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## CAUCHY PROBLEM FOR DIFFERENTIAL EQUATION WITH CAPUTO DERIVATIVE

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*Dedicated to Professor Ivan H. Dimovski  
on the occasion of his 70th birthday*

### Abstract

The paper is devoted to the study of the Cauchy problem for a nonlinear differential equation of complex order with the Caputo fractional derivative. The equivalence of this problem and a nonlinear Volterra integral equation in the space of continuously differentiable functions is established. On the basis of this result, the existence and uniqueness of the solution of the considered Cauchy problem is proved. The approximate-iterative method by Dzijadyk is used to obtain the approximate solution of this problem. Two numerical examples are given.

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*Key Words and Phrases:* differential equation of fractional order, Caputo derivative, existence and uniqueness theorem, approximate-iterative method

### 1. Introduction

Let  $I_{a+}^{\alpha}g$  and  $D_{a+}^{\alpha}y$  be the Riemann-Liouville fractional integrals and derivatives of a complex order  $\alpha \in \mathbf{C}$  ( $Re(\alpha) > 0$ ) on a finite interval  $[a, b]$  of the real line  $\mathbf{R} = (-\infty; \infty)$ :

$$(I_{a+}^{\alpha}g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g(t)dt}{(x-t)^{1-\alpha}}, \quad (\alpha \in \mathbf{C}, Re(\alpha) > 0), \quad (1.1)$$

$$(D_{a+}^{\alpha}y)(x) \equiv (D^n I_{a+}^{n-\alpha}y)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y(t)dt}{(x-t)^{1-n+\alpha}} \quad (1.2)$$

$$\left(n = [Re(\alpha)] + 1; D = \frac{d}{dx}\right),$$

where  $[Re(\alpha)]$  is an integer part of  $Re(\alpha)$  [14, §§ 2.3, 2.4]. Denote by  ${}^cD_{a+}^{\alpha}y$  a modified fractional derivative defined by

$$({}^cD_{a+}^{\alpha}y)(x) = \left(D_{a+}^{\alpha} \left[ y(x) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j \right] \right) (x) \quad (\alpha \in \mathbf{C}, Re(\alpha) > 0), \quad (1.3)$$

$$n = [Re(\alpha)] + 1 \text{ for } \alpha \notin \mathbf{N} = \{1, 2, \dots\}, \quad n = \alpha \text{ for } \alpha \in \mathbf{N}. \quad (1.4)$$

If  $\alpha > 0$ ,  $n - 1 < \alpha \leq n$  ( $n \in \mathbf{N}$ ) and  $y(x) \in C^n[a, b]$  is a function,  $n$  times continuously differentiable on  $[a, b]$ , then for  $\alpha = n \in \mathbf{N}$  the Caputo derivative  ${}^cD_{a+}^{\alpha}y$  coincides with the usual derivative of order  $n$ :

$$({}^cD_{a+}^n y)(x) = (D^n y)(x) \quad \left(n \in \mathbf{N}; D = \frac{d}{dx}\right), \quad (1.5)$$

while for  $n - 1 < \alpha < n$  the operator  ${}^cD_{a+}^{\alpha}$  is represented as a composition of the Riemann-Liouville fractional integration operator  $I_{a+}^{n-\alpha}$  and the differentiation operator  $D^n$ :

$$({}^cD_{a+}^{\alpha}y)(x) = (I_{a+}^{n-\alpha} D^n y)(x) \quad \left(n - 1 < \alpha < n, n \in \mathbf{N}; D = \frac{d}{dx}\right). \quad (1.6)$$

Expression (1.6) was introduced by Caputo [1] in connection with the problems of elasticity (see also [2]-[3]), and therefore the constructions (1.3) and (1.6) are called Caputo derivatives of order  $\alpha \in \mathbf{C}$ ; see [5], [11], [13, § 2.4.1].

Boundary value problems for the so-called differential equations of fractional order, in which an unknown function is under the sign of fractional derivative, were studied by many authors; see historical remarks and results in the monograph [14, §§ 42 – 43] and in the survey paper [8]. The interest to such problems is arisen by their applications in problems of physics, mechanics and other applied sciences; see [5], [12], [13]. Related to the above problems, boundary value problems with the Riemann-Liouville fractional derivative (1.2) were more extensively investigated (see for example [14, §§ 42], [8]), while problems involving the Caputo fractional derivative (1.3) were studied less.

The present paper deals with investigation of the Cauchy problem for the nonlinear differential equation of order  $\alpha \in \mathbf{C}$  ( $Re(\alpha) > 0$ ):

$$({}^c D_{a+}^\alpha y)(x) = f[x, y(x)] \quad (\alpha \in \mathbf{C}, Re(\alpha) > 0) \quad (1.7)$$

with the initial conditions

$$y^{(k)}(a) = b_k \quad b_k \in \mathbf{C} \quad (k = 0, 1, \dots, n - 1; n = [Re(\alpha)] + 1). \quad (1.8)$$

When  $\alpha = m \in \mathbf{N}$ , then in accordance with (1.5), the problem (1.7)-(1.8) takes the form of the Cauchy problem for the ordinary differential equation of order  $m$ :

$$y^{(m)}(x) = f[x, y(x)], \quad y^{(k)}(a) = b_k \in \mathbf{C} \quad (k = 0, 1, \dots, m - 1), \quad (1.9)$$

which is well studied; for example, see [15, p. 113-121]. Note also that the Cauchy-type problem for the model nonlinear fractional differentiation of the form (1.7), in which the Caputo derivative  ${}^c D_{a+}^\alpha y$  is replaced by the Riemann-Liouville fractional derivative  $D_{a+}^\alpha y$ , in the weighted space  $C_{n-\alpha}[a, b]$  of continuous functions  $y(x)$  such that  $(x - a)^\alpha y(x) \in C[a, b]$ , was investigated in [7]. In particular, when  $\alpha \in \mathbf{N}$ , the problem considered was reduced to the Cauchy problem (1.9), and the existence and uniqueness result for such a problem in the space  $C^n[a, b]$  was proved.

We study the Cauchy problem (1.7)-(1.8) with complex  $\alpha \in \mathbf{C}$  ( $Re(\alpha) > 0$ ,  $\alpha \notin \mathbf{N}$ ), in a Banach space  $C^{n-1}[a, b]$  of  $n - 1$  times continuously differentiable functions:

$$C^{n-1}[a, b] = \left\{ g : \|g\|_{C^{n-1}} = \sum_{k=0}^{n-1} \|g^{(k)}\|_C, \quad n = [Re(\alpha)] + 1 \right\}, \quad (1.10)$$

$$C^0[a, b] = C[a, b],$$

provided that a function  $f[x, y]$  maps from  $[a, b] \times Y$  ( $Y \subset \mathbf{R}$ ) into  $\mathbf{R}$ , and for any fixed  $y \in \bar{Y}$  it is continuous function with respect to  $x$  on  $[a, b]$ . First we prove the existence of a unique solution  $y(x) \in C^{n-1}[a, b]$  of the Cauchy problem (1.7)-(1.8) on the basis of the equivalence of this problem and the nonlinear Volterra integral equation

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} (x - a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]}{(x - t)^{1-\alpha}} dt \quad (Re(\alpha) > 0, \alpha \notin \mathbf{N}). \quad (1.11)$$

Then we apply the approximate-iterative method (AI-method) by Dzyadyk [4] to deduce the approximate solution of (1.7)-(1.8).

The paper is organized as follows. In Section 2 we use properties of the Riemann-Liouville fractional integrals and derivatives (1.1) and (1.2) to prove the equivalence of the Cauchy problem (1.7)-(1.8) and the nonlinear Volterra integral equation (1.11) in the space  $C^{n-1}[a, b]$ . Using this fact and applying the method of successive approximation, in Section 3 we establish the existence of a unique solution of (1.7)-(1.8). Section 4 and 5 deal with application of Dzyadyk's method to construct the approximate solution of the Cauchy problem (1.7)-(1.8) with  $\alpha > 1$ , and to derive the estimate between the exact and approximate solutions. Numerical results are given in Section 6.

## 2. Equivalence of Cauchy problem and Volterra integral equation

In this section we prove that the Cauchy problem (1.7)-(1.8) and the Volterra integral equation (1.12) are equivalent in the space  $C^{n-1}[a, b]$ , in the sense that if  $y(x) \in C^{n-1}[a, b]$  satisfies one of these relations, it also satisfies the other one.

We need the following auxiliary assertion.

LEMMA 1. *If  $\alpha \in \mathbf{C}$  ( $\operatorname{Re}(\alpha) > 0$ ,  $\alpha \notin \mathbf{N}$ ) and  $n = [\operatorname{Re}(\alpha)] + 1$ , then the Riemann-Liouville fractional integration operator  $I_{a+}^{\alpha}$  is bounded from  $C[a, b]$  to  $C^{n-1}[a, b]$ :*

$$\|I_{a+}^{\alpha}g\|_{C^{n-1}} \leq K\|g\|_C, \quad K = \sum_{k=0}^{n-1} \frac{(b-a)^{\operatorname{Re}(\alpha)-k}}{|\Gamma(\alpha-k)|[\operatorname{Re}(\alpha)-k]}. \quad (2.1)$$

P r o o f. Let  $g(t) \in C[a, b]$ . By [7, Corollary 1],  $D^k I_{a+}^{\alpha}g = I_{a+}^{\alpha-k}g$  for  $k = 0, 1, \dots, n-1$ . Using (1.10) and taking [6, Lemma 11] (see also [7, (3.11)]) into account, we have for any  $x \in [a, b]$ :

$$\|I_{a+}^{\alpha}g\|_{C^{n-1}} = \sum_{k=0}^{n-1} \|I_{a+}^{\alpha-k}g\|_C \leq \|g\|_C \sum_{k=0}^{n-1} \frac{(b-a)^{\operatorname{Re}(\alpha)-k}}{|\Gamma(\alpha-k)|[\operatorname{Re}(\alpha)-k]},$$

which yields (2.1). ■

**THEOREM 1.** Let  $\alpha \in \mathbf{C}$  ( $Re(\alpha) > 0$ ,  $\alpha \notin \mathbf{N}$ ), and  $n = [Re(\alpha)] + 1$ . Let  $f[x, y] : [a, b] \times Y \rightarrow \mathbf{R}$  ( $Y \subset \mathbf{R}$ ) be a continuous function with respect to  $x$  on  $[a, b]$  for any fixed  $y \in \bar{Y}$ .

If  $y(x) \in C^{n-1}[a, b]$ , then  $y(x)$  satisfies the relations (1.7) and (1.8) if and only if  $y(x)$  satisfies the Volterra integral equation (1.11).

**P r o o f.** First we prove *Necessity*: Let  $y(x) \in C^{n-1}[a, b]$  satisfy (1.7)-(1.8). Since  $f[x, y] \in C[a, b]$  for any  $y \in \bar{Y}$ , then (1.7) means that there exists  $({}^c D_{a+}^\alpha y)(x) \in C[a, b]$ . By (1.2) and (1.3)

$$({}^c D_{a+}^\alpha y)(x) = \left( \frac{d}{dx} \right)^n \left( I_{a+}^{n-\alpha} \left[ y(t) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j \right] \right) (x), \quad (2.2)$$

where  $n = [Re(\alpha)] + 1$ , and therefore on the basis of [6, Lemma 2] (see also [7, Lemma 3] with  $\gamma = 0$ ),

$$\left( I_{a+}^{n-\alpha} \left[ y(t) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j \right] \right) (x) \in C^n[a, b].$$

Applying [6, Lemma 5] (see also [7, Corollary 3 with  $\gamma = 0$ ]) to  $g(t) = y(t) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j$  and using the formula (3.17) from [7], we find

$$\begin{aligned} (I_{a+}^\alpha {}^c D_{a+}^\alpha y)(x) &= \left( I_{a+}^\alpha D_{a+}^\alpha \left[ y(t) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j \right] \right) (x) \\ &= y(x) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (x-a)^j - \sum_{k=1}^n \frac{y_{n-\alpha}^{(n-k)}(a)}{\Gamma(\alpha - k + 1)} (x-a)^{\alpha-k}, \end{aligned} \quad (2.3)$$

where

$$y_{n-\alpha}(x) = \left( I_{a+}^{n-\alpha} \left[ y(t) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (t-a)^j \right] \right) (x). \quad (2.4)$$

Integrating by parts in (2.3) and then differentiating the equation obtained, and using the relation (3.16) from [6, Lemma 4] (see also [7, Corollary 1] with  $\beta = 1$ ), we have

$$y'_{n-\alpha}(x) = \frac{d}{dx} \left( I_{a+}^{n+1-\alpha} \left[ y'(t) - \sum_{j=1}^{n-1} \frac{y^{(j)}(a)}{(j-1)!} (t-a)^{j-1} \right] \right) (x)$$

$$= \left( I_{a+}^{n-\alpha} \left[ y'(t) - \sum_{j=1}^{n-1} \frac{y^{(j)}(a)}{(j-1)!} (t-a)^{j-1} \right] \right) (x).$$

Repeating this process  $(n-k)$  ( $k = 0, 1, \dots, n-1$ ) times, we come to the formula

$$y_{n-\alpha}^{(n-k)}(x) = \left( I_{a+}^{n-\alpha} \left[ y^{(n-k)}(t) - \sum_{j=n-k}^{n-1} \frac{y^{(j)}(a)}{(j-n+k)!} (t-a)^{j-n+k} \right] \right) (x). \quad (2.5)$$

Changing the variable  $t = a + s(x-a)$ , we obtain for  $k = 1, 2, \dots, n$

$$y_{n-\alpha}^{(n-k)}(x) = \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha)} \int_0^1 (1-s)^{n-\alpha-1} \left\{ y^{(n-k)}[a+s(x-a)] - \sum_{j=n-k}^{n-1} \frac{y^{(j)}(a)}{(j-n+k)!} [s(x-a)]^{j-n+k} \right\} ds. \quad (2.6)$$

Since  $\operatorname{Re}(\alpha) < n$  and  $y^{(n-k)}(x) \in C[a, b]$  ( $k = 1, \dots, n$ ), then from (2.6) it follows that  $y_{n-\alpha}^{(n-k)}(a) = 0$  ( $k = 1, \dots, n$ ), and hence (2.3) takes the form

$$(I_{a+}^{\alpha} {}^c D_{a+}^{\alpha} y)(x) = y(x) - \sum_{j=0}^{n-1} \frac{y^{(j)}(a)}{j!} (x-a)^j. \quad (2.7)$$

Let now

$$M = \max_{(x,y) \in [a,b] \times \bar{Y}} |f[x, y]| < \infty. \quad (2.8)$$

By Lemma 1,  $I_{a+}^{\alpha} f[t, y(t)] \in C^{n-1}[a, b]$  and, in accordance with (2.1), there holds the estimate

$$|I_{a+}^{\alpha} f[t, y(t)]|_{C^{n-1}[a,b]} \leq M \sum_{k=0}^{n-1} \frac{(b-a)^{\operatorname{Re}(\alpha)-k}}{|\Gamma(\alpha-k)|[\operatorname{Re}(\alpha)-k]}. \quad (2.9)$$

Applying the operator  $I_{a+}^{\alpha}$  to the both sides of (1.7) and using (2.7) and (1.8), we obtain the equation (1.11), and hence necessity is proved.

Now we prove *Sufficiency*: Let  $y(x) \in C^{n-1}[a, b]$  satisfies the equation (1.11). Show that  $y(x)$  satisfies the initial relations (1.8). Differentiating

both sides of (1.11) and using [6, Lemma 4] (see also [7, Corollary 1]) we have ( $k = 1, \dots, n - 1$ ):

$$y^{(k)}(x) = \sum_{j=k}^{n-1} \frac{b_j}{(j-k)!} (x-a)^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{1-\alpha+k}} dt. \quad (2.10)$$

Making the change of variable  $t = a + s(x-a)$  in integrals of (1.11) and (2.10), we find for  $k = 0, 1, \dots, n - 1$ :

$$y^{(k)}(x) = \sum_{j=k}^{n-1} \frac{b_j}{(j-k)!} (x-a)^{j-k} + \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k)} \int_0^1 \frac{f[a+s(x-a), y(a+s(x-a))]}{(1-s)^{1-\alpha+k}} ds.$$

Taking a limit, as  $x \rightarrow a+$ , and using the continuity of  $f$ , we come to the relations (1.8).

Applying the operator  $D_{a+}^\alpha$  to both sides of (1.11), taking into account [6, Lemma 4], the initial conditions (1.8), [6, Lemma 4] (see also [7, Corollary 1]) and (1.3), we obtain the relation (1.7). Thus sufficiency is proved which completes the proof of theorem. ■

**COROLLARY 1.** *Let  $\alpha \in \mathbf{C}$ ,  $0 < Re(\alpha) < 1$ , and let  $f[x, y] : [a, b] \times Y \rightarrow \mathbf{R}$  ( $Y \subset \mathbf{R}$ ) be a continuous function with respect to  $x$  on  $[a, b]$  for any fixed  $y \in \bar{Y}$ .*

*If  $y(x) \in C^{n-1}[a, b]$ , then  $y(x)$  satisfies the relations*

$$({}^c D_{a+}^\alpha y)(x) = f[x, y(x)] \quad (0 < Re(\alpha) < 1), \quad y(a) = b \in \mathbf{C} \quad (2.11)$$

*if only if  $y(x)$  satisfies the Volterra integral equation*

$$y(x) = b + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{1-\alpha}} dt \quad (0 < Re(\alpha) < 1). \quad (2.12)$$

### 3. Existence and uniqueness of the solution of the Cauchy problem and connection between the exact and approximate solutions

In this section we establish the existence and uniqueness of the solution of the Cauchy problem (1.7)-(1.8) in the space of functions  $\mathbf{C}^{n-1, \alpha}[a, b]$  defined by

$$\mathbf{C}^{n-1, \alpha}[a, b] = \{y \in C^{n-1}[a, b], \quad {}^c D_{a+}^\alpha y \in C[a, b], \quad n = [Re(\alpha)] + 1\}, \quad (3.1)$$

under conditions of Theorem 1 and an additional Lipschitz condition of  $f[x, y]$  with respect to  $y$ : there exists  $L > 0$  such that for any  $x \in [a, b]$  and any  $y_1, y_2 \in Y$  there holds the inequality

$$|f[x, y_1] - f[x, y_2]| \leq L|y_1 - y_2| \quad (L > 0). \quad (3.2)$$

**THEOREM 2.** *Let  $\alpha \in \mathbf{C}$  ( $\operatorname{Re}(\alpha) > 0$ ,  $\alpha \notin \mathbf{N}$ ),  $n = [\operatorname{Re}(\alpha)] + 1$ , let  $f[x, y] : [a, b] \times Y \rightarrow R$  ( $Y \subset R$ ) satisfy the conditions of Theorem 1, and let the relation (3.2) hold.*

*Then there exists a unique solution  $y(x)$  of the Cauchy problem (1.7)-(1.8) in the space  $C^{n-1, \alpha}[a, b]$ .*

**P r o o f.** First we show that there exists a unique solution  $y(x) \in C^{n-1}[a, b]$  of (1.7)-(1.8). By Theorem 1 it is sufficient to prove the existence of an unique solution  $y(x) \in C^{n-1}[a, b]$  of the nonlinear Volterra integral equation (1.11). To this end we use the known method for nonlinear Volterra integral equations; for example, see [9]. The equation (1.11) has a sense in any interval  $[a, x_1] \subset [a, b]$ . Choose  $x_1$  such that there holds the inequality

$$L \frac{(x_1 - a)^{\operatorname{Re}(\alpha)}}{|\Gamma(\alpha)| \operatorname{Re}(\alpha)} < 1 \quad (3.3)$$

and prove the existence of an unique solution  $y(x) \in C^{n-1}[a, x_1]$  of the equation (1.11) on the interval  $[a, x_1]$ . For this we apply the method of successive approximations and set

$$y_0(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} (x - a)^j, \quad (3.4)$$

$$y_\nu(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_{\nu-1}(t)]}{(x-t)^{1-\alpha}} dt \quad (\nu \in \mathbf{N}). \quad (3.5)$$

We show that  $y_\nu(x) \in C^{n-1}[a, b]$ . From (3.4) it follows that  $y_0(x) \in C^{n-1}[a, b]$ . Differentiating (3.5)  $k$  ( $k = 1, \dots, n-1$ ) times and using [6, Lemma 4] (see also [7, Corollary 1]), we have

$$y_\nu^{(k)}(x) = y_0^{(k)}(x) + \frac{1}{\Gamma(\alpha - k)} \int_a^x \frac{f[t, y_{\nu-1}(t)] dt}{(x-t)^{1-(\alpha-k)}} \quad (k = 1, \dots, n-1; \nu \in \mathbf{N}). \quad (3.6)$$



By (3.4) and Lemma 1,  $y_\nu(x) \in C^{n-1}[a, b]$ .

Now we estimate  $\|y_\nu(x) - y_{\nu-1}(x)\|_{C^{n-1}}$  for  $m \in \mathbf{N}$ . By (3.4) and (3.5), using the inequality (2.1) and the relation (2.8), we find

$$\|y_1(t) - y_0(t)\|_{C^{n-1}[a, x_1]} \leq M \sum_{k=0}^{n-1} \frac{(x_1 - a)^{Re(\alpha-k)}}{|\Gamma(\alpha - k)|Re(\alpha - k)}. \tag{3.7}$$

Using again (2.1) and taking into account (3.2) and (3.7), we deduce

$$\begin{aligned} \|y_2(x) - y_1(x)\|_{C^{n-1}[a, x_1]} &= \sum_{k=0}^{n-1} \left\| I_{a+}^{\alpha-k} (f[x, y_1(x)] - f[x, y_0(x)]) \right\|_{C[a, x_1]} \\ &\leq \sum_{k=0}^{n-1} \frac{(x_1 - a)^{Re(\alpha-k)}}{|\Gamma(\alpha - k)|Re(\alpha - k)} \|y_1(x) - y_0(x)\|_{C[a, x_1]} \\ &\leq LM \sum_{k=0}^{n-1} \frac{(x_1 - a)^{Re(\alpha-k)}}{|\Gamma(\alpha - k)|Re(\alpha - k)} \frac{(x_1 - a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)}. \end{aligned}$$

Repeating such an estimate  $\nu$  times, we arrive at the inequality

$$\|y_\nu - y_{\nu-1}\|_{C^{n-1}[a, x_1]} \leq M \sum_{k=0}^{n-1} \frac{(x_1 - a)^{Re(\alpha-k)}}{|\Gamma(\alpha - k)|Re(\alpha - k)} \left( L \frac{(x_1 - a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^{\nu-1}.$$

By (3.3), from here it follows that the sequence  $\{y_\nu(x)\}$  tends to a certain limit function  $y(x) \in C^{n-1}[a, x_1]$ :

$$\lim_{\nu \rightarrow \infty} \|y_\nu(x) - y(x)\|_{C^{n-1}[a, x_1]} = 0. \tag{3.8}$$

By (2.1) (with  $b = x_1$  and  $g(t)$  being replaced by  $f[t, y_\nu(t)] - f[t, y(t)]$ ) and the Lipschitz condition (3.2), we have

$$\begin{aligned} &\left\| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_\nu(t)] dt}{(x-t)^{1-\alpha}} - \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}} \right\|_{C^{n-1}[a, x_1]} \\ &\leq L \sum_{k=0}^{n-1} \frac{(x_1 - a)^{Re(\alpha-k)}}{|\Gamma(\alpha - k)|Re(\alpha - k)} \|y_\nu(x) - y(x)\|_{C^{n-1}[a, x_1]}, \end{aligned}$$

and hence

$$\lim_{\nu \rightarrow \infty} \left\| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_\nu(t)] dt}{(x-t)^{1-\alpha}} - \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}} \right\|_{C^{n-1}[a, x_1]} = 0. \quad (3.9)$$

It follows from (3.8) and (3.9) that  $y(x)$  is the solution of the equation (1.11) in the space  $C^{n-1}[a, x_1]$ .

Now show that this solution  $y(x)$  is a unique. We suppose that there exist two solutions  $y_1(x)$   $y_2(x)$  of the equation (1.11) on  $[a, x_1]$ . Substituting them into (1.11) and applying (2.1) and (3.2), we have

$$|y_1(x) - y_2(x)| = |I_{a+}^\alpha (f[t, y_1(t)] - f[t, y_2(t)])| \leq L \frac{(x_1 - a)^{Re(\alpha)}}{|\Gamma(\alpha)| Re(\alpha)} |y_1(t) - y_2(t)|. \quad (3.10)$$

This relation yields

$$L \frac{(x_1 - a)^{Re(\alpha)}}{|\Gamma(\alpha)| Re(\alpha)} \geq 1,$$

which contradicts the assumption (3.3). Thus there exists a unique solution  $y(x) = y_1(x) \in C^{n-1}[a, x_1]$  on the interval  $[a, x_1]$ .

Next consider the interval  $[x_1, x_2]$ , where  $x_2 = x_1 + h_1$ , and  $h_1 > 0$  are such that  $x_2 < b$ . Rewrite the equation (1.11) in the form

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^x \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}} + \sum_{k=0}^{n-1} \frac{b_k}{k!} (x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^{x_1} \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}}, \quad x \in [x_1, x_2]. \quad (3.11)$$

Since the function  $y(x)$  is uniquely defined on the interval  $[a, x_1]$ , the last integral can be considered as the known function, and we rewrite the last equation in the form

$$y(x) = y_0^*(x) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^x \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}}, \quad (3.12)$$

where

$$y_0^*(x) = \sum_{k=0}^{n-1} \frac{b_k}{k!} (x-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^{x_1} \frac{f[t, y(t)] dt}{(x-t)^{1-\alpha}} \quad (3.13)$$

is the known function. Taking the same arguments as above, we deduce that there exist a unique solution  $y(x) = y_2(x) \in C^{n-1}[x_1, x_2]$  of the equation

(1.11) on the interval  $[x_1, x_2]$ . Taking the next interval  $[x_2, x_3]$ , where  $x_3 = x_2 + h_2$  and  $h_2 > 0$  are such that  $x_3 < b$ , and repeating this process, we obtain that there exists a unique solution  $y(x) \in C^{n-1}[a, b]$  of the equation (1.11) such that  $y(x) = y_k(x) \in C^{n-1}[x_{k-1}, x_k]$  ( $k = 1, 2, \dots, l$ ), where  $a = x_1 < x_2 < \dots < x_l = b$ .

Thus, there exists a unique solution  $y(x) \in C^{n-1}[a, b]$  of the equation (1.11).

To prove that this solution  $y(x) \in C^{n-1,\alpha}[a, b]$ , by (3.1) it is sufficient to show that  ${}^c D_{a+}^\alpha y \in C[a, b]$ . In accordance with (3.8), there holds the relation

$$\lim_{\nu \rightarrow \infty} \|y_\nu(x) - y(x)\|_{C^{n-1}[a,b]} = 0. \tag{3.14}$$

Using (1.7) and the Lipschitz condition (3.2), we obtain

$$\begin{aligned} \|({}^c D_{a+}^\alpha y_\nu)(x) - ({}^c D_{a+}^\alpha y)(x)\|_{C[a,b]} &= \|f[x, y_\nu(x)] - y[x, y(x)]\|_{C[a,b]} \\ &\leq L \|y_\nu(x) - y(x)\|_{C[a,b]}. \end{aligned}$$

Hence, by (3.14),  ${}^c D_{a+}^\alpha y \in C[a, b]$ .

Thus, there exists a unique solution  $y(x)$  of the integral equation (1.11) and hence of the Cauchy problem (1.7)-(1.8) in the space in the space  $C^{n-1,\alpha}[a, b]$ . This completes the proof of theorem. ■

Now we give an estimate between the exact solution  $y(x)$  of the Cauchy problem (1.7)-(1.8) and the approximate functions  $y_m(x)$  given in (3.5).

**THEOREM 3.** *Let  $\alpha \in C$  ( $Re(\alpha) > 0$ ,  $\alpha \notin \mathbf{N}$ ),  $n = [Re(\alpha)] + 1$ , let a function  $f[x, y] : [a, b] \times Y \rightarrow \mathbf{R}$  ( $Y \subset \mathbf{R}$ ) satisfy the conditions of Theorem 2, and let the inequality*

$$L \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)| Re(\alpha)} < 1 \tag{3.15}$$

be valid.

Then any function  $y_\nu(x)$  ( $\nu \in \mathbf{N}$ ) defined by (3.5), approximates the solution  $y(x)$  in such a way that there holds the estimate

$$|y(x) - y_\nu(x)| \leq \left( \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)| Re(\alpha)} \right)^{\nu+1} \frac{ML^\nu}{1 - L \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)| Re(\alpha)}}, \tag{3.16}$$

where  $M$  and  $L$  are given in (2.8) and (3.2), respectively.

P r o o f. By (3.5) and (2.9) (for  $n = 1$ ), we have

$$|y_1(x) - y_0(x)| \leq \frac{M(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)}.$$

Using this estimate and (3.2), we find

$$|y_2(x) - y_1(x)| \leq \left| \frac{L}{\Gamma(\alpha)} \int_a^x \frac{y_1(t) - y_0(t)}{(x-t)^{1-\alpha}} dt \right| \leq LM \left( \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^2.$$

Continuing this process, for any  $\nu \in \mathbf{N}$  we obtain the estimate:

$$|y_\nu(x) - y_{\nu-1}(x)| \leq M \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \left( L \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^{\nu-1} \quad (\nu \in \mathbf{N}). \quad (3.17)$$

Applying (3.17) and taking the condition (3.15) into account, for any  $x \in [a, b]$  we have

$$\begin{aligned} |y(x) - y_\nu(x)| &= \lim_{k \rightarrow \infty} |y_{\nu+k}(x) - y_\nu(x)| \\ &= |[y_{\nu+1}(x) - y_\nu(x)] + [y_{\nu+2}(x) - y_{\nu+1}(x)] + \dots| \\ &\leq ML^\nu \left( \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^{\nu+1} + ML^{\nu+1} \left( \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^{\nu+2} + \dots \\ &= ML^\nu \left( \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^{\nu+1} \left[ 1 + L \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} + \left( L \frac{(b-a)^{Re(\alpha)}}{|\Gamma(\alpha)|Re(\alpha)} \right)^2 + \dots \right]. \end{aligned}$$

From here we deduce the estimate (3.16), which completes the proof of theorem.  $\blacksquare$

From (3.16) it follows that  $y_m(x)$  converges sufficiently fast to the solution  $y(x)$  of the Cauchy problem (1.7)-(1.8). However, as a rule, the iterative process given by (3.5) is difficult to use for an effective construction of  $y_m(x)$  because of an integration operation.

In the next two sections we present a method without an integration operation, which under the conditions on  $f[x, y]$  given in Section 2, allows us to obtain almost the same results that can be deduced by the method of successive approximations.

**4. AI-method. Auxiliary results**

This and next sections deal with an approximate solution of the Cauchy problem for the fractional differential equation (1.7) of order  $\alpha > 1$  with the initial conditions (1.8). For this we apply the iterative-approximate method developed in [4, p. 98-120], to an approximate solution of the nonlinear Volterra integral equation (1.11). To this end, we rewrite the equation (1.11) in the form

$$y(x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_a^x f[t, y(t)]K(x, t)dt \quad (\alpha > 1), \tag{4.1}$$

where

$$g(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} (x - a)^j, \tag{4.2}$$

$$K(x, t) = (x - t)^{\alpha-1}. \tag{4.3}$$

In this section we present some auxiliary results. We shall use standard Lagrange interpolation polynomials  $L_n$ , obtained by the mapping of standard Lagrange polynomials  $L_n^*$ , constructed at interpolation points

$$\xi_j^{(l)} = -\cos \frac{j\pi}{l} \quad (j = 0, \dots, l),$$

from  $[-1, 1]$  into  $[a, b]$ ; see [10, p.399]. This mapping is carried out by the formula

$$t = a + \frac{(b - a)(1 + \xi)}{2},$$

and therefore  $L_n$  is an interpolation polynomial constructed at interpolation points

$$t_j^{(l)} = a + \frac{(b - a)(1 + \xi_j^{(l)})}{2} \quad (j = 0, \dots, l). \tag{4.4}$$

It is known [4, p.111] the estimate

$$\frac{2}{\pi} \ln(n - 1) \leq \|L_n^*\| \leq \frac{2}{\pi} \ln n + 1, \tag{4.5}$$

and the relation

$$\|L_n\| = \|L_n^*\|. \tag{4.6}$$

Since the interpolation points  $\xi_j^{(l)} = -\cos \frac{j\pi}{l}$  ( $j = 0, \dots, l$ ) are roots of the polynomial

$$\overset{\circ}{U}_{l+1}(x) = (1-x^2)U_{l-1}(x) = \sqrt{1-x^2} \sin(l \arccos x),$$

where  $U_{l-1}(x)$  is the Chebyshev polynomial of second kind of power  $l-1$  [4, (I.1.27)], then the fundamental polynomials  $l_j^{*(l)}(\xi)$  can be represented in the form [4, (I.3.20)]

$$\begin{aligned} l_j^{*(l)}(\xi) &= l_j^{*(l)}(l; \xi) = \frac{\overset{\circ}{U}_{l+1}(\xi)}{(\xi - \xi_j^{(l)}) \overset{\circ}{U}'_{l+1}(\xi_j^{(l)})} \\ &= \frac{\varepsilon_j}{l} \left[ 1 - (-1)^{l-j} T_l(\xi) + 2 \sum_{\nu=1}^l (-1)^\nu \frac{\cos(j\nu\pi)}{l} T_\nu(\xi) \right], \end{aligned} \quad (4.7)$$

where  $\varepsilon_0 = \varepsilon_l = \frac{1}{2}$  and  $\varepsilon_j = 1$  for any  $j = 1, 2, \dots, l-1$  and  $T_\nu(\xi)$  is the Chebyshev polynomial of first kind.

On the basis of these fundamental polynomials for fixed natural  $l \in \mathbf{N}$  and  $m \in \mathbf{N}$  we construct the matrix of numbers

$$a_{ij}^{(l,m)} = \int_{-1}^{\xi_i^{(l)}} l_j^{*(l)}(\xi) d\xi \quad (i = 0, \dots, l; j = 0, \dots, m),$$

see explicit expressions below in Lemma 3.

Denote by  $A$  an integral operator in the right-hand side of (4.1):

$$(Ay)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f[t, y(t)] K(x, t) dt. \quad (4.8)$$

Suppose that  $K(x, t) \equiv 0$  for  $x \leq t$ , and define the interpolation polynomial operator  $\bar{A}$  by

$$(\bar{A}y)(x) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} \sum_{j=0}^m f \left[ t_j^{(m)}, y \left( t_j^{(m)} \right) \right] K \left( t_i^{(l)}, t_j^{(m)} \right) l_j^{(m)} dt \cdot l_i^{(l)}(x). \quad (4.9)$$

Here the interpolation points  $t_i^{(l)}$  and  $t_j^{(m)}$  are defined by (4.4), while  $l_i^{(l)}(x)$  and  $l_j^{(m)}$  are fundamental Lagrange polynomials at points of interpolation  $t_i^{(l)}$  and  $t_j^{(m)}$ , respectively.

Using the iterative process with respect to  $\nu$ , construct functions  $\bar{y}_\nu(x)$  of the form

$$\bar{y}_0 = g(x), \quad \bar{y}_\nu = g(x) + \bar{A}\bar{y}_{\nu-1}, \quad \nu = 1, 2, \dots \quad (4.10)$$

LEMMA 2.  $\bar{y}_\nu(x)$  are algebraic polynomials of the form

$$\begin{aligned} \bar{y}_0(x) = g(x), \bar{y}_\nu(x) = g(x) + \frac{b-a}{2\Gamma(\alpha)} \sum_{i=0}^l \sum_{j=0}^m a_{ij}^{(l,m)} \\ \times f \left[ t_j^{(m)}, \bar{y}_{\nu-1} \left( t_j^{(m)} \right) \right] K \left( x_i^{(l)}, t_j^{(m)} \right) l_i^{(l)}(x), \end{aligned} \quad (4.11)$$

where  $l_i^{(l)}$  and  $l_j^{(m)}$  are fundamental Lagrange polynomials at interpolation points  $t_i^{(l)}$  and  $t_j^{(m)}$ , respectively.

P r o o f. For  $\nu = 0$  the lemma follows from (4.10) and (4.2).

Let  $\nu \in \mathbf{N}$ . Making the change of variable  $t = a + \frac{b-a}{2}(\xi + 1)$ , and using the notation  $l_j^{(m)}(t) := l_j^{*(m)} \left[ -1 + \frac{2}{b-a}(t-a) \right]$ , we have

$$\begin{aligned} \int_a^{t_i^{(l)}} l_j^{(m)}(t) dt &= \int_a^{t_i^{(l)}} l_j^{*(m)} \left[ -1 + \frac{2}{b-a}(t-a) \right] dt \\ &= \frac{b-a}{2} \int_{-1}^{\xi_i^{(l)}} l_j^{*(m)}(\xi) d\xi = \frac{b-a}{2} a_{ij}^{(l,m)} \quad (i = 0, \dots, l; j = 0, \dots, m). \end{aligned} \quad (4.12)$$

Thus lemma is proved. ■

The next assertion yields the explicit form for  $a_{ij}^{(l,m)}$ .

LEMMA 3. Let  $l, m \in \mathbf{N}$  ( $i = 0, \dots, l; j = 0, \dots, m$ ). Then

$$\begin{aligned} a_{ij}^{(l,m)} = \frac{\varepsilon_j}{m} \left\{ 1 - C_i^{(l)} + \frac{1}{2} C_j^{(m)} (1 - C_{2i}^{(l)}) \right. \\ \left. + \sum_{\nu=2}^m \varepsilon_\nu C_{j\nu}^{(m)} \left[ \frac{C_{(\nu-1)i}^{(l)}}{\nu-1} - \frac{C_{(\nu+1)i}^{(l)}}{\nu+1} - \frac{2}{\nu^2-1} \right] \right\}, \end{aligned}$$

where  $\varepsilon_0 = \varepsilon_m = \frac{1}{2}$ ,  $\varepsilon_\nu = 1$  for  $\nu = 1, \dots, m-1$ , and  $C_k^{(s)} = \cos\left(\frac{k\pi}{s}\right)$ .

P r o o f. Using (4.7) and taking into account that for any  $\nu = 2, 3, \dots$  there holds the equality [4, p.105]

$$\int_0^x T_\nu(\xi) d\xi = \int_{\arccos x}^{\frac{\pi}{2}} \cos(\nu s) \sin(s) ds = \frac{1}{2} \left\{ \frac{T_{\nu+1}(x)}{\nu+1} - \frac{T_{\nu-1}(x)}{\nu-1} \right\} + \bar{c}_\nu,$$

$\bar{c}_\nu = \text{const}$ , we have

$$\begin{aligned}
a_{ij}^{(l,m)} &= \int_{\xi_i^{(l)}}^{\xi_i^{(l)}} l_j^{*(m)}(\xi) d\xi = \frac{\varepsilon_j}{m} \left\{ \xi - \xi^2 \cos\left(\frac{j\pi}{m}\right) \right. \\
&+ \sum_{\nu=2}^m (-1)^\nu \frac{\cos(j\pi\nu)}{m} \left( \frac{T_{\nu+1}(\xi)}{\nu+1} - \frac{T_{\nu-1}(\xi)}{\nu-1} \right) \\
&\quad \left. - \frac{(-1)^{m-j}}{2} \left[ \frac{T_{m+1}(\xi)}{m+1} - \frac{T_{m-1}(\xi)}{m-1} \right] \right\} \Big|_{\xi_i^{(l)}}^{\xi_i^{(l)}} \\
&= \left\{ 1 - \cos\frac{i\pi}{l} - \cos\left(\frac{j\pi}{m}\right) \left( \cos^2\left(\frac{i\pi}{l}\right) - 1 \right) \right. \\
&+ \sum_{\nu=2}^m (-1)^\nu \frac{\cos(j\nu\pi)}{m} \left[ \frac{\cos\left[(\nu+1)\left(1-\frac{i}{l}\right)\pi\right]}{\nu+1} - \frac{\cos\left[(\nu-1)\left(1-\frac{i}{l}\right)\pi\right]}{\nu-1} \right. \\
&\quad \left. \left. - (-1)^{\nu+1} \left( \frac{1}{\nu+1} - \frac{1}{\nu-1} \right) \right] - \frac{(-1)^{m-j}}{2} \left[ \frac{\cos\left[(m+1)\left(1-\frac{i}{l}\right)\pi\right]}{m+1} \right. \right. \\
&\quad \left. \left. - \frac{\cos\left[(m-1)\left(1-\frac{i}{l}\right)\pi\right]}{m-1} + (-1)^m \left( \frac{1}{m+1} - \frac{1}{m-1} \right) \right] \right\} \\
&= \frac{\varepsilon_j}{m} \left\{ 1 - \cos\left(\frac{i\pi}{l}\right) - \cos\left(\frac{j\pi}{m}\right) \left( \cos^2\left(\frac{i\pi}{l}\right) - 1 \right) \right. \\
&- \sum_{\nu=2}^m \cos\frac{j\nu\pi}{m} \left[ \frac{\cos[(\nu+1)i\pi/l]}{\nu+1} - \frac{\cos[(\nu-1)i\pi/l]}{\nu-1} + \frac{2}{\nu^2-1} \right] \\
&\quad \left. - \frac{(-1)^j}{2} \left[ \frac{\cos[(m-1)i\pi/l]}{m-1} - \frac{\cos[(m+1)i\pi/l]}{m+1} - \frac{2}{m^2-1} \right] \right\} \\
&= \frac{\varepsilon_j}{m} \left\{ 1 - \cos\left(\frac{i\pi}{l}\right) - \frac{1}{2} \cos\left(\frac{j\pi}{m}\right) \left( 1 - \cos\left(\frac{2i\pi}{l}\right) \right) \right. \\
&\quad \left. + \sum_{\nu=2}^m \varepsilon_\nu \cos\left(\frac{j\pi\nu}{m}\right) \left[ \frac{\cos[(\nu-1)i\pi/l]}{\nu-1} - \frac{\cos[(\nu+1)i\pi/l]}{\nu+1} - \frac{2}{\nu^2-1} \right] \right\}.
\end{aligned}$$

Thus the lemma is proved.  $\blacksquare$

Denote by  $W_2^{(r)} = W_2^{(r)}(\mu; a, b)$  ( $r \in \mathbf{N}$ ) a class of functions  $f(x)$ , having absolutely continuous derivative of order  $r-1$  and derivative of order  $r$  on  $[a, b]$  and satisfying the condition

$$\int_a^b \left[ f^{(r)}(t) \right] dt \leq \mu^2. \quad (4.13)$$



For a given  $x \in [a, b]$  we use the notation

$$\sup_{f \in W_2^{(r)}} |f(x) - L_l(f; x)| = E_{L_l}(W_2^{(r)}, x),$$

where  $L_l(f; x)$  is the Lagrange interpolation polynomial at points of interpolation (4.4).

It is known [16, p. 26], that for a fixed  $x \in [a, b]$

$$E_{L_l}(W_2^{(r)}, x) = \frac{\mu}{(r-1)!} J^{\frac{1}{2}}, \tag{4.14}$$

where

$$J = \int_a^b H_r^2(t, x) dt,$$

$$H_r(t, x) = K_r(x-t) - \sum_{v=0}^l K_r(t_v^{(l)} - t) l_v(x), \quad K_r(u) = \begin{cases} (u)^{\alpha-1}, & u > 0; \\ 0, & u \leq 0. \end{cases}$$

It is clear that  $J = J_1 - 2J_2 + J_3$ , where

$$J_1 = \int_a^b K_r^2(x-t) dt,$$

$$J_2 = \int_a^b K_r(x-t) \sum_{v=0}^l K_r(t_v^{(l)} - t) l_v(x) dt,$$

$$J_3 = \int_a^b \sum_{k=0}^l \sum_{v=0}^l K_r(t_k^{(l)} - t) K_r(t_v^{(l)} - t) l_k(x) l_v(x) dt.$$

Then, by using (4.5) we have

$$\|J_1\|_C \leq \frac{(b-a)^{2r-1}}{2r-1},$$

$$\|J_2\|_C \leq \frac{(b-a)^r}{r} (b-a)^{r-1} \|L_l\|_C \leq \frac{(b-a)^{2r-1}}{r} \left( \frac{2}{\pi} \ln l + 1 \right),$$

$$\|J_3\|_C \leq (b-a)(b-a)^{r-1}(b-a)^{r-1} \|L_l\|_C^2 \leq (b-a)^{2r-1} \left( \frac{2}{\pi} \ln l + 1 \right)^2,$$

and hence

$$\|J\|_C \leq (b-a)^{2r-1} \left[ \frac{1}{2r-1} + \frac{2}{r} \left( \frac{2}{\pi} \ln l + 1 \right) + \left( \frac{2}{\pi} \ln l + 1 \right)^2 \right]. \quad (4.15)$$

We also need two other assertions. The first one follows from [4, p.113].

LEMMA 4. *Let  $B$  be the Banach space, let  $T : B \rightarrow B$  and  $\bar{T} : B \rightarrow B$  be operators from  $B$  into  $B$ , and let  $T$  be a contractive operator in a ball  $K = \{\psi \in B : \|\psi\| \leq H\}$  ( $H > 0$ ):*

$$\forall \psi_1, \psi_2 \in K : \|T\psi_1 - T\psi_2\| \leq q\|\psi_1 - \psi_2\|, \quad q = \text{const} < 1,$$

and let  $\bar{T}K \subset K$ .

Let  $\{\varphi_\nu\}_{\nu=0}^\infty$  and  $\{\bar{\varphi}_\nu\}_{\nu=0}^\infty$  be sequences such that

$$\varphi_0 = \bar{\varphi}_0 \in K, \quad \varphi_{\nu+1} = T\varphi_\nu, \quad \bar{\varphi}_{\nu+1} = \bar{T}\bar{\varphi}_\nu, \quad \nu = 0, 1, \dots,$$

and let

$$\delta = \sup_{\psi \in K} \|T\psi - \bar{T}\psi\|.$$

Then

$$\|\varphi_\nu - \bar{\varphi}_\nu\| \leq \delta \frac{1 - q^\nu}{1 - q}, \quad \nu = 0, 1, \dots$$

LEMMA 5. *Let  $\alpha > 1$ , and let  $f[x, y] : [a, b] \times Y \rightarrow R$  ( $Y \subset R$ ) satisfies the conditions of Theorem 1. Then*

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{1-\alpha}} dt \in W_2^{[\alpha]}(\mu; a, b), \quad \mu = \frac{M(b-a)^{\alpha-[\alpha]+\frac{1}{2}}}{\Gamma(\alpha-[\alpha]+1)},$$

where  $M$  is given by (2.8).

P r o o f. Set

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{1-\alpha}} dt. \quad (4.16)$$

Using [6, Lemma 4] (see also [7, Corollary 1]) we have

$$F^{([\alpha]-1)}(x) = \frac{1}{\Gamma(\alpha-[\alpha]+1)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{[\alpha]-\alpha}} dt,$$

and, in accordance with (2.8) and [14, (1.4)],  $F^{([\alpha]-1)}(x) \in AC[a, b]$ , where  $AC[a, b]$  is the space of functions absolutely continuous on  $[a, b]$ .

Further, by [6, Lemma 4],

$$F^{([\alpha])}(x) = \frac{1}{\Gamma(\alpha - [\alpha])} \int_a^x \frac{f[t, y(t)]dt}{(x - t)^{1-\alpha+[\alpha]}}$$

and taking (2.8) into account, we have

$$\int_a^b [F^{([\alpha])}(t)]^2 dt \leq \frac{(b - a)M^2}{(\Gamma(\alpha - [\alpha]))^2} \left( \frac{(b - a)^{\alpha-[\alpha]}}{\alpha - [\alpha]} \right)^2 = \frac{(b - a)^{2(\alpha-[\alpha])+1}M^2}{(\Gamma(\alpha - [\alpha] + 1))^2}.$$

This completes the proof of lemma. ■

### 5. AI-method. Main theorem

Let  $\alpha > 1$  and  $r, n, m \in \mathbf{N}$ . We introduce the following notations:

$$\begin{aligned} \delta_{lm} &= \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} S_l + \frac{\alpha(b - a)^{r+\frac{1}{2}-\alpha}}{(r - 1)!} \left( \frac{2}{\pi} \ln l + 1 \right) S_m, \\ S_l &= \left( \frac{1}{2[\alpha] - 1} + \frac{2}{[\alpha]} \left( \frac{2}{\pi} \ln l + 1 \right) + \left( \frac{2}{\pi} \ln l + 1 \right)^2 \right)^{\frac{1}{2}}, \\ S_m &= \left( \frac{1}{2r - 1} + \frac{2}{r} \left( \frac{2}{\pi} \ln m + 1 \right) + \left( \frac{2}{\pi} \ln m + 1 \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

There holds the following statement.

**THEOREM 4.** *Let  $\alpha > 1, K_H = \{y \in R, |y| < H, H > 0\}$ , and let  $f[t, y] : [a, b] \times K_H \rightarrow R$  be function satisfying the conditions of Theorem 3 and such that for fixed  $x \in [a, b]$  and  $y \in K_H$ ,*

$$f[t, y](x - t)^{\alpha-1} \in W_2^{(r)}(\mu; a, b) \quad (r \in N, \mu > 0). \tag{5.2}$$

Let

$$q = \frac{L(b - a)^\alpha}{\Gamma(\alpha + 1)} < 1, \tag{5.3}$$

where  $L$  is given in (3.2), and for some  $\varepsilon > 0$

$$\max \left\{ \frac{M(b - a)^\alpha}{\Gamma(\alpha + 1)} (1 + \varepsilon), \frac{\mu(b - a)^\alpha}{\Gamma(\alpha + 1)} (1 + \varepsilon) \right\} \leq H, \tag{5.4}$$

where  $M$  is defined by (2.8).

Then the sequence of polynomials  $\bar{y}_\nu(x)$ , constructed by using the AI-algorithm in (4.9), approximates the solution  $y(x)$  of the Cauchy problem (1.7)-(1.8) in such a way that for any natural  $l, m \in \mathbf{N}$  such that

$$\delta_{lm} < \varepsilon, \quad (5.5)$$

there holds the inequality

$$\|y(x) - \bar{y}_\nu(x)\|_C \leq D \frac{q^\nu + \delta_{lm}(1 - q^\nu)}{1 - q}, \quad (5.6)$$

where

$$D = \max \left\{ \frac{M(b-a)^\alpha}{\Gamma(\alpha+1)}, \frac{\mu(b-a)^\alpha}{\Gamma(\alpha+1)} \right\}. \quad (5.7)$$

**P r o o f.** By Theorem 2, the Cauchy problem (1.7)-(1.8) has a unique solution  $y(x) \in C^{n-1}[a, b]$ .

Denote by  $C_H[a, b]$  the following subset of  $C[a, b]$ :

$$K \equiv C_H[a, b] = \{y(x) \in C[a, b], \|y(x)\|_C \leq H\}.$$

According to (4.8) and (4.9), for any  $y(x) \in C_H[a, b]$  we have

$$\begin{aligned} |(Ay)(x) - (\bar{A}y)(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^x f[t, y(t)]K(x, t)dt \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} \sum_{j=0}^m f \left[ t_j^{(m)}, y \left( t_j^{(m)} \right) \right] K \left( t_i^{(l)}, t_j^{(m)} \right) l_j^{(m)} dt \cdot l_i^{(l)}(x) \right| \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \int_a^x f[t, y(t)]K(x, t)dt - \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} f[t, y(t)]K \left( t_i^{(l)}, t \right) dt \cdots l_i^{(l)}(x) \right| \\ &\quad + \left| \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} f[t, y(t)]K \left( t_i^{(l)}, t \right) dt \cdot l_i^{(l)}(x) \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} \sum_{j=0}^m f \left[ t_j^{(m)}, y \left( t_j^{(m)} \right) \right] K \left( t_i^{(l)}, t_j^{(m)} \right) l_j^{(m)} dt \cdot l_i^{(l)}(x) \right| = I_1 + I_2. \end{aligned} \quad (5.8)$$

Let  $L_l(F; x)$  be the Lagrange interpolation polynomial, constructed at the points of interpolations (4.4), of a function  $F(x)$  defined by (4.16). Using the notation (4.3), Lemma 4 and the inequality (4.15) we have

$$\begin{aligned}
 I_1 &= \left| \frac{1}{\Gamma(\alpha)} \int_a^x f[t, y(t)] K(x, t) dt - \frac{1}{\Gamma(\alpha)} \sum_{i=0}^l \int_a^{t_i^{(l)}} f[t, y(t)] K(t_i^{(l)}, t) dt \cdot l_i^{(l)}(x) \right| \\
 &= |F(x) - L_l(F, x)| < \frac{M(b-a)^{\alpha - [\alpha] + \frac{1}{2}} (b-a)^{[\alpha] - \frac{1}{2}}}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} \\
 &\quad \times \left( \frac{1}{2[\alpha] - 1} + \frac{2}{[\alpha]} \left( \frac{2}{\pi} \ln l + 1 \right) + \left( \frac{2}{\pi} \ln l + 1 \right)^2 \right)^{\frac{1}{2}} \\
 &= \frac{M(b-a)^\alpha}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} S_l. \tag{5.9}
 \end{aligned}$$

Further, taking (5.2) and (4.15) into account, we obtain

$$\begin{aligned}
 I_2 &= \frac{1}{\Gamma(\alpha)} \left| \sum_{i=0}^l \int_a^{t_i^{(l)}} f[t, y(t)] K(t_i^{(l)}, t) dt \cdot l_i^{(l)}(x) \right. \\
 &\quad \left. - \sum_{i=0}^l \int_a^{t_i^{(l)}} \sum_{j=0}^m f[t_j^{(m)}, y(t_j^{(m)})] K(t_i^{(l)}, t_j^{(m)}) l_j^{(m)} dt \cdot l_i^{(l)}(x) \right| \\
 &\leq \frac{(b-a)}{\Gamma(\alpha)} \|L_l\|_C \max_{0 \leq i \leq l} \left| f[t, y(t)] K(t_i^{(l)}, t) - L_m(f \cdot K, t) \right| \\
 &\leq \frac{(b-a) \|L_l\|_C M'(b-a)^{r - \frac{1}{2}}}{\Gamma(\alpha) (r-1)!} \left( \frac{1}{2r-1} + \frac{2}{r} \left( \frac{2}{\pi} \ln m + 1 \right) \right. \\
 &\quad \left. + \left( \frac{2}{\pi} \ln m + 1 \right)^2 \right)^{\frac{1}{2}} \leq \frac{(b-a)^{r + \frac{1}{2}} \mu}{\Gamma(\alpha) (r-1)!} \left( \frac{2}{\pi} \ln l + 1 \right) S_m, \tag{5.10}
 \end{aligned}$$

where  $S_m$  is given by (5.1). Substituting (5.9) and (5.10) into (5.8), we deduce the relation

$$\begin{aligned}
 \|(Ay)(x) - (\bar{A}y)(x)\|_C &\leq \frac{M(b-a)^\alpha}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} S_l \\
 &\quad + \frac{(b-a)^{r + \frac{1}{2}} \mu}{\Gamma(\alpha) (r-1)!} \left( \frac{2}{\pi} \ln l + 1 \right) S_m. \tag{5.11}
 \end{aligned}$$

To estimate  $\|y_\nu - \bar{y}_\nu\|_C$ , we apply Lemma 3. For this we prove that  $A$  is a contractive operator in  $C[a, b]$ . By (4.8) and (3.2), we have

$$\begin{aligned} \|(Ay_1)(t) - (Ay_2)(t)\|_C &= \left\| \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_1(t)] - f[t, y_2(t)]}{(x-t)^{1-\alpha}} dt \right\|_C \\ &\leq \frac{L(b-a)^\alpha}{\Gamma(\alpha+1)} \|y_1(x) - y_2(x)\|_C. \end{aligned}$$

In accordance with the condition (5.3),

$$q = \frac{L(b-a)^\alpha}{\Gamma(\alpha+1)} < 1,$$

and, therefore,  $A$  is a contractive operator in  $C[a, b]$ .

It follows from (5.11) that

$$\begin{aligned} \delta = \sup_{y \in C_H[a, b]} \|Ay - \bar{A}y\|_C &\leq \frac{M(b-a)^\alpha}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} S_l \\ &\quad + \frac{(b-a)^{r+\frac{1}{2}}\mu}{\Gamma(\alpha)(r-1)!} \left( \frac{2}{\pi} \ln l + 1 \right) S_m. \end{aligned} \quad (5.12)$$

Using (2.8) and (5.11), for any  $y \in C_H[a, b]$  we have

$$\begin{aligned} \|\bar{A}y\|_C &\leq \|Ay\|_C + \|\bar{A}y - Ay\|_C \leq \frac{M(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{M(b-a)^\alpha}{\Gamma(\alpha - [\alpha] + 1)([\alpha] - 1)!} S_l \\ &\quad + \frac{(b-a)^{r+\frac{1}{2}}\mu}{\Gamma(\alpha)(r-1)!} \left( \frac{2}{\pi} \ln l + 1 \right) S_m \leq D(1 + \delta_{lm}). \end{aligned}$$

From here taking into account the condition (5.4), being valid for any  $l, m \in \mathbf{N}$  such that  $\delta_{lm} < \epsilon$ , we obtain the estimate

$$\|\bar{A}y\|_C \leq D(1 + \epsilon) \leq H,$$

which means  $\bar{A}(C_H[a, b]) \subset C_H[a, b]$ . Therefore, in accordance with Lemma 3 and estimates (5.3) and (5.12),

$$\|y_\nu - \bar{y}_\nu\|_C \leq D\delta_{lm} \frac{1 - q^\nu}{1 - q}. \quad (5.13)$$

Applying Theorem 3 and the estimates (4.5) and (5.13), we obtain

$$\begin{aligned} \|y - \bar{y}_\nu\|_C &\leq \|y - y_\nu\|_C + \|y_\nu - \bar{y}_\nu\|_C \\ &\leq l e \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \frac{Mq^\nu}{1-q} + D\delta_{lm} \frac{1 - q^\nu}{1 - q} \leq D \frac{q^\nu + \delta_{lm}(1 - q^\nu)}{1 - q}. \end{aligned}$$

This yields the estimate (5.6), and theorem is proved.  $\blacksquare$

### 6. Numerical examples

In this section we present two numerical examples.

EXAMPLE 1. Consider the Cauchy problem for the differential equation with the Caputo fractional derivative of order 3/2:

$$\left({}^c D_{0+}^{\frac{3}{2}} y\right)(x) = y^2(x) + \frac{4\sqrt{x}}{\sqrt{\pi}} - x^4, \quad y(0) = 0, \quad y'(0) = 0. \quad (6.1)$$

It is directly verified that  $y(x) = x^2$  is the exact solution of this problem.

We apply AI-method to approximate solution of (6.1) by taking  $l = 4$ ,  $m = 15$ ,  $\nu = 10$  and  $x \in [0; 0,01]$ . As a result we obtain the following polynomial  $y_{10}(x)$  approximating the solution of the Cauchy problem (6.1):

$$y_{10}(x) = -0,000134983x + 1,08366x^2 - 13,2886x^3 + 625,571x^4. \quad (6.2)$$

The exact solution  $y(x)$  and the approximate solution  $y_{10}(x)$  together with their difference  $y(x) - y_{10}(x)$  are presented in Table 1.

$x$	0	0,002	0,004
$y(x)$	0	0,000004	0,000016
$y_{10}(x)$	0	$3,96839 \cdot 10^{-6}$	0,0000161084
$y_{10}(x) - y(x)$	0	$-3,16134 \cdot 10^{-8}$	$1,08355 \cdot 10^{-7}$
$x$	0,006	0,008	0,01
$y(x)$	0,000036	0,000064	0,001
$y_{10}(x)$	0,0000361424	0,0000640332	$9,36017 \cdot 10^{-7}$
$y_{10}(x) - y(x)$	$1,42384 \cdot 10^{-7}$	$3,31702 \cdot 10^{-8}$	$-6,39832 \cdot 10^{-8}$

Table 1

EXAMPLE 2. Consider the Cauchy problem for the differential equation with the Caputo fractional derivative of order 5/2:

$$\left({}^c D_{0+}^{\frac{5}{2}} y\right)(x) = y^2(x) + \frac{105\sqrt{\pi}}{16}x - x^7, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0. \quad (6.3)$$

It is directly verified that  $y(x) = \sqrt{x^7}$  is the exact solution of this problem.

We apply AI-method to approximate solution of (6.3) by taking  $l = 4$ ,  $m = 15$ ,  $\nu = 10$  and  $x \in [0; 0,01]$ . As a result we obtain the following polynomial  $y_{10}(x)$  approximating the solution of the Cauchy problem (6.3):

$$y_{10}(x) = 8,90217 \cdot 10^{-8}x - 0,000112757x^2 + 0,0691963x^3 + 4,11891x^4. \quad (6.4)$$

The exact solution  $y(x)$  and the approximate solution  $y_{10}(x)$  together with their difference  $y(x) - y_{10}(x)$  are presented in Table 2.

$x$	0	0,002	0,004
$y(x)$	0	$3,57771 \cdot 10^{-10}$	$4,04772 \cdot 10^{-9}$
$y_{10}(x)$	0	$3,46489 \cdot 10^{-10}$	$4,03498 \cdot 10^{-9}$
$y_{10}(x) - y(x)$	0	$-1,12816 \cdot 10^{-11}$	$-1,27323 \cdot 10^{-11}$
$x$	0,006	0,008	0,01
$y(x)$	$1,67313 \cdot 10^{-8}$	$4,57947 \cdot 10^{-8}$	$\approx 1 \cdot 10^{-7}$
$y_{10}(x)$	$1,67594 \cdot 10^{-8}$	$4,57953 \cdot 10^{-8}$	$\approx 1 \cdot 10^{-7}$
$y_{10}(x) - y(x)$	$2,81094 \cdot 10^{-11}$	$6,38602 \cdot 10^{-13}$	$-3,62857 \cdot 10^{-14}$

Table 2.

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