Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 7, NUMBER 4 (2004) ISSN 1311-0454

FROM DIFFERENCES TO DERIVATIVES

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Abstract

A relation showing that the Grünwald-Letnikov and generalized Cauchy derivatives are equal is deduced confirming the validity of a well known conjecture. Integral representations for both direct and reverse fractional differences are presented. From these the fractional derivative is readily obtained generalizing the Cauchy integral formula.

Mathematics Subject Classification: 26A33
Key Words and Phrases: fractional difference; Grünwald-Letnikov derivative; generalized Cauchy derivative

1. Introduction

In [1] Diaz and Osler presented a brief insight into the fractional differences. They proposed an integral formulation for the differences and conjectured about the possibility of using them for defining fractional derivatives. This problem was also discussed in a round table held at the International Conference on "Transform Methods & Special Functions, Varna'96" as stated by Kiryakova [3].

In this paper, the validity of such conjecture is proved. In fact, we will enounce and later prove the following statement [5]:

The Grünwald-Letnikov fractional derivative is equal to the generalized Cauchy derivative.

We must refer that we will not address the existence problem. We are mainly interested in obtaining generalization for a well known formalism.

For the proof, we begin by introducing the fractional differences, considering two cases: direct and reverse[1]. From these we obtain the direct and reverse Grünwald-Letnikov fractional derivatives. For those differences, integral representations will be proposed following a Diaz and Osler's idea. The integration path is a U shaped contour that encircles all the poles and "closes" at infinite. From these representations we obtain the fractional differintegrals by using the asymptotic properties of the Gamma function. As we will show there are infinite ways of computing a fractional derivative: we only have to choose a branch cut line and fix a suitable integration path. If α is a positive integer, we have an integrand with a pole: this is the well known Cauchy formula. It is important to remark that accordingly to the equality of the Cauchy and the Grünwald-Letnikov derivatives, the integration path has necessarily a form like we referred above, unless α is a positive integer. This means that the derivative definitions based in finite straight line segments are not valid.

A theorem stating the main result of the paper is presented in Section 2. In Section 3, we will introduce the general formulation for the differences, considering two cases: direct and reverse. For these, integral representations will be proposed. From these representations we obtain in Section 4 the derivative integrals by using the properties of the Gamma function. At last we will present some conclusions.

2. Main result

The main result of this paper can be stated in a theorem that we will present below. Consider the U shaped contour represented in Figure 1.

Let $f(z)$ be a complex variable function analytic in the region inside and continuous on that contour. The generalized Cauchy derivative is given by [4-6]:

$$\mathrm{D}_d^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_d} f(w) \frac{1}{(w-z)^{\alpha+1}} dw, \tag{1}$$

where we assume that the branch cut line is inside the above referred analyticity region, and C_d is a U shaped contour encircling the branch cut line. By now, we consider a branch cut line (with the corresponding contour) in

[1] We avoid the common used backward and forward since can be misleading.

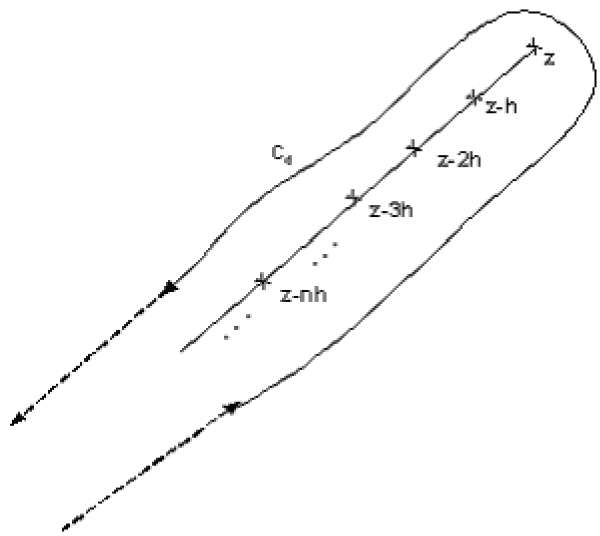

Figure 1: U shaped contour

the left hand complex plane or in the lower imaginary half axis. Now, we introduce the direct Grünwald-Letnikov derivative given by:

$$\mathrm{D}_d^\alpha f(z) = \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(z-kh)}{h^\alpha}, \tag{2}$$

where h is any complex in the right hand complex plane. Making a substitution $h \to -h$, we obtain the reverse Grünwald-Letnikov derivative.

$$\mathrm{D}_r^\alpha f(z) = (-1)^\alpha \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(z+kh)}{h^\alpha}. \tag{3}$$

Both expressions agree with the usual derivative definition when α is a positive integer. Moreover, expression (2) corresponds to the so-called left-hand sided Grünwald-Letnikov fractional derivative while (3) has the extra factor $(-1)^\alpha$, when compared with the right-hand sided Grünwald-Letnikov fractional derivative [5]. We maintain that this factor must be retained and call the pairs defined by (2) and (3) as Grünwald-Letnikov fractional derivatives[2]. We assume also that the points $z-nh, n=0,1,\ldots$, are on the branch cut line.

THEOREM . *Under the above stated conditions we have:*

$$\lim_{h\to 0^+}\frac{\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\ k\end{array}\right)f(z-kh)}{h^\alpha}=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_d}f(w)\frac{1}{(w-z)^{\alpha+1}}. \quad (4)$$

This theorem remains valid for the reverse Grünwald-Letnikov fractional derivative provided that we choose a branch cut line in the right hand complex plane or in the upper imaginary half axis (see Figure 3). This result is very important since it establishes a bridge between two different formulations both generalizing classical results. Other definitions that are not logically deduced from (4), although allowing similar results, should be considered as pseudo-derivatives. Although all the above relations are formally valid for any $\alpha \in R$ (or even for $\alpha \in C$), in the following we will speak always of derivative. Before we prove the theorem, we will present the integral representations for the fractional differences.

3. Differences

3.1. Definitions

Let $f(z)$ be a function of complex variable and introduce $\triangle_d$ and $\triangle_r$ as finite "direct" and "reverse" differences defined by:

$$\triangle_d f(z) = f(z) - f(z-h) \quad (5)$$

and

$$\triangle_r f(z) = f(z+h) - f(z) \quad (6)$$

with $h \in \mathbf{C}$ and, as before, we assume that $\mathrm{R}e(h) > 0$. The repeated use of the above definitions leads to

$$\triangle_d^N f(z) = \sum_{k=0}^{N}(-1)^k\left(\begin{array}{c}N\\ k\end{array}\right)f(z-kh) \quad (7)$$

[2]These derivatives were proposed first by Liouville [2]. So, they should be called Liouville-Grünwald-Letnikov derivatives.

and

$$\triangle_r^N f(z) = (-1)^N \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(z+kh) \tag{8}$$

where $\binom{N}{k}$ are the binomial coefficients. These definitions are readily extended to the fractional order case [1]:

$$\triangle_d^\alpha f(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh) \tag{9}$$

and

$$\triangle_r^\alpha f(z) = (-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh). \tag{10}$$

This formulation remains valid in the negative integer order case. Let $\alpha = -N$ (N a positive integer). As it is well known from the Z Transform theory, the following relation holds if $k \geq 0$

$$\sum_{n=0}^{\infty} (n+1)_k \, q^n = \frac{k!}{(1-q)^{k+1}} \quad \text{for } |q| < 1. \tag{11}$$

Introducing the Pochhammer symbol for the shifted factorial, $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ and putting $k = N-1$, we obtain easily:

$$(1-z)^{-N} = \sum_{n=0}^{\infty} \frac{(n+1)_{N-1}}{(N-1)!} z^n \quad \text{for } |z| < 1 \tag{12}$$

leading to

$$\triangle_d^{-N} f(z) = \sum_{n=0}^{\infty} \frac{(n+1)_{N-1}}{(N-1)!} \cdot f(z-nh). \tag{13}$$

For the reverse case, we have:

$$\begin{aligned} \triangle_r^{-N} f(z) &= (z-1)^{-N} \\ &= (-1)^N \sum_{n=0}^{\infty} \frac{(n+1)_{N-1}}{(N-1)!} \cdot f(z+nh). \end{aligned} \tag{14}$$

As

$$(n+1)_{N-1} = \frac{(n+N-1)!}{n!} = \frac{(N-1)!(N)_n}{n!} \tag{15}$$

and

$$\frac{(-a)_n}{n!} = (-1)^n \begin{pmatrix} a \\ n \end{pmatrix}, \tag{16}$$

we have

$$(n+1)_{N-1} = (N-1)!\,(-1)^n \begin{pmatrix} -N \\ n \end{pmatrix}. \tag{17}$$

So, we can write:

$$\triangle_d^{-N} f(z) = \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -N \\ n \end{pmatrix} f(z-nh). \tag{18}$$

For the anti-causal case, we have:

$$\triangle_r^{-N} f(z) = (-1)^N \sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -N \\ n \end{pmatrix} f(z+nh). \tag{19}$$

As it can be seen, these expressions are the ones we obtain by putting $\alpha = -N$ into (9) and (10). So, the relations (9) and (10) are representations for the differences of any order.

3.2. Integral representations

3.2.1. Positive integer order

Consider first the positive integer order case. Assume that $f(z)$ is analytic inside and on a closed integration path that includes the points $t = z - kh$ in the direct case and $t = z + kh$ in the corresponding reverse case, with $k = 0, 1, \ldots, N$. The results stated in (9) and (10) can be interpreted in terms of the residue theorem[3]. In fact, they can be considered as $\frac{1}{2\pi i}\sum$ residues in the computation of the integral of a function with poles at $t = z - kh$ and $t = z + kh$, $k = 0, 1, 2, \ldots$. As it can be seen by direct verification, we have [1]:

$$\sum_{k=0}^{N} (-1)^k \begin{pmatrix} N \\ k \end{pmatrix} f(z-kh) = \frac{N!}{2\pi i h} \cdot \int_{C_d} \frac{f(w)}{\prod_{k=0}^{N} \left(\frac{w-z}{h} + k\right)} dw \tag{20}$$

[3]It is important to remark that the poles are simple and that this case can be deduced without the use of the derivative notion.

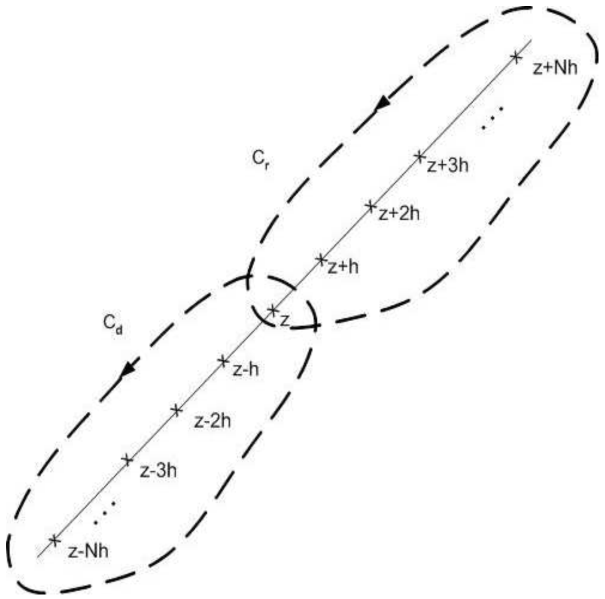

Figure 2: integration paths and poles for the integral representation of integer order differences

and

$$\sum_{k=0}^{N} (-1)^k \left(\begin{array}{c} N \\ k \end{array} \right) f(z+kh) = -\frac{N!}{2\pi ih} \cdot \int_{C_r} \frac{f(w)}{\prod_{k=0}^{N} \left(\frac{z-w}{h} + k\right)} dw. \tag{21}$$

We must remark that the binomial coefficients appear naturally when computing the residues.

Introducing the Pochhammer symbol, we can rewrite the above formulae as:

$$\triangle_d^N f(z) = \frac{N!}{2\pi ih} \int_{C_d} \frac{f(w)}{\left(\frac{w-z}{h}\right)_{N+1}} dw \tag{22}$$

and

$$\triangle_r^N f(z) = \frac{(-1)^{N+1} N!}{2\pi ih} \int_{C_r} \frac{f(w)}{\left(\frac{z-w}{h}\right)_{N+1}} dw. \tag{23}$$

Attending to the relation between the Pochhammer symbol and the Gamma function:

$$\Gamma(z+n) = (z)_n \Gamma(z), \tag{24}$$

we can write:

$$\triangle_d^N f(z) = \frac{N!}{2\pi i h} \int_{C_d} f(w) \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h} + N + 1\right)} dw \tag{25}$$

and

$$\triangle_r^N f(z) = \frac{(-1)^{N+1} N!}{2\pi i h} \int_{C_r} f(w) \frac{\Gamma\left(\frac{z-w}{h}\right)}{\Gamma\left(\frac{z-w}{h} + N + 1\right)} dw. \tag{26}$$

This is correct and is coherent with (20) and (21), because the Gamma function $\Gamma(z)$ has poles at the negative integers ($z = -n$). The corresponding residues are equal to $(-1)^n /n!$, [5]. Although both the Gamma functions have infinite poles, outside the contour they cancel out and the integrand is analytic. We should also remark that the direct and reverse differences are not formally equal.

3.2.2. Fractional order

Consider the fractional order differences defined in (9) and (10). It is not hard to see that we are in presence of a situation similar to the positive integer case, excepting the fact of having infinite poles. So we have to use an integration path that encircles all the poles. This can be done with a U shaped contour like those shown in Figure 3.

With the suitable adaptations, we obtain from (25) and (26):

$$\triangle_d^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i h} \int_C f(w) \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h} + \alpha + 1\right)} dw \tag{27}$$

and

$$\triangle_r^\alpha f(z) = \frac{(-1)^{\alpha+1} \Gamma(\alpha+1)}{2\pi i h} \int_C f(w) \frac{\Gamma\left(\frac{z-w}{h}\right)}{\Gamma\left(\frac{z-w}{h} + \alpha + 1\right)} dw. \tag{28}$$

Remark that one turns into the other with the substitution $h \to -h$, justifying the assumption we made in Section 2.

3.2.3. Two properties

In the following, we shall be concerned with the fractional order case and will consider the direct case only. The other one is similar.

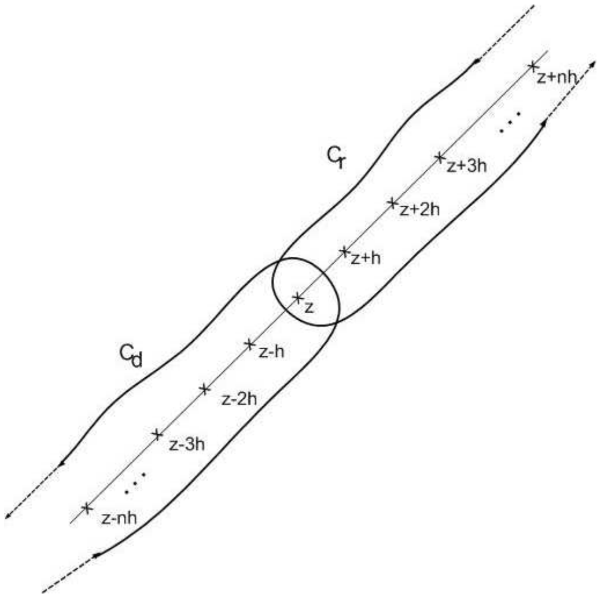

Figure 3: integration paths and poles for the integral representation of fractional order differences

3.2.3.1. Repeated differencing

We are going to study the effect of a sequential application of the difference operator $\triangle$. We have

$$\triangle_d^{\beta}\left[\triangle_d^{\alpha} f(z)\right]=\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{(2\pi i h)^2}\int_C\int_C f(w)\frac{\Gamma\left(\frac{w-s}{h}\right)}{\Gamma\left(\frac{w-s}{h}+\alpha+1\right)}dw\frac{\Gamma\left(\frac{s-z}{h}\right)}{\Gamma\left(\frac{s-z}{h}+\beta+1\right)}ds. \tag{29}$$

Permuting the integrations, we have:

$$\triangle_d^{\beta}\left[\triangle_d^{\alpha} f(z)\right]=\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{(2\pi i h)^2}\int_C f(w)\int_C\frac{\Gamma\left(\frac{w-s}{h}\right)}{\Gamma\left(\frac{w-s}{h}+\alpha+1\right)}\frac{\Gamma\left(\frac{s-z}{h}\right)}{\Gamma\left(\frac{s-z}{h}+\beta+1\right)}dsdw. \tag{30}$$

As, by the residue theorem

$$
\begin{aligned}
& \frac{\Gamma(\beta+1)}{2\pi i h} \int_C \frac{\Gamma\left(\frac{w-s}{h}\right)}{\Gamma\left(\frac{w-s}{h}+\alpha+1\right)} \frac{\Gamma\left(\frac{s-z}{h}\right)}{\Gamma\left(\frac{s-z}{h}+\beta+1\right)} ds \\
= & \frac{1}{h} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\beta+1)\cdot\Gamma\left(\frac{w-z}{h}+n\right)}{\Gamma\left(\frac{w-z}{h}+\alpha+1+n\right)\Gamma(\beta-n+1)} \\
= & \frac{1}{h} \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h}+\alpha+1\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{w-z}{h}\right)_n (-\beta)_n}{\left(\frac{w-z}{h}+\alpha+1\right)_n} \\
= & \frac{1}{h} \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h}+\alpha+1\right)} {}_2F_1\left(\frac{w-z}{h}, -\beta, \frac{w-z}{h}+\alpha+1, 1\right), \qquad (31)
\end{aligned}
$$

where ${}_2F_1$ is the Gauss hypergeometric function. If $\alpha+\beta+1>0$, we have:

$$
\begin{aligned}
& \frac{\Gamma(\beta+1)}{2\pi i h} \int_C \frac{\Gamma\left(\frac{w-s}{h}\right)}{\Gamma\left(\frac{w-s}{h}+\alpha+1\right)} \frac{\Gamma\left(\frac{s-z}{h}\right)}{\Gamma\left(\frac{s-z}{h}+\beta+1\right)} ds \\
= & \frac{1}{h} \frac{\Gamma\left(\frac{w-z}{h}\right)\Gamma(\alpha+\beta+1)}{\Gamma\left(\frac{w-z}{h}+\alpha+\beta+1\right)\Gamma(\alpha+1)}, \qquad (32)
\end{aligned}
$$

leading to the conclusion that:

$$
\triangle_d^{\beta}\left[\triangle_d^{\alpha} f(z)\right] = \frac{\Gamma(\alpha+\beta+1)}{2\pi i h} \int_C f(w) \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h}+\alpha+\beta+1\right)} dw \qquad (33)
$$

and

$$
\triangle_d^{\beta}\left[\triangle_d^{\alpha} f(z)\right] = \triangle_d^{\alpha+\beta} f(z), \qquad (34)
$$

provided that $\alpha+\beta+1>0$. It is not difficult to see that the above operation is commutative. The condition $\alpha+\beta+1>0$ is restrictive, since we cannot have $\beta \le -\alpha-1$. However, we must remark that (27) and (28) are valid for every $\alpha \in \mathbf{R}$. The same happens with $(\alpha+\beta)$ in (33). This means that we can use (33) with every $\alpha, \beta \in \mathbf{R}$, at least formally.

3.2.3.2. Inversion

Putting $\alpha = -\beta$ into (33), we obtain:

$$
\triangle_d^{-\alpha}\left[\triangle_d^{\alpha} f(z)\right] = \triangle_d^{\alpha}\left[\triangle_d^{-\alpha} f(z)\right] = \frac{1}{2\pi i} \int_C f(w) \frac{1}{w-z} dw = f(z) \qquad (35)
$$

as we would expect. So the operation of differencing is invertible. This means that we can write:

$$
f(z) = \frac{\Gamma(\alpha+1)}{2\pi i h} \int_C \triangle_d^{\alpha} f(w) \frac{\Gamma\left(\frac{w-z}{h}\right)}{\Gamma\left(\frac{w-z}{h}-\alpha+1\right)} dw \qquad (36)
$$

in the direct case. In the reverse case, we will have:

$$f(z)=\frac{\Gamma(\alpha+1)}{2\pi ih}\int_C \triangle_r^\alpha f(w)\frac{\Gamma\left(\frac{z-w}{h}\right)}{\Gamma\left(\frac{z-w}{h}-\alpha+1\right)}dw, \tag{37}$$

according to (27).

4. Derivatives

4.1. On Grünwald-Letnikov derivatives

The Grünwald-Letnikov derivatives are obtained from (9) and (10) by dividing them by h^α and performing the limit $h \to 0^+$ [4]. Although we are not concerned here with existence problems, we must refer that in general we can have the direct derivative without existing the reverse one and *vice-versa*. For example, let us apply both definitions to the function $f(z) = e^{az}$. If $\mathrm{Re}(a) > 0$, expression (2) converges to $\mathrm{D}_d^\alpha f(z) = a^\alpha e^{az}$, while (3) diverges. On the other hand, if $f(z) = e^{-az}$ equation (2) diverges while (3) converges to $\mathrm{D}_r^\alpha f(z) = (-a)^\alpha e^{-az}$. This suggests that (2) and (3) should be adopted for right and left functions[5], respectively in agreement with Liouville reasoning [2]. In particular, for the functions such that $f(z) = 0$ for $\mathrm{Re}(z) < 0$ or $f(z) = 0$ for $\mathrm{Re}(z) > 0$.

4.2. Generalizing the Cauchy formula

The ratio of two gamma functions $\frac{\Gamma(s+a)}{\Gamma(s+b)}$ has an interesting expansion [6]:

$$\frac{\Gamma(s+a)}{\Gamma(s+b)}=s^{a-b}\left[1+\sum_1^N c_k s^{-k}+\mathcal{O}\left(s^{-N-1}\right)\right] \tag{38}$$

as $|s| \to \infty$, uniformly in every sector that excludes the negative real half-axis. The coefficients in the series can be expressed in terms of Bernoulli polynomials. Their knowledge is not important here.

Consider (27) and (28) again. Let $|h| < \epsilon \in \mathbf{R}$, where ϵ is a small number. This allows us to write:

$$\triangle_d^\alpha f(z)=\frac{\Gamma(\alpha+1)}{2\pi ih}\int_{C_d} f(w)\frac{1}{\left(\frac{w-z}{h}\right)^{\alpha+1}}dw+h^{\alpha+2}g_1(h) \tag{39}$$

---

[4] 0^+ means that $\mathrm{Re}(h) > 0$

[5] We say that $f(t)$ is a right [left] function if $f(-\infty) = 0$ [$f(+\infty) = 0$].

and

$$\triangle_r^\alpha f(z) = \frac{(-1)^{\alpha+1}\,\Gamma(\alpha+1)}{2\pi i h}\int_{C_r} f(w)\frac{1}{\left(\frac{z-w}{h}\right)^{\alpha+1}}dw + h^{\alpha+2}g_2(h), \tag{40}$$

where C_d and C_r are the contours represented in Figure 3. The $g_1(h)$ and $g_2(h)$ are functions that assume a finite value near zero. So, the fractional incremental ratia are given by:

$$\frac{\triangle_d^\alpha f(z)}{h^\alpha} = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_d} f(w)\frac{1}{(w-z)^{\alpha+1}}dw \tag{41}$$

and

$$\frac{\triangle_r^\alpha f(z)}{h^\alpha} = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_r} f(w)\frac{1}{(w-z)^{\alpha+1}}dw. \tag{42}$$

Allowing $h \to 0^+$, we obtain the direct and reverse generalized Cauchy derivatives:

$$\mathrm{D}_d^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_d} f(w)\frac{1}{(w-z)^{\alpha+1}}dw \tag{43}$$

and

$$\mathrm{D}_r^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C_r} f(w)\frac{1}{(w-z)^{\alpha+1}}dw. \tag{44}$$

If $\alpha = N$, both the derivatives are equal and coincide with the usual Cauchy definition. In the fractional case we have different solutions, since we are using a different integration path. Remark that (43) and (44) are formally the same. They differ only in the integration path. Thus we can define a generalized Cauchy derivative by:

$$\mathrm{D}^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C} f(w)\frac{1}{(w-z)^{\alpha+1}}dw, \tag{45}$$

where C is any half straight line starting at z.

5. Conclusion

Let γ be a half straight line beginning at any $z \in \mathbf{C}$ and let $h \in \gamma$. The Grünwald-Letnikov fractional derivative is equal to the generalized Cauchy derivative provided that it is computed along an integration path surrounding the above straight line.

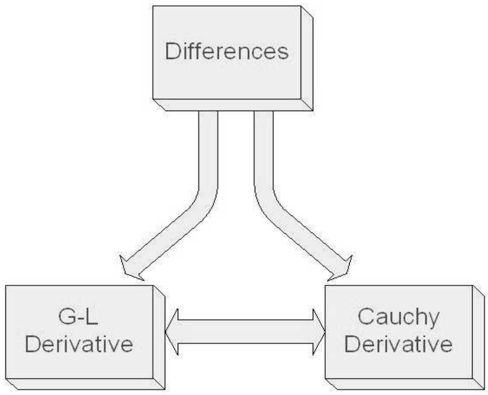

Figure 4: Grünwald-Letnikov derivative *vs* Cauchy derivative

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