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# ON THE UNIFORM CONVERGENCE OF PARTIAL DUNKL INTEGRALS IN BESOV-DUNKL SPACES

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### Abstract

In this paper we prove that the partial Dunkl integral $S_T(f)$ of $f$ converges to $f$, as $T \to +\infty$ in $L^\infty(\nu_\mu)$ and we show that the Dunkl transform $\mathcal{F}_\mu(f)$ of $f$ is in $L^1(\nu_\mu)$ when $f$ belongs to a suitable Besov-Dunkl space. We also give sufficient conditions on a function $f$ in order that the Dunkl transform $\mathcal{F}_\mu(f)$ of $f$ is in a $L^p$-space.

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## 1. Introduction

Dunkl operators are differential-difference operators introduced in 1989, by C. Dunkl [3]. On the real line, these operators, which are denoted by $\Lambda_\mu$, depend on a real parameter $\mu > -\frac{1}{2}$ and they are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. For $\mu > -\frac{1}{2}$, Dunkl kernel $E_\mu$ is defined as the unique solution of a differential-difference equation related to $\Lambda_\mu$ and satisfying $E_\mu(0) = 1$. This kernel is used to define Dunkl transform $\mathcal{F}_\mu$ which was introduced by C. Dunkl in [4]. More complete results concerning this transform were later obtained by M.F.E de Jeu [2]. Rösler in [9] shows that Dunkl kernels verify a product formula. This allows us to define Dunkl translation operators $\tau_x$, $x \in \mathbb{R}$. As a result we have a Dunkl convolution.

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If $\alpha > 0, p \geq 1$ and $1 \leq r < +\infty$ a Besov-Dunkl space, denoted by $BD^{p,r}_{\mu,\alpha}$ is a subspace of functions $f \in L^p(\nu_\mu)$ satisfying

$$\int_0^{+\infty} \left(\frac{w_{p,\mu}(f)(t)}{t^\alpha}\right)^r \frac{dt}{t} < +\infty,$$

where $w_{p,\mu}(f,t)$ is the $L^p(\nu_\mu)$ norm $\|\tau_t(f) + \tau_{-t}(f) - 2f\|_{p,\mu}, t \in \mathbb{R}$ (see [7]).

The goal of this paper is to prove that the partial Dunkl integral $S_T(f)$ of $f$ defined by

$$S_T(f)(x) = \int_{-T}^{T} E_\mu(ixy)\mathcal{F}_\mu(f)(y)d\nu_\mu(y), \quad x \in \mathbb{R}, T > 0,$$

converges to $f$, as $T \to +\infty$, in $L^\infty(\nu_\mu)$ and to show that the Dunkl transform $\mathcal{F}_\mu(f)$ of $f$ on $\mathbb{R}$, is in $L^1(\nu_\mu)$, when $f$ belongs to Besov-Dunkl space $BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$ for $\frac{4(\mu+1)}{2\mu+3} < p \leq 2$.

The contents of this paper are as follows.

In Section 2, we collect some results about harmonic analysis associated with Dunkl operator. In Section 3, we study some properties of partial Dunkl integral $S_T$, that will be useful to establish the uniform convergence of $S_T(f)$ to $f$. In Section 4, using a Hardy-Littlewood inequality for Dunkl transform we prove the absolute integrability of $\mathcal{F}_\mu(f)$.

Analogous results have been obtained by Giang and Móricz in [6] for a classical Fourier transform on $\mathbb{R}$. Later, Betancor and Rodríguez-Mesa in [1] have established similar results in Lipshitz-Hankel spaces involving Hankel transform on $(0, \infty)$.

In the sequel $c$ represents a suitable positive constant which is not necessarily the same in each occurrence.

## 2. Preliminaries

We consider the differential-difference operator defined for a $C^\infty$ function $f$, on $\mathbb{R}$, by

$$\Lambda_\mu f(x) = \frac{df}{dx}(x) + \frac{2\mu+1}{x}\left[\frac{f(x) - f(-x)}{2}\right],$$

called Dunkl operator.

For $\lambda \in \mathbb{C}$, the initial problem

$$\Lambda_\mu f(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},$$

has a unique solution $E_\mu(\lambda\,.)$ called Dunkl kernel given by

$$E_\mu(\lambda x) = j_\mu(i\lambda x) + \frac{\lambda x}{2(\mu+1)} j_{\mu+1}(i\lambda x), \quad x \in \mathbb{R},$$

where $j_\mu$ is the normalized Bessel function of the first kind and order $\mu$, given by

$$j_\mu(\lambda x) = \begin{cases} 2^\mu \Gamma(\mu+1)\frac{J_\mu(\lambda x)}{(\lambda x)^\mu} & \text{if} \quad \lambda x \neq 0 \\ 1 & \text{if} \quad \lambda x = 0 \end{cases},$$

where $J_\mu$ is the Bessel function of first kind and order $\mu$ (see [13]).

We have for $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$

$$|E_\mu(-i\lambda x)| \leq 1. \tag{1}$$

Let $\nu_\mu$ the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\nu_\mu(x) = \frac{|x|^{2\mu+1}}{2^{\mu+1}\Gamma(\mu+1)}dx$$

For every $1 \leq p \leq \infty$, we denote by $L^p(\nu_\mu)$ the space of complex-valued functions $f$, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\mu} = \left(\int_{\mathbb{R}} |f(x)|^p d\nu_\mu(x)\right)^{1/p} < +\infty, \qquad \text{if } p < +\infty$$

and

$$\|f\|_{\infty,\mu} = ess \sup_{x\in\mathbb{R}} |f(x)| < +\infty.$$

The Dunkl transform $\mathcal{F}_\mu$ which was introduced by C. Dunkl in [4], is defined for $f \in L^1(\nu_\mu)$ by :

$$\mathcal{F}_\mu(f)(x) = \int_{\mathbb{R}} E_\mu(-ixy)f(y)d\nu_\mu(y), \quad x \in \mathbb{R},$$

and we have (see [2])

$$\|\mathcal{F}_\mu(f)\|_{\infty,\mu} \leq \|f\|_{1,\mu}. \tag{2}$$

For all $x, y, z \in \mathbb{R}$, consider

$$W_\mu(x,y,z) = \frac{(\Gamma(\mu+1))^2}{2^{\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x})\Delta_\mu(x,y,z)$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R} - \{0\}, z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\mu(x,y,z) = \begin{cases} \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\mu-1/2}}{|xyz|^{2\mu}} & \text{if } |z| \in A_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where $A_{x,y} = [||x| - |y||, |x| + |y|]$.

The kernel $W_\mu$ (see [9]), is even and we have

$$W_\mu(x,y,z) = W_\mu(y,x,z) = W_\mu(-x,z,y) = W_\mu(-z,y,-x). \tag{3}$$

In the sequel we consider the signed measure $\gamma_{x,y}$, on $\mathbb{R}$, given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\mu(x,y,z)d\nu_\mu(z) & \text{if } x,y \in \mathbb{R}\setminus\{0\} \\ d\delta_x(z) & \text{if } y=0 \\ d\delta_y(z) & \text{if } x=0 \end{cases}.$$

According to [9], the Dunkl kernel $E_\mu$, satisfies the following product formula

$$E_\mu(ixt)E_\mu(iyt) = \int_{\mathbb{R}} E_\mu(itz)d\gamma_{x,y}(z), \quad t \in \mathbb{R}. \tag{4}$$

The Dunkl translation operator $\tau_x$ (see [8], [10], [11]), given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z)d\gamma_{x,y}(z), \quad x,y \in \mathbb{R},$$

satisfies the following properties

$$\tau_x(f)(y) = \tau_y(f)(x)\ ; \quad \tau_0(f)(x) = f(x), \tag{5}$$

$$\|\tau_x f\|_{p,\mu} \leq 3\|f\|_{p,\mu}, \quad f \in L^p(\nu_\mu), \tag{6}$$

$$\mathcal{F}_\mu(\tau_x(f))(y)) = \mathcal{F}_\mu(f)(y)E_\mu(ixy). \tag{7}$$

The Dunkl convolution $f * g$ (see [9]), of the measurable functions $f$ and $g$ on $\mathbb{R}$, is defined by

$$(f * g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y)g(y)d\nu_\mu(y), \quad x \in \mathbb{R}.$$

Let $T \in ]0,+\infty]$, $1 < p \leq 2$ and $f \in L^p(\nu_\mu)$. For $\beta > \mu + \frac{1}{2}$, the Bochner-Riesz mean $\sigma_T^\beta(f)$ of $f$ is defined by:

$$\sigma_T^\beta(f)(x) = \int_{-T}^{T} E_\mu(ixy)(1-\frac{y^2}{T^2})^\beta \mathcal{F}_\mu(f)(y)d\nu_\mu(y), \quad x\in\mathbb{R}.$$

According to [7, (15)], we can write

$$S_T(f)(x) = (f\,*\,\varphi_T)(x) = \int_{\mathbb{R}} \tau_x(f)(-y)\varphi_T(y)d\nu_\mu(y), \quad x\in\mathbb{R},$$

where

$$\varphi_T(x) = \frac{T^{2(\mu+1)}}{2^{\mu+1}\Gamma(\mu+2)} j_{\mu+1}(xT), \quad x\in\mathbb{R}.$$

By Proposition 4 and Remark 2, § 3.1 of [7], for $\beta > \mu + \frac{1}{2}$, we have

$$\sigma_T^\beta(x) = (f\,*\,\phi_{T,\beta}(x) = \int_{\mathbb{R}} \tau_x(f)(-y)\phi_{T,\beta}(y)d\nu_\mu(y), \quad x\in\mathbb{R},$$

where

$$\phi_{T,\beta}(x) = \frac{\Gamma(\beta+1)T^{2(\mu+1)}}{2^{\mu+1}\Gamma(\mu+\beta+2)} j_{\mu+\beta+1}(xT), \quad x\in\mathbb{R}.$$

### 3. Uniform convergence of partial Dunkl integrals

In this section we establish that $S_T(f)\to f$, as $T\to+\infty$ in $L^\infty(\nu_\mu)$, provided that $f\in BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$. Before proving this result we need establish some useful lemmas.

Lemma 1. *If $f\in L^p(\nu_\mu)$ for some $1<p\le 2$ and $\beta>\mu+\frac{1}{2}$ then for every $0<T\le T_1<\infty$, we have*

$$S_T(S_{T_1}(f))(x) = S_{T_1}(S_T(f))(x) = S_T(f)(x), \quad x\in\mathbb{R}, \tag{8}$$

*and*

$$\sigma_T^\beta(S_{T_1}(f))(x) = \sigma_T^\beta(f)(x), \quad x\in\mathbb{R}. \tag{9}$$

P r o o f. Let $0<T\le T_1<\infty$. Define

$$h(z) = \chi_{[-T,T]}(z) \quad \text{and} \quad g(z) = \chi_{[-T_1,T_1]}(z)E_\mu(iyz), \quad y,z\in\mathbb{R},$$

then

$$\mathcal{F}_\mu(h)(z) \;=\; \int_{-T}^{T} E_\mu(-izt)d\nu_\mu(t) = \int_{-T}^{T} j_\mu(zt)d\nu_\mu(t) = \varphi_T(z),$$

and

$$\mathcal{F}_\mu(g)(z) = \int_{-T_1}^{T_1} E_\mu(-izt)E_\mu(iyt)d\nu_\mu(t),$$

by (3), (4) and using Fubini's theorem, we get

$$\begin{aligned}\mathcal{F}_\mu(g)(z) &= \int_{\mathbb{R}}\int_{-T_1}^{T_1} E_\mu(itx)d\nu_\mu(t)d\gamma_{y,-z}(x)\\ &= \int_{\mathbb{R}} \varphi_{T_1}(x)d\gamma_{y,-z}(x)\\ &= \tau_y(\varphi_{T_1})(-z)\end{aligned}$$

thus, Parseval's formula gives

$$\int_{\mathbb{R}} \mathcal{F}_\mu(g)(z)\overline{\mathcal{F}_\mu(h)(z)}d\nu_\mu(z) = \int_{\mathbb{R}} g(z)\overline{h(z)}d\nu_\mu(z)$$

it follows that

$$\int_{\mathbb{R}} \tau_y(\varphi_{T_1})(-z)\varphi_T(z)d\nu_\mu(z) = \int_{-T}^{T} E_\mu(iyz)d\nu_\mu(z)$$

we conclude by (5) that

$$(\varphi_{T_1} * \varphi_T)(z) = \varphi_T(z), \quad 0 < T \leq T_1$$

since the convolution product is associative and commutative (see [9]) we have

$$S_T(S_{T_1}(f)) = f \ * \ (\varphi_{T_1} * \varphi_T) = S_{T_1}(S_T(f)) = f \ * \ \varphi_T = S_T(f),$$

so (8) is established.

From [7, (21)], we have

$$\sigma_T^\beta(f)(x) = \frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t S_t(f)(x)dt, \quad x \in \mathbb{R},$$

so using (8), we obtain

$$\sigma_T^\beta(S_{T_1}(f))(x) = \sigma_T^\beta(f)(x), \quad x \in \mathbb{R}$$

hence (9) follows. ∎

Lemma 2. *If $f \in L^p(\nu_\mu)$ for some $\frac{4(\mu+1)}{2\mu+3} < p < \frac{4(\mu+1)}{2\mu+1}$ and $\beta > \mu + \frac{1}{2}$, then*

$$\|S_T(f)\|_{p,\mu} \leq c\|f\|_{p,\mu}, \quad T > 0$$

*and*

$$\lim_{T\to+\infty} \|S_T(f) - f\|_{p,\mu} = 0. \tag{10}$$

P r o o f. Since $f \in L^p(\nu_\mu)$, according to [7, Proposition 6] and (9), we have

$$\|S_T(f) - f\|_{p,\mu} \leq c\,\|\sigma_T^\beta(f) - f\|_{p,\mu}, \tag{11}$$

and

$$\|\sigma_T^\beta(f)\|_{p,\mu} \leq c\,\|f\|_{p,\mu}\ ; \quad \|S_T(f)\|_{p,\mu} \leq c\,\|f\|_{p,\mu}. \tag{12}$$

In [11], it was proved that $\sigma_T^\beta(f)(x) \to f(x)$ as $T \to +\infty$, almost everywhere $x \in \mathbb{R}$. By taking into account (11) and (12) we have $\|S_T(f) - f\|_{p,\mu} \to 0$, as $T \to +\infty$. ■

LEMMA 3. *If $f \in L^p(\nu_\mu)$ for some $\frac{4(\mu+1)}{2\mu+3} < p \leq 2$ and $T > 0$. Then for $x, \delta \in \mathbb{R}$, we have*

$$|\tau_\delta(S_T(f))(x) + \tau_{-\delta}(S_T(f))(x) - 2S_T(f)(x)| \leq c\,T^{\frac{2(\mu+1)}{p}} w_{p,\mu}(f)(\delta). \tag{13}$$

P r o o f. By using Fubini's theorem we obtain:

$$\begin{aligned}
\tau_\delta(S_T(f))(x) &= \int_{\mathbb{R}} \int_{-T}^{T} E_\mu(iyz)\mathcal{F}_\mu(f)(y)d\nu_\mu(y)d\gamma_{\delta,x}(z) \\
&= \int_{-T}^{T} \mathcal{F}_\mu(f)(y) \int_{\mathbb{R}} E_\mu(iyz)d\gamma_{\delta,x}(z)d\nu_\mu(y) \\
&= \int_{-T}^{T} \mathcal{F}_\mu(f)(y)E_\mu(i\delta y)E_\mu(ixy)d\nu_\mu(y)
\end{aligned}$$

by (7), we get

$$\tau_\delta(S_T(f))(x) = \int_{-T}^{T} E_\mu(ixy)\mathcal{F}_\mu(\tau_\delta(f))(y)d\nu_\mu(y) = S_T(\tau_\delta(f))(x).$$

Put

$$k(x) = [\tau_\delta(f) + \tau_{-\delta}(f) - 2f](x), \quad x \in \mathbb{R},$$

we can write from (1)

$$\begin{aligned}
|S_T(k)(x)| &= |[\tau_\delta(S_T(f))(x) + \tau_{-\delta}(S_T(f)) - 2S_T(f)](x)| \\
&\leq c\int_{-T}^{T} |\mathcal{F}_\mu(k)(y)||y|^{2\mu+1}dy.
\end{aligned}$$

Moreover, by invoking the Hölder inequality we have

$$|S_T(k)(x)| \leq c\left(\int_{-T}^{T} |y|^{2\mu+1}dy\right)^{1/p} \left(\int_{-T}^{T} |\mathcal{F}_\mu(k)(y)|^q|y|^{2\mu+1}dy\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Hence we obtain

$$|S_T(k)(x)| \le c\,T^{\frac{2(\mu+1)}{p}} \|\mathcal{F}_\mu(k)\|_{q,\mu},$$

according to (2) and Plancherel's Theorem for Dunkl transform (see [2]), we can assert by the Marcinkiewicz interpolation theorem (see [12]), that

$$|S_T(k)(x)| \le c\,T^{\frac{2(\mu+1)}{p}} \|k\|_{p,\mu},$$

which proves (13). ∎

Lemma 4. *If $f \in BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$ for some $\frac{4(\mu+1)}{2\mu+3} < p \le 2$ and $\beta > \mu + \frac{1}{2}$, then there exist $A \in ]1,\infty[$ such that for every $0 < T \le T_1 \le AT$,*

$$\|S_{T_1}(f) - S_T(f)\|_{\infty,\mu} \le c \int_{\frac{1}{T}}^{\frac{A}{T}} \frac{w_{p,\mu}(f(t))}{t^{\frac{2(\mu+1)}{p}}} \frac{dt}{t}. \tag{14}$$

P r o o f. Define

$$L(x) = S_{T_1}(f)(x) - S_T(f)(x), x \in \mathbb{R},$$

by (9) we have

$$\sigma_T^\beta(L)(x) = 0, \quad x \in \mathbb{R}.$$

According to [5, (19) p. 49], we have

$$\int_{\mathbb{R}} \phi_{T,\beta}(t) d\nu_\mu(t) = 1,$$

hence we can write

$$\begin{aligned} 2L(x) &= \int_{\mathbb{R}} [2L(x) - (\tau_t(L)(x) + \tau_{-t}(L)(x))] \phi_{T,\beta}(t) d\nu_\mu(t) \\ &= \int_{|t|\le \frac{1}{T}} + \int_{\frac{1}{T}\le |t| \le \frac{A}{T}} + \int_{\frac{A}{T}\le |t|} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{15}$$

Since

$$|\phi_{T,\beta}(t)| \le \frac{\Gamma(\beta+1)T^{2(\mu+1)}}{2^{\mu+1}\Gamma(\mu+\beta+2)}, \quad t \in \mathbb{R},$$

by (6), we get

$$\begin{aligned}
|I_1| &\leq 8\frac{\Gamma(\beta+1)T^{2(\mu+1)}}{2^{\mu+1}\Gamma(\mu+\beta+2)}\|L\|_{\infty,\mu}\int_{-1/T}^{1/T} d\nu_\mu(t)\\
&\leq 8\frac{\Gamma(\beta+1)T^{2(\mu+1)}2}{2^{2(\mu+1)}\Gamma(\mu+1)\Gamma(\mu+\beta+2)}\|L\|_{\infty,\mu}\int_0^{1/T} t^{2\mu+1}dt\\
&\leq 8\frac{\Gamma(\beta+1)}{2^{2(\mu+1)}\Gamma(\mu+2)\Gamma(\mu+\beta+2)}\|L\|_{\infty,\mu}.
\end{aligned}$$

In the other hand, we have

$$|I_3| \leq 8\|L\|_{\infty,\mu}\frac{2^\beta\Gamma(\beta+1)T^{2(\mu+1)}}{T^{\mu+\beta+1}2^{\mu+1}\Gamma(\mu+1)}\int_{\frac{A}{T}\leq|t|}\frac{|J_{\mu+\beta+1}(tT)|}{|t|^{\beta-\mu}}dt.$$

Since there exists a constant $C_0 > 0$ such that

$$|J_{\mu+\beta+1}(tT)| \leq \frac{C_0}{|tT|^{1/2}},\ t\neq 0,$$

we obtain

$$\begin{aligned}
|I_3| &\leq 8C_0\|L\|_{\infty,\mu}\frac{2^\beta\Gamma(\beta+1)T^{2(\mu+1)}}{T^{\mu+\beta+3/2}2^{\mu+1}\Gamma(\mu+1)}\int_{A/T\leq|t|}\frac{dt}{|t|^{\beta-\mu+\frac{1}{2}}}\\
&\leq 8C_0\|L\|_{\infty,\mu}\frac{2^{\beta+1}\Gamma(\beta+1)}{2^{\mu+1}\Gamma(\mu+1)T^{\beta-(\mu+\frac{1}{2})}}\int_{A/T}^{+\infty}\frac{dt}{t^{\beta+\frac{1}{2}-\mu}}\\
&\leq 8C_0\frac{2^{\beta+1}\Gamma(\beta+1)A^{\mu+\frac{1}{2}-\beta}}{2^{\mu+1}\Gamma(\mu+1)(\beta-\mu-\frac{1}{2})}\|L\|_{\infty,\mu}.
\end{aligned}$$

If we choose $A\in]1,+\infty[$ such that

$$\frac{8\Gamma(\beta+1)}{2^{2(\mu+1)}\Gamma(\mu+2)\Gamma(\mu+\beta+2)}+8C_0\frac{2^{\beta+1}\Gamma(\beta+1)A^{\mu+\frac{1}{2}-\beta}}{2^{\mu+1}\Gamma(\mu+1)(\beta-\mu-\frac{1}{2})}=\rho<2, \quad (16)$$

then

$$|I_1|+|I_3|\leq\rho\|L\|_{\infty,\mu}. \quad (17)$$

To estimate $I_2$, we can see that

$$|I_2|\leq\int_{\frac{1}{T}\leq|t|\leq\frac{A}{T}}|2S_{T_1}(f)(x)-[\tau_t\left(S_{T_1}(f)\right)(x)+\tau_{-t}\left(S_{T_1}(f)\right)(x)]|\,|\phi_{T,\beta}(t)|d\nu_\mu(t)$$

$$+\int_{\frac{1}{T}\leq|t|\leq\frac{A}{T}}|2S_T(f)(x)-[\tau_t\left(S_T(f)\right)(x)+\tau_{-t}\left(S_T(f)\right)(x)]||\phi_{T,\beta}(t)|d\nu_\mu(t),$$

by (13)

$$\begin{aligned}
|I_2| &\leq c\int_{\frac{1}{T}\leq|t|\leq\frac{A}{T}}[T_1^{\frac{2(\mu+1)}{p}}+T^{\frac{2(\mu+1)}{p}}]w_{p,\mu}(f)(t)|\phi_{T,\beta}(t)|d\nu_\mu(t)\\
&\leq c\int_{\frac{1}{T}\leq|t|\leq\frac{A}{T}}T^{\frac{2(\mu+1)}{p}}w_{p,\mu}(f)(t)|\phi_{T,\beta}(t)|d\nu_\mu(t)\\
&\leq c\int_{1/T}^{A/T}T^{\frac{2(\mu+1)}{p}}w_{p,\mu}(f)(t)|\phi_{T,\beta}(t)|t^{2\mu+1}dt\\
&\leq c\int_{1/T}^{A/T}T^{\frac{2(\mu+1)}{p}}T^{2(\mu+1)}w_{p,\mu}(f)(t)t^{2(\mu+1)}\frac{dt}{t}\\
&\leq c\int_{1/T}^{A/T}A^{2(\mu+1)}T^{\frac{2(\mu+1)}{p}}w_{p,\mu}(f)(t)\frac{dt}{t}\leq c\int_{1/T}^{A/T}\frac{w_{p,\mu}(f)(t)}{t^{\frac{2(\mu+1)}{p}}}\frac{dt}{t}.
\end{aligned}$$

Thus using (15) and (17), we conclude that

$$\|L\|_{\infty,\mu}\leq c\int_{1/T}^{A/T}\frac{w_{p,\mu}(f)(t)}{t^{\frac{2(\mu+1)}{p}}}\frac{dt}{t},$$

so (14) is established. ■

Theorem 1. *If* $f\in BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$ *for some* $\frac{4(\mu+1)}{2\mu+3}<p\leq 2$ *and* $\beta>\mu+\frac{1}{2}$, *then*

$$\|S_T(f)-f\|_{\infty,\mu}\to 0,\quad \text{as}\quad T\to\infty.$$

P r o o f. Let $0<T\leq T_1$ and $n$ a nonnegative integer for which $A^nT\leq T_1\leq A^{n+1}T$ where $A$ is the positive constant given by (16).

By (14) we can write

$$\|S_{T_1}(f)-S_T(f)\|_{\infty,\mu}\leq\|S_{T_1}(f)-S_{A^nT}(f)\|_{\infty,\mu}$$

$$+\sum_{k=0}^{n-1}\|S_{A^{k+1}T}(f)-S_{A^kT}(f)\|_{\infty,\mu}\leq c\int_0^{+\infty}\frac{w_{p,\mu}(f)(t)}{t^{\frac{2(\mu+1)}{p}}}\frac{dt}{t}.$$

If we replace $f$ by $S_{T_1}(f) - S_T(f)$ and apply (8), we find that

$$\|S_{T_1}(f) - S_T(f)\|_{\infty,\mu} \leq c \int_0^{+\infty} \frac{w_{p,\mu}(S_{T_1}(f) - S_T(f))(t)}{t^{\frac{2(\mu+1)}{p}}} \frac{dt}{t},$$

since

$$\begin{aligned} w_{p,\mu}(S_{T_1}(f) - S_T(f))(t) &\leq \|\tau_t(S_{T_1}(f) - f)\|_{p,\mu} + \|\tau_{-t}(S_{T_1}(f) - f)\|_{p,\mu} \\ &+ 2\|S_{T_1}(f) - f\|_{p,\mu} \\ &+ \|\tau_t(S_T(f) - f)\|_{p,\mu} + \|\tau_{-t}(S_T(f) - f)\|_{p,\mu} \\ &+ 2\|S_T(f) - f\|_{p,\mu} \\ &\leq 8[\|S_{T_1}(f) - f\|_{p,\mu} + \|S_T(f) - f\|_{p,\mu}], \end{aligned}$$

by (10) for any $t \in \mathbb{R}_+$, we get

$$\lim_{T,T_1 \to +\infty} w_{p,\mu}(S_{T_1}(f) - S_T(f))(t) = 0,$$

the dominated convergence theorem implies that

$$\|S_{T_1}(f) - S_T(f)\|_{\infty,\mu} \to 0 \quad \text{as } T_1, T \to \infty.$$

Hence we conclude that

$$\|S_T(f) - f\|_{\infty,\mu} \to 0 \quad \text{as } T \to \infty,$$

which proves the result. ■

## 4. Absolute integrability of Dunkl transform of functions in Besov-Dunkl spaces

We now prove that the Dunkl transform $\mathcal{F}_\mu(f)$ of $f$ is in $L^1(\nu_\mu)$ when $f$ belongs to *a* suitable Besov-Dunkl space. Before proving this result we need establish a useful lemma.

LEMMA 5. *If $f \in L^p(\nu_\mu)$ for some $1 < p \leq 2$, then*

$$\int_{\mathbb{R}} |t|^{2(\mu+1)(p-2)} |\mathcal{F}_\mu(f)(t)|^p d\nu_\mu(t) \leq c \int_{\mathbb{R}} |f(t)|^p d\nu_\mu(t). \tag{18}$$

P r o o f. To see (18) we will make use of the Marcinkiewicz interpolation theorem (see [12]).

Consider the operator

$$T(f)(x) = |x|^{2(\mu+1)}\mathcal{F}_\mu(f)(x), \quad x \in \mathbb{R}.$$

For every $f \in L^2(\nu_\mu)$, we have

$$\left(\int_{\mathbb{R}} |T(f)(x)|^2 \frac{d\nu_\mu(x)}{|x|^{4(\mu+1)}}\right)^{1/2} = \|\mathcal{F}_\mu(f)\|_{2,\mu} = \|f\|_{2,\mu}, \tag{19}$$

hence $T$ is an operator of Strong (2,2) type between the spaces $(\mathbb{R}, d\nu_\mu(x)))$ and $(\mathbb{R}, \frac{d\nu_\mu(x)}{|x|^{4(\mu+1)}})$. Moreover, according to (2), we can write for $\lambda \in ]0, +\infty)$ and $f \in L^1(\nu_\mu)$

$$\begin{aligned}\int_{\{x\in\mathbb{R}, |T(f)(x)|>\lambda\}} \frac{d\nu_\mu(x)}{|x|^{4(\mu+1)}} &\leq \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{|x|>(\frac{\lambda}{\|f\|_{1,\mu}})^{\frac{1}{2(\mu+1)}}} |x|^{-2\mu-3}dx \\ &\leq \frac{1}{2^{\mu}\Gamma(\mu+1)} \int_{(\frac{\lambda}{\|f\|_{1,\mu}})^{\frac{1}{2(\mu+1)}}}^{+\infty} x^{-2\mu-3}dx \leq c\,\frac{\|f\|_{1,\mu}}{\lambda}.\end{aligned} \tag{20}$$

Hence $T$ is an operator of weak (1,1) type between the spaces under consideration. By (19), (20) and the Marcinkiewicz interpolation theorem (see [12]), we can assert that for $1 < p \leq 2$, $T$ is an operator of strong $(p,p)$ type between the spaces $(\mathbb{R}, d\nu_\mu(x)))$ and $(\mathbb{R}, \frac{d\nu_\mu(x)}{|x|^{4(\mu+1)}})$. We conclude that

$$\begin{aligned}\int_{\mathbb{R}} |T(f)(x)|^p \frac{d\nu_\mu(x)}{|x|^{4(\mu+1)}} &= \int_{\mathbb{R}} |x|^{2(\mu+1)(p-2)}|\mathcal{F}_\mu(f)(x)|^p d\nu_\mu(x) \\ &\leq c\int_{\mathbb{R}} |f(x)|^p d\nu_\mu(x),\end{aligned}$$

thus we obtain the result. ∎

Theorem 2. *If $f \in BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$ for some $\frac{4(\mu+1)}{2\mu+3} < p \leq 2$, then*

$$\mathcal{F}_\mu(f) \in L^1(\nu_\mu).$$

P r o o f. Let $f \in BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$, since $f \in L^p(\nu_\mu)$ we have

$$\mathcal{F}_\mu(\tau_\delta(f) + \tau_{-\delta}(f) - 2f)(t) = \mathcal{F}_\mu(f)(t)[E_\mu(it\delta) + E_\mu(-it\delta) - 2],$$

$\delta \in (0, \infty)$ and a.e. $t \in \mathbb{R}$.

Then according to (18) it follows that

$$\begin{aligned}&\int_{\mathbb{R}} |\mathcal{F}_\mu(f)(t)|^p |E_\mu(it\delta) + E_\mu(-it\delta) - 2|^p |t|^{2(\mu+1)(p-2)} d\nu_\mu(t)\\ &\leq c \int_{\mathbb{R}} |(\tau_\delta(f) + \tau_{-\delta}(f) - 2f)(t)|^p d\nu_\mu(t) \leq c\,[w_{p,\mu}(f)(\delta)]^p.\end{aligned}$$

Moreover there exist $a, b \in (0,\infty)$, such that $|j_\mu(t\delta) - 1| \geq a(t\delta)^2$ for each $0 < |t\delta| < b$. Hence we can write

$$\begin{aligned}|E_\mu(-it\delta) + E_\mu(it\delta) - 2| &\geq 2|j_\mu(t\delta) - 1|\\ &\geq 2a(t\delta)^2 \quad \text{for each } 0 < |t\delta| < b,\end{aligned}$$

it follows that

$$\delta^{2p} \int_{|t|\leq \frac{b}{\delta}} |t|^{2(\mu+1)(p-2)+2p} |\mathcal{F}_\mu(f)(t)|^p d\nu_\mu(t) \leq c\,[w_{p,\mu}(f)(\delta)]^p.$$

By Hölder's inequality, for $q$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$\begin{aligned}\int_{|t|\leq \frac{b}{\delta}} |t||\mathcal{F}_\mu(f)(t)| d\nu_\mu(t) &\leq \Big(\int_{|t|\leq \frac{b}{\delta}} |t|^{2(\mu+1)(p-2)+2p} |\mathcal{F}_\mu(f)(t)|^p d\nu_\mu(t)\Big)^{1/p}\\ &\times \Big(\int_{|t|\leq \frac{b}{\delta}} |t|^{(2\mu+1)(q-2)-2} d\nu_\mu(t)\Big)^{1/q}\end{aligned}$$

$$\leq c\,\frac{w_{p,\mu}(f)(\delta)}{\delta^2} \times \frac{1}{\delta^{\frac{2(\mu+1)}{p}-1}} \leq c\,\frac{w_{p,\mu}(f)(\delta)}{\delta^{\frac{2(\mu+1)}{p}}} \times \frac{1}{\delta}.$$

Integrating with respect to $\delta$ over $\mathbb{R}_+$ and applying Fubini's theorem, it yields

$$\int_{\mathbb{R}} |\mathcal{F}_\mu(f)(t)| d\nu_\mu(t) \leq c \int_0^\infty \frac{w_{p,\mu}(f)(\delta)}{\delta^{\frac{2(\mu+1)}{p}}} \frac{d\delta}{\delta} < \infty,$$

by the definition of $BD^{p,1}_{\mu,\frac{2(\mu+1)}{p}}$. Thus, we obtain the desired result. ■

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