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ON THE GENERALIZED CONFLUENT HYPERGEOMETRIC FUNCTION AND ITS APPLICATION

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Dedicated to Professor Megumi Saigo,
on the occasion of his 70th birthday

Abstract

This paper is devoted to further development of important case of Wright's hypergeometric function and its applications to the generalization of Γ-, B-, ψ-, ζ-, Volterra functions.

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1. Introduction

In many areas of applied mathematics and in particular, in applied analysis and differential equations various types of special functions become essential tools for scientists and engineers. Further studying of the special functions is prospective and very useful for different branches of science. The diversity of the problems generating special functions has led to a quick increase in a number of special functions used in applications, see for example, [1]-[3],[5],[11],[15].

It should be noted that the special and degenerated cases of the hypergeometric functions, in particular, the Bessel, Legendre, Whittaker functions, the classical orthogonal polynomials such as the Jacobi, Laguerre, Kravchuk, Hermite polynomials etc. are expressed in terms of the ${}_2F_1(a,b;c;z)$. Examples are: the Bessel function

$$J_\nu = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\ {}_0F_1(\nu+1; -\frac{z^2}{4}),$$

the Kravchuk polynomial

$$k_n(x) = q^n \binom{x}{n}\ {}_2F_1\left(-n, x-N; x-n; -\frac{p}{q}\right), \qquad \text{etc.}$$

The special functions are kernels of many integral transforms, see e.g. [3],[6],[9],[11]. In turn, the theory of integral transforms is an effective mathematical instrument for solving many problems in analysis, boundary value problems of the mathematical physics, etc. The Fourier, Laplace, Mehler-Fock, Mellin transforms can be applied in many problems. Some new integral transforms have been introduced recently. Here it should be noted the appearance of the remarkable book about **H**-transform [9]. This book deals with integral transforms involving Fox's H-functions as kernels, and their applications. The H-function is defined by a Mellin-Barnes type integral with the integrand containing products and quotients of the Euler gamma functions.

The continuous development of the mathematical physics, mechanics of solid medium, quantum mechanics, probability theory, aerodynamics, biomedicine, theory of modeling, partition theory, combinatorics, astronomy, heat conduction and others has led to the generalization and the creation of new classes of special functions [1],[10], [11],[12],[14]. In our article we consider the generalized (in the sense of Wright [20],[21]) confluent hypergeometric functions and its applications to the generalization of the Γ-, B-, ψ-, ζ-, Volterra functions. These functions arise in such areas of applications as heat conduction, communication systems, electro-optics, approximation theory, probability theory, electric circuit, nuclear and molecular physics, diffraction and plasma wave problems, etc, see e.g. [5].

As it is known, Buchholz [4] constructed the theory of confluent hypergeometric functions on the base of the Whittaker functions, Tricomi took the Kummer function for construction of the confluent hypergeometric functions, Slater [16] applied both above methods.

2. Main results

Let us introduce the τ,β-generalized confluent hypergeometric function:

$${}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\ {}_1\Psi_1 \begin{bmatrix} (c,\tau); & zt^\tau \\ (c,\beta); & \end{bmatrix} dt, \quad (1)$$

where $\Re c > \Re a > 0,\ \tau \in \mathbb{R},\ \tau > 0,\ \beta \in R,\ \beta > 0, \tau - \beta < 1,\ \Gamma(a)$ is the classical gamma-function, ${}_1\Psi_1$ is the Fox-Wright function [7]-[8],[13]. As $\beta = \tau$ in (1), we have the function ${}_1\Phi_1^{\tau}(z)$ [17]-[18]; as $\tau = \beta = 1$, (1) gives the classical confluent hypergeometric function [7].

With the help of the series representation of ${}_1\Phi_1^{\tau,\beta}(z)$, application of the properties of $\Gamma-$, $B-$functions, the legality of interchanging the order of integration and summation, we obtain some relations for ${}_1\Phi_1^{\tau,\beta}(z)$. Let us lit below some of them:

$$\frac{d^n}{dz^n}\ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{\Gamma(c)\Gamma(a+n\tau)}{\Gamma(a)\Gamma(c+n\beta)}\ {}_1\Phi_1^{\tau,\beta}(a+n\tau;c+n\beta;z), \tag{2}$$

$$\frac{d}{dz}\Big[z^{c-1}\ {}_1\Phi_1^{\tau,\beta}(a;c;z^\beta) = z^{c-2}\ {}_1\Phi_1^{\tau,\beta}(a;c-1;z^\beta), \tag{3}$$

$$\left(\frac{d}{dz}\right)^m\Big[z^{c-1}\ {}_1\Phi_1^{\tau,\beta}(a;c;z^\beta) = z^{c-m-1}\ {}_1\Phi_1^{\tau,\beta}(a;c-m;z^\beta), \tag{4}$$

$${}_1\Phi_1^{\tau,\beta}(a;c;z) = z\,{}_1\Phi_1^{\tau,\beta}(a;c+\beta;z) + \frac{1}{\Gamma(c)}, \tag{5}$$

$${}_1\Phi_1^{\tau,\beta}(a;c;\frac{1}{x^\tau}) = x^a\frac{\Gamma(a+\tau)}{\Gamma(a)\Gamma(\tau)}\int_x^\infty \frac{(t-x)^{\tau-1}}{t^{a+\tau}}\ {}_1\Phi_1^{\tau,\beta}\left(a+\tau;c;\frac{1}{t^\tau}\right)dt, \tag{6}$$
$$(|x|>1),$$

$${}_1\Phi_1^{\tau,\beta}(a;c;\frac{1}{x^\beta}) = \frac{\Gamma(c)x^{c-\beta}}{\Gamma(c-\beta)\Gamma(\beta)}\int_x^\infty \frac{(t-x)^{\beta-1}}{t^{c}}\ {}_1\Phi_1^{\tau,\beta}\left(a;c-\beta;\frac{1}{t^\beta}\right)dt, \tag{7}$$
$$(|x|>1,\ \Re(c-\beta)>0),$$

$$\int_0^z t^{c-1}\ {}_1\Phi_1^{\tau,\beta}(a;c;t^\beta)dt = z^c\ {}_1\Phi_1^{\tau,\beta}(a;c+1;z^\beta), \tag{8}$$

$$\int_0^1 (1-t)^{c-1}t^{a-1}\ {}_1\Phi_1^{\tau,\beta}(a;c;z^{\tau+\beta}(1-t)^\beta)dt = B(a,c)\ {}_1\Phi_1^{\tau,\beta}(a;a+c;z^{\tau+\beta}), \tag{9}$$
$$(B(a,c) - \text{beta-function}),$$

$$\int_0^z (z-t)^{\alpha-1}t^{c-1}\ {}_1\Phi_1^{\tau,\beta}(a;c;t^\beta)dt = \frac{\Gamma(\alpha)z^{\alpha+c-1}}{\Gamma(\alpha+c)}\ {}_1\Phi_1^{\tau,\beta}(a;\alpha+c;z^\beta), \tag{10}$$

$$\frac{\Gamma(a+1)}{\tau\Gamma(c)}\ {}_1\Phi_1^{\tau,\beta}(a+1;c;z) - \frac{\Gamma(a+1)}{\tau\Gamma(c)}\ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{\Gamma(a+\tau)z}{\Gamma(c+\beta)}\ {}_1\Phi_1^{\tau,\beta}(a+\tau;c+\beta;z), \tag{11}$$

$$\int_0^z {}_1\Phi_1^{\tau,\beta}(a+\tau;c+\beta;t)dt = \frac{\Gamma(a)\Gamma(c+\beta)}{\Gamma(c)\Gamma(a+\tau)}\,{}_1\Phi_1^{\tau,\beta}(a+1;c;z)$$
$$-\frac{\tau}{a}\,z\,{}_1\Phi_1(a+\tau;c+\beta;z) - \frac{\Gamma(a)\Gamma(c+\beta)}{\Gamma(c)\Gamma(a+\tau)}\,. \tag{12}$$

Let us prove, for example, the last formula (12):

$$\int_0^z {}_1\Phi_1^{\tau,\beta}(a+\tau;c+\beta;t)dt = \int_0^z \frac{\Gamma(a)}{\Gamma(c)}\,\frac{\Gamma(c+\beta)}{\Gamma(a+\tau)}\frac{d}{dz}\Big[\,{}_1\Phi_1^{\tau,\beta}(a;c;t)\Big]dt.$$

Calculating the integral and using (11) we get:

$$\frac{\Gamma(a)}{\Gamma(c)}\,\frac{\Gamma(c+\beta)}{\Gamma(a+\tau)}\left\{\frac{\tau\Gamma(c)}{\Gamma(a+1)}\left[\frac{\Gamma(a+1)}{\Gamma(c)\tau}\,{}_1\Phi_1^{\tau,\beta}(a+1;c;z)\right.\right.$$
$$\left.\left.-\frac{\Gamma(a+\tau)}{\Gamma(c+\beta)}z\,{}_1\Phi_1^{\tau,\beta}(a+\tau;c+\beta;z)\right]-1\right\}$$
$$= \frac{\Gamma(a)\Gamma(c+\beta)}{\Gamma(c)\Gamma(a+\tau)}\,{}_1\Phi_1^{\tau,\beta}(a+1;c;z) - \frac{\tau\Gamma(a)z}{\Gamma(a+1)}\,{}_1\Phi_1^{\tau,\beta}(a+\tau;c+\beta;z)$$
$$-\frac{\Gamma(a)\Gamma(c+\beta)}{\Gamma(c)\Gamma(a+\tau)}\,.$$

LEMMA 1. *Let $\tau\in\mathbb{R},\tau>0,\beta\in\mathbb{R},\beta>0,c>0$, then the following integral relation is valid:*

$${}_1\Phi_1^{\tau,\beta}(a;c;z^\beta)={}_1\Phi_1^{\tau,2\beta}\Big(a;c;z^{2\beta}\Big)+\frac{z^{1-c}}{\Gamma(\beta)}\int_0^z t^{c-1}(z-t)^{\beta-1}\,{}_1\Phi_1^{\tau,2\beta}\Big(a;c;t^{2\beta}\Big)dt. \tag{13}$$

P r o o f. Let us consider the following expression:

$$A = \int_0^z t^{c-1}\,{}_1\Phi_1^{\tau,\beta}\Big(a;c;t^{2\beta}\Big)\left[1+\frac{(z-t)^\beta}{\Gamma(1+\beta)}\right]dt. \tag{14}$$

Using the series representation of ${}_1\Psi_1$ [7], the Leibnitz formula and relation (8), we obtain:

$$A = \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+\tau n)}{\Gamma(c+2\beta n)}\int_0^z t^{2n\beta+c-1}\left[1+\frac{(z-t)^\beta}{\Gamma(1+\beta)}\right]dt$$
$$= z^c\,{}_1\Phi_1^{\tau,\beta}(a;c+1;z^\beta). \tag{15}$$

After differentiation of (15) and some transformations, we get (13). ∎

With the help of ${}_1\Phi_1^{\tau,\beta}(a;c;z)$ we may consider some generalizations of special functions.

Let us define the $(\tau,\beta)-$**generalized gamma − function**, by means of the following expression:

$$ {}_{\tau,\beta}\Gamma_a^c(\alpha;\gamma,w;b) \equiv \tilde{\Gamma}(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t^w}\ {}_1\Phi_1^{\tau,\beta}(a;c;-bt^{-\gamma})dt, \qquad (16) $$

where $\Re c > \Re a > 0,\ \Re\alpha > 0,\ \tau \in \mathbb{R},\ \tau > 0,\ \beta \in \mathbb{R},\ \beta > 0, \tau - \beta < 1,$ $b > 0,\ \gamma \geq 1,\ w > 0$, and ${}_1\Phi_1^{\tau,\beta}$ is defined by (1). As $\beta = \tau,\ w = \gamma = 1$ in (16), we have the $\tau-$generalized gamma-function, [17].

THEOREM 1. *Let* $\Re c > \Re a > 0,\ \tau \in \mathbb{R},\ \tau > 0,\ a + \tau n \neq 0, -1, -2, ...$ *as* $n = 0, 1, 2, ...$*;* a *and* c *are such that* $\Gamma(a + \tau n),\ \Gamma(c + \tau n)$ *are finite as* $n = 0, 1, 2, ...; \Re\alpha > 0,\ b > 0,\ w > 0,\ \gamma \geq 1$. *Then there holds the formula:*

$$ {}_\tau\Gamma_a^c(\alpha;\gamma,w;b) = \frac{\Gamma(c)}{\Gamma(a)w}\sum_{n=0}^\infty \frac{(-b)^n}{n!}\,\frac{\Gamma(a+n\tau)}{\Gamma(c+n\beta)}\,\Gamma\left(\frac{\alpha-\gamma n}{w}\right). \qquad (17) $$

P r o o f. In view of the representation of ${}_1\Phi_1^\tau(z)$ [17], interchanging the order of integration and summation, we oftain (17). ■

The (τ,β)**-generalized beta-function** we may define in the following form:

$$ {}_{\tau,\beta}B_a^c(\alpha;\mu;b) \equiv \tilde{B} = \int_0^1 t^{\alpha-1}(1-t)^{\mu-1}\ {}_1\Phi_1^{\tau,\beta}\left(a;c;-\frac{b}{t(1-t)}\right)dt, \qquad (18) $$

where $\Re c > \Re a > 0,\ \Re b > 0,\ \Re\alpha > 0,\ \Re\mu > 0,\ \tau \in \mathbb{R},\ \tau > 0,\ \beta \in \mathbb{R},$ $\beta > 0,\ \tau - \beta < 1,\ {}_1\Phi_1^{\tau,\beta}(z)$ is the function (1).

Using integral representation (18), the corresponding substitutions and transformations we can evaluate some integrals which are missing in the known literature, in particular:

$$ \int_0^1 (1-t^2)^{\alpha-1}\exp\left(-\frac{4b}{1-t^2}\right)dt = 4^{\alpha-1}\ {}_\tau B_a^a(\alpha;\alpha;b), \qquad (19) $$

$$ \int_0^\infty \frac{\cos(2zt)}{ch(\pi t)}\ {}_1\Phi_1^\tau\left(a;c;-4bch^2(\pi t)\right)dt = \frac{1}{2\pi}\ {}_\tau B_a^c\left(\frac{1}{2}+\frac{iz}{\pi},\ \frac{1}{2}-\frac{iz}{\pi};\ b\right), \qquad (20) $$

where $\Re\alpha > 0,\ \Re c > Rea > 0,\ \Re b > 0,\ \tau > 0,\ |Imz| < \frac{\pi}{2}$.

The (τ,β)**- generalized psi-function** has the following form:

$$ {}_{\tau,\beta}\psi_a^c(\alpha;\gamma,w;b) \equiv \tilde{\psi}(\alpha) = \frac{d\ln\tilde{\Gamma}(\alpha)}{d\alpha} = \frac{1}{\tilde{\Gamma}(\alpha)}\,\frac{d\tilde{\Gamma}(\alpha)}{d\alpha} $$

$$= \frac{1}{\tilde{\Gamma}(\alpha)} \int_0^\infty t^{\alpha-1} (\ln t) e^{-t^w} \, {}_1\Phi_1^{\tau,\beta}(a; c; -bt^{-\gamma}) dt, \tag{21}$$

where $\Re c > \Re a > 0$, $\Re \alpha > 0, \tau \in \mathbb{R}$, $\tau > 0$, $\beta \in \mathbb{R}, \beta > 0$, $b > 0$, $\gamma \geq 1$, $w > 0$, ${}_1\Phi_1^{\tau,\beta}(a; c; z)$ is the function (1).

When $\beta = \tau$, $w = 1$ the Dirichlet formula [7] is valid in the following form:

$$\tilde{\psi}(\alpha) = \int_0^\infty \left[e^{-x} - (1+x)^{-\alpha} \, \frac{\tilde{\Gamma}(\alpha, b(1+x)^\gamma)}{\tilde{\Gamma}(\alpha; b)} \right] x^{-1} dx. \tag{22}$$

The (τ, β)**- generalized Hurwitz** ζ**- function** is defined as follows:

$${}_{\tau,\beta}\zeta_b(\alpha, q; w, \gamma; a, c) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-qt}}{(1-e^{-t})^w} \, {}_1\Phi_1^{\tau,\beta}(a; c; -bt^{-\gamma}) dt, \tag{23}$$

where $\Re c > \Re a > 0$, $\Re q > 0$, $\Re \alpha > 0$, $\tau \in \mathbb{R}$, $\tau > 0$, $\beta \in \mathbb{R}, \beta > 0$, $\gamma > 0$, $w \geq 1$, $\Re b > 0$, ${}_1\Phi_1^{\tau,\beta}(z)$ is the function (1).

Theorem 2. *Let* $\Re \alpha > 0, \Re c > \Re a > 0$, $\Re q > 0$, $\gamma > 0$, $w \geq 1$, $\tau \in \mathbb{R}$, $\tau > 0$. *Then there holds the formula:*

$${}_\tau\zeta_b(\alpha, q; w, \gamma; a, c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(\alpha)} \sum_{n=0}^\infty \frac{(-b)^n}{n!} \frac{\Gamma(a+\tau n)}{\Gamma(c+\tau n)} \, {}_\tau\zeta_0(\alpha - \gamma n, q; w) \Gamma(\alpha - \gamma n), \tag{24}$$

where

$${}_\tau\zeta_0 = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-qt}}{(1-e^{-t})^w} dt = \frac{1}{\Gamma(w)} \sum_{n=0}^\infty \frac{\Gamma(w+n)}{(n+q)^\alpha} \cdot \frac{1}{n!}. \tag{25}$$

The proof of this theorem is similar to the proof of Theorem 1.

We now give the following **generalization of the Volterra functions** [19]:

$${}_{\tau,\beta}\, \nu_a^c(b; x) = \int_0^\infty \frac{x^t}{\tilde{\Gamma}(t+1; b)} dt, \tag{26}$$

$${}_{\tau,\beta}\, \nu_a^c(b; x; \alpha) = \int_0^\infty \frac{x^{\alpha+t}}{\tilde{\Gamma}(\alpha+t+1; b)} dt, \tag{27}$$

$${}_{\tau,\beta}\, \mu_a^c(b; x; \gamma) = \int_0^\infty \frac{x^t t^\gamma}{\Gamma(\gamma+1)\tilde{\Gamma}(t+1; b)} dt, \tag{28}$$

$${}_{\tau,\beta}\, \mu_a^c(b; x; \gamma; \alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\gamma}{\Gamma(\gamma+1)\tilde{\Gamma}(\alpha+t+1; b)}, \tag{29}$$

where $x \neq 0$; x, α, γ are parameters, $\Re\gamma > 0$, $\tilde{\Gamma}$ is the (τ, β)-generalized gamma function (16).

Let us notice that:
- all integrals in (26)-(29) are convergent as $x \neq 0$, $\Re\gamma > -1$, α is arbitrary;
- all functions (26)-(29) are analytical functions with branch points $x = 0$ and $x = \infty$;
- all functions (27)-(29) are entire functions of α.

LEMMA 2. *If $x \neq 0$, x is arbitrary, then for ${}_{\tau,\beta}\mu_a^c(b; x; \gamma; \alpha)$ the following holds for $\Re\beta > -m - 1$:*

$$ {}_{\tau,\beta}\,\mu_a^c(b; x; \gamma; \alpha) = \frac{(-1)^m}{\Gamma(\gamma + m + 1)} \int_0^\infty t^{\gamma+m} \frac{d^m}{dt^m} \left[\frac{x^{\alpha+t}}{{}_{\tau,\beta}\Gamma_a^c(\alpha + t + 1; b)} \right] dt. \tag{30} $$

The proof is done by integration by parts m-times and taking into account formulae (29),(16),(1),(2),(4).

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