_1(a, p; c; \delta, \lambda)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}, \quad \lambda > 0 \quad (53)$$

This function is non-negative and satisfies the condition $\int_0^\infty g_4(\lambda) d\lambda = 1$ by virtue of the result (14). Thus (53) represents a continuous distribution of Appell and confluent hypergeometric function type, which yields for $\delta = 0$ and $\gamma = a$, the probability density function defined in [6]. As in previous section we let X to have a conditional Poisson distribution with parameter λ .

THEOREM 7. *The unconditional pmf of X is given by*

$$f_4(x) = \frac{(\gamma)_x (\alpha+1)^\gamma}{x! (\alpha+2)^{x+\gamma}} \frac{F_1\left(a, p, x+\gamma; c; \delta, \frac{1}{\alpha+2}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}, \quad x = 0, 1, 2, \dots, \quad (54)$$

whose characteristic function, for any real t , is given by

$$\Psi_X(t) = E[e^{itX}] = \left(\frac{\alpha+1}{\alpha+2-e^{it}}\right)^\gamma \frac{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+2-e^{it}}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}, \quad (55)$$

its factorial moment is

$$E[X(X-1)\dots(X-r+1)] = \frac{(\gamma)_r}{(\alpha+1)^r} \frac{F_1\left(a, p, \gamma+r; c; \delta, \frac{1}{\alpha+1}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}, \quad (56)$$

and its r -th moment is given by

$$E[X^r] = \sum_{n=0}^r S(r, n) \frac{(\gamma)_r}{(\alpha+1)^r} \frac{F_1\left(a, p, \gamma+r; c; \delta, \frac{1}{\alpha+1}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}. \quad (57)$$

P r o o f. From (53) and (34) the unconditional pmf of X is

$$\begin{aligned}
 f_4(x) &= \int_0^\infty f_2(x/\lambda) g_4(\lambda) d\lambda \\
 &= \frac{(\alpha + 1)^\gamma}{x! \Gamma(\gamma)} \frac{1}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)} \int_0^\infty \lambda^{x+\gamma-1} e^{-(\alpha+2)\lambda} \Phi_1(a, p; c; \delta, \lambda) d\lambda \\
 &= \frac{(\gamma)_x (\alpha + 1)^\gamma}{x! (\alpha + 2)^{x+\gamma}} \frac{F_1\left(a, p, x + \gamma; c; \delta, \frac{1}{\alpha+2}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}, \quad x = 0, 1, 2, \dots
 \end{aligned}$$

Using (35) the characteristic function of X is

$$\begin{aligned}
 \Psi_X(t) &= E [e^{itX}] = E [E [e^{itX} / \Lambda]] = E [e^{-\Lambda(1-e^{it})}] \\
 &= \left(\frac{\alpha + 1}{\alpha + 2 - e^{it}} \right)^\gamma \frac{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+2-e^{it}}\right)}{F_1\left(a, p, \gamma; c; \delta, \frac{1}{\alpha+1}\right)}.
 \end{aligned}$$

From (36) the factorial moment of X is

$$\begin{aligned}
 E [X (X - 1) \dots (X - r + 1)] &= E [E [X (X - 1) \dots (X - r + 1) / \Lambda]] = E [\Lambda^r] \\
 &= \frac{(\gamma)_r (b)_r}{(q)_r (\alpha + 2)^r} \frac{F_2\left(\gamma + r, a, b + r; p, q + r; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}{F_2\left(\gamma, a, b; p, q; \frac{1}{\alpha+2}, \frac{1}{\alpha+2}\right)}.
 \end{aligned}$$

Finally, the r -th moment of X is obtained as before. ■

Now we state our last result.

THEOREM 8. *The distribution of X satisfies a three-term recurrence relation*

$$\begin{aligned}
 (x + 1) f_4(x+1) &= \sum_{k=0}^\infty A_k \frac{(\gamma + x - 1)(c + k - \gamma - x)}{x (\alpha + 1)(\alpha + 2)} v_4(x - 1) F(x + \gamma - 1) \\
 &\quad + \sum_{k=0}^\infty \frac{A_k}{\alpha + 1} \left[\frac{a + k - \gamma - x}{\alpha + 2} + 2(\gamma + x) - (c + k) \right] v_4(x) F(x + \gamma),
 \end{aligned} \tag{58}$$

where

$$A_k = \frac{(a)_k (p)_k \delta^k}{(c)_k k!}. \tag{59}$$

P r o o f. Using the recurrence relation (9), for $b = x + \gamma$, $z = \frac{1}{\alpha+2}$, we get

$$(\gamma + x) F(x + \gamma + 1) = \frac{\alpha + 2}{\alpha + 1} \{ (c + k - \gamma - x) F(x + \gamma - 1) + \left[\frac{a + k - \gamma - x}{\alpha + 2} + 2(\gamma + x) - (c + k) \right] F(x + \gamma) \}, \quad (60)$$

where

$$F(\gamma + x) = {}_2F_1 \left(a + k, \gamma + x; c + k; \frac{1}{\alpha + 2} \right).$$

Rewrite $f_4(x)$, given by (54), as

$$f_4(x) = v_4(x) F_1 \left(a, p, x + \gamma; c; \delta, \frac{1}{\alpha + 2} \right), \quad x = 0, 1, 2, \dots, \quad (61)$$

where

$$F_1 \left(a, p, x + \gamma + 1; c; \delta, \frac{1}{\alpha + 2} \right) = \sum_{k=0}^{\infty} A_k {}_2F_1 \left(a + k, \gamma + x + 1; c + k; \frac{1}{\alpha + 2} \right) \quad (62)$$

and

$$v_4(x) = \frac{(\gamma)_x (\alpha + 1)^\gamma}{x! (\alpha + 2)^{x+\gamma}} \frac{1}{F_1 \left(a, p, \gamma; c; \delta, \frac{1}{\alpha + 1} \right)}, \quad x = 0, 1, 2, \dots,$$

from which we get

$$v_4(x + 1) = \frac{(\gamma + x)}{(x + 1) (\alpha + 2)} v_4(x) = \frac{(\gamma + x - 1)_2}{(x)_2 (\alpha + 2)^2} v_4(x - 1). \quad (63)$$

Using (60), (61) and (63), we obtain, for $x = 1, 2, \dots$

$$\begin{aligned} (x + 1) f_4(x + 1) &= (x + 1) v_4(x + 1) F_1 \left(a, p, x + \gamma + 1; c; \delta, \frac{1}{\alpha + 2} \right) \\ &= v_4(x + 1) \frac{(x + 1) (\alpha + 2)}{(\gamma + x) (\alpha + 1)} \sum_{k=0}^{\infty} A_k \{ [c + k - (\gamma + x)] F(x + \gamma - 1) \\ &\quad + \left[\frac{a + k - \gamma - x}{\alpha + 2} + 2(\gamma + x) - (c + k) \right] F(x + \gamma) \} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} A_k \frac{(\gamma + x - 1)(c + k - \gamma - x)}{x(\alpha + 1)(\alpha + 2)} v_4(x - 1) F(x + \gamma - 1) \\
&+ \sum_{k=0}^{\infty} \frac{A_k}{\alpha + 1} \left[\frac{a + k - \gamma - x}{\alpha + 2} + 2(\gamma + x) - (c + k) \right] v_4(x) F(x + \gamma),
\end{aligned}$$

which completes the proof. \blacksquare

R e m a r k (Added during proofreading ¹). For further results of this type, please see: Y. Ben Nakhi and S.L. Kalla, On a generalized mixture distribution. *Applied Mathematics and Computation* **169** (2005), 943-952.

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