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ON INTEGRAL MEANS FOR FRACTIONAL CALCULUS OPERATORS OF MULTIVALENT FUNCTIONS

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Dedicated to Professor Megumi Saigo,
on the occasion of his 70th birthday

Abstract

Integral means inequalities are obtained for the fractional derivatives and the fractional integrals of multivalent functions. Relevant connections with various known integral means inequalities are also pointed out.

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1. Introduction

Let $\mathcal{A}_{p,n}$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $p(z)$ be given by

$$p(z) = z^p + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{sj-(s-1)p} \quad (j \geqq n+p \quad ; n \in \mathbb{N}; \quad m \geq 2). \tag{1.2}$$

A function $f(z)$ belonging to $\mathcal{A}_{p,n}$ is called multivalently starlike of order α in $\mathbb{U}$, if it satisfies the inequality:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha\,(0 \leq \alpha < p)$. Also, a function $f(z) \in \mathcal{A}_{p,n}$ is said to be multivalently convex of order α in $\mathbb{U}$, if it satisfies the inequality

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha\,(0 \leq \alpha < p)$. We denote by $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$ the class of functions $f(z) \in \mathcal{A}_{p,n}$ which are multivalently starlike of order α and multivalently convex of order α, respectively. We note that

$$f(z) \in \mathcal{K}_{p,n}(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*_{p,n}(\alpha)$$

For functions $f(z)$ belonging to the classes $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$, Owa [4] has shown the following coefficient inequalities.

Theorem 1.1. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)\,|a_k| \leq p-\alpha \tag{1.3}$$

for some $\alpha\,(0 \leq \alpha < p)$, then $f(z) \in \mathcal{S}^*_{p,n}(\alpha)$.

Theorem 1.2. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)\,|a_k| \leq p-\alpha \tag{1.4}$$

for some $\alpha\,(0 \leq \alpha < p)$, then $f(z) \in \mathcal{K}_{p,n}(\alpha)$.

In this paper, we investigate the integral means inequalities for the fractional derivatives and for the fractional integrals of multivalent functions.

We shall make use of the following definitions of fractional calculus (cf. Owa[6]; see also Srivastava and Owa [7]).

Definition 1. The fractional integral of order λ is defined, for a function $f\,(z)$, by

$$D_z^{-\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}}d\xi \qquad (\lambda > 0),$$

where the function $f\,(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^\lambda = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\lambda}d\xi \qquad (0 \le \lambda < 1),$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed, as in Definition 1, above.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z) \qquad (0 \le \lambda < 1; n \in \mathbb{N}_0 = \mathbb{N} \cup 0).$$

It readily follows from Definitions 1 and 2 that

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\lambda+1)} z^{k+\lambda} \qquad (\lambda > 0) \tag{1.5}$$

and

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \qquad (0 \le \lambda < 1), \tag{1.6}$$

respectively.

Further, we need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad \text{(cf. Duren [1])}.$$

Theorem 1.3. (Littlewood [3]) *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then for $\mu > 0$and $z = re^{i\theta} (0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

2. Integral means inequalities

First we prove the integral means inequalities for the fractional derivatives.

THEOREM 2.1. *Let $f(z) \in \mathcal{A}_{p,n}$, $p(z)$ be given by (1.2), $p > \lambda$, and suppose that*

$$\sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} |a_k|$$
$$\leqq \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)\Gamma(n+p+1-\lambda-\delta)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(n+p-\lambda)\Gamma(p-\lambda+1-\delta)} |b_{sj-(s-1)p}| \tag{2.1}$$

for $\lambda = 0$ or 1 $(0 \leq \delta, \nu < 1)$ and $2 \leq \lambda \leq n$ $(0 < \delta, \nu < 1)$, where $(k-\lambda)_{\lambda+1}$ denotes the Pochhammer symbol defined by $(k-\lambda)_{\lambda+1} = (k-\lambda)(k-\lambda+1)\dots k$. Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{\lambda+\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| \frac{\Gamma(p-\lambda+1-\nu)}{\Gamma(p-\lambda+1-\delta)} z^{\nu-\delta} D_z^{\lambda+\nu} p(z) \right|^{\mu} d\theta \quad (\mu > 0). \tag{2.2}$$

P r o o f. By means of the fractional derivative formula (1.6) and Definition 3, we find from (1.1) that

$$D_z^{\lambda+\delta} f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1-\delta)} z^{p-\lambda-\delta} \left[1 + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)\Gamma(k-\lambda+1-\delta)} a_k z^{k-p} \right]$$
$$= \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1-\delta)} z^{p-\lambda-\delta} \left[1 + \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)} \Phi(k) a_k z^{k-p} \right],$$

where

$$\Phi(k) = \frac{\Gamma(k-\lambda)}{\Gamma(k+1-\lambda-\delta)} \quad \left\{ \begin{array}{ll} \lambda = 0 \text{ or } 1 & (0 \leq \delta < 1) \\ 2 \leq \lambda \leq n & (0 < \delta < 1) \end{array} ; k \geq n+p,\, n \in \mathbb{N} \right\}.$$

Since $\Phi(k)$ is a decreasing function of k, we have

$$0 < \Phi(k) \leq \Phi(n+p) = \frac{\Gamma(n+p-\lambda)}{\Gamma(n+p+1-\lambda-\delta)}$$
$$\left\{ \begin{array}{ll} \lambda = 0 \text{ or } 1 & (0 \leq \delta < 1) \\ 2 \leq \lambda \leq n & (0 < \delta < 1) \end{array} ; k \geq n+p,\, n \in \mathbb{N} \right\}.$$

Similarly, by using (1.2),(1.6) and Definition 3, we obtain

$$D_z^{\lambda+\nu} p(z) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1-\nu)} z^{p-\lambda-\nu}$$
$$\times \left[1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(p+1)\Gamma(sj-(s-1)p-\lambda+1-\nu)} b_{sj-(s-1)p} z^{s(j-p)} \right].$$

Thus we have

$$\frac{\Gamma(p-\lambda+1-\nu)}{\Gamma(p-\lambda+1-\delta)} z^{\nu-\delta} D_z^{\lambda+\nu} p(z) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1-\delta)} z^{p-\lambda-\delta} \times \left[1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(p+1)\Gamma(sj-(s-1)p-\lambda+1-\nu)} b_{sj-(s-1)p} z^{s(j-p)}\right].$$

For $z = re^{i\theta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left|1 + \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)} \Phi(k) a_k z^{k-p}\right|^{\mu} d\theta$$
$$\leq \int_0^{2\pi} \left|1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} b_{sj-(s-1)p} z^{s(j-p)}\right|^{\mu} d\theta \ (\mu > 0).$$

By applying Theorem 1.3, it suffices to show that

$$1 + \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)} \Phi(k) a_k z^{k-p}$$
$$\prec 1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} b_{sj-(s-1)p} z^{s(j-p)}. \quad (2.3)$$

By setting

$$1 + \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)} \Phi(k) a_k z^{k-p}$$
$$= 1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} b_{sj-(s-1)p} \{w(z)\}^{s(j-p)},$$

we find that

$$\{w(z)\}^{s(j-p)} = \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)}$$
$$\times \Phi(k) a_k z^{k-p} \frac{1}{\sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} b_{sj-(s-1)p}}$$

which readily yields $w(0)=0$. Therefore, we have

$$
\begin{aligned}
|w(z)|^{s(j-p)} &\leq \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1} \frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)} \\
&\qquad \times \Phi(k)\,|a_k|\,|z|^{k-p} \frac{1}{\sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} \left|b_{sj-(s-1)p}\right|} \\
&\leq |z|^n \frac{\Phi(n+p)\frac{\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)}}{\sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} \left|b_{sj-(s-1)p}\right|} \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1}\,|a_k| \\
&= |z|^n \frac{\frac{\Gamma(n+p-\lambda)\Gamma(p-\lambda+1-\delta)}{\Gamma(p+1)\Gamma(n+p+1-\lambda-\delta)}}{\sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p-\lambda+1-\nu)}{\Gamma(sj-(s-1)p-\lambda+1-\nu)\Gamma(p+1)} \left|b_{sj-(s-1)p}\right|} \sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1}\,|a_k| \\
&\qquad \leq |z|^n < 1
\end{aligned}
$$

by means of the hypothesis (2.1) of Theorem 2.1.

In light of the last inequality above, we have the subordination (2.3) which evidently proves Theorem 2.1. ∎

Remark 1. By applying the Hölder inequality to the right hand side of the inequality (2.2) of Theorem 2.1, we have the following

$$
\begin{aligned}
\int_0^{2\pi} |F(z)|^{\mu}\, d\theta &\leq \left\{ \int_0^{2\pi} |F(z)|^{\mu\cdot\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \left\{ \int_0^{2\pi} 1^{\frac{2}{2-\mu}} d\theta \right\}^{\frac{2-\mu}{2}} \\
&= \left\{ \int_0^{2\pi} |F(z)|^2 d\theta \right\}^{\frac{\mu}{2}} (2\pi)^{\frac{2-\mu}{2}},
\end{aligned}
$$

where $0<\mu<2$ and

$$F(z) = \frac{\Gamma(p-\lambda+1-\nu)}{\Gamma(p-\lambda+1-\delta)} z^{\nu-\delta} D^{\lambda+\nu} p(z).$$

This remark is applicable to the following results.

Next we have the following integral means inequalities for the fractional integrals.

Theorem 2.2. *Let $f(z) \in \mathcal{A}_{p,n}$ and $p(z)$ given by* (1.2) *and supposed that*

$$\sum_{k=p+n}^{\infty} k\,|a_k|$$
$$\leqq \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(p+1+\nu)\Gamma(n+p+1+\delta)}{\Gamma(sj-(s-1)p+1+\nu)\Gamma(n+p)\Gamma(p+1+\delta)} \left|b_{sj-(s-1)p}\right|$$

for δ and $\nu > 0$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{-\delta} f(z)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|\frac{\Gamma(p+1+\nu)}{\Gamma(p+1+\delta)} z^{\delta-\nu} D_z^{-\nu} p(z)\right|^{\mu} d\theta \qquad (\mu > 0).$$

P r o o f. By virtue of the fractional integral formula (1.5) and the fractional derivative formula (1.6), replacing δ with $-\delta(\delta > 0)$ and ν with $-\nu(\nu > 0)$ and putting $\lambda = 0$ in Theorem 2.1, we complete the proof. ■

When $\delta = \nu$, Theorem 2.1 readily yields the following.

Corollary 2.1. *Let $f(z) \in \mathcal{A}_{p,n}$, $p(z)$ be given by (1.2), $p > \lambda$, and suppose that*

$$\sum_{k=p+n}^{\infty} (k-\lambda)_{\lambda+1}\,|a_k|$$
$$\leq \sum_{k=p+n}^{\infty} \frac{\Gamma(sj-(s-1)p+1)\Gamma(n+p+1-\lambda-\delta)}{\Gamma(sj-(s-1)p-\lambda-\delta+1)\Gamma(n+p-\lambda)} \left|b_{sj-(s-1)p}\right|$$

for $0 \leq \lambda \leq n$ and $0 \leq \delta < 1$, where $(k-\lambda)_{\lambda+1}$ denotes the Pochhammer symbol defined by $(k-\lambda)_{\lambda+1} = (k-\lambda)(k-\lambda+1)\dots k$.

Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{\lambda+\delta} f(z)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|D_z^{\lambda+\delta} p(z)\right|^{\mu} d\theta \qquad (\mu > 0).$$

Also, from Theorem 2.2, when $\delta = \nu$, we have the following:

Corollary 2.2. *Let $f(z) \in \mathcal{A}_{p,n}$ and $p(z)$ be given by (1.2), and supposed that*

$$\sum_{k=p+n}^{\infty} k\,|a_k| \leq \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(n+p+1+\delta)}{\Gamma(sj-(s-1)p+1+\delta)\Gamma(n+p)} \left|b_{sj-(s-1)p}\right|.$$

Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{-\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| D_z^{-\delta} p(z) \right|^{\mu} d\theta \qquad (\mu > 0).$$

As the special case $\lambda = 0$, Corollary 2.1 readily yields the following corollary.

Corollary 2.3. *Let* $f(z) \in \mathcal{A}_{p,n}$ *and* $p(z)$ *be given by (1.2), and supposed that*

$$\sum_{k=p+n}^{\infty} k\,|a_k| \leq \sum_{s=1}^{m} \frac{\Gamma(sj-(s-1)p+1)\Gamma(n+p+1-\delta)}{\Gamma(sj-(s-1)p+1-\delta)\Gamma(n+p)} \left| b_{sj-(s-1)p} \right|$$
$$(0 \leq \delta < 1; j \geq n+p; n \in \mathbb{N}).$$

Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| D_z^{\delta} p(z) \right|^{\mu} d\theta \qquad (\mu > 0).$$

Also, from Corollary 2.1, when $\lambda = 1$*, we have the following.*

Corollary 2.4. *Let* $f(z) \in \mathcal{A}_{p,n}$ *and* $p(z)$ *be given by (1.2), and supposed that*

$$\sum_{k=p+n}^{\infty} k(k-1)\,|a_k| \leq \sum_{k=p+n}^{\infty} \frac{\Gamma(sj-(s-1)p+1)\Gamma(n+p-\delta)}{\Gamma(sj-(s-1)p-\delta)\Gamma(n+p-1)} \left| b_{sj-(s-1)p} \right|$$
$$(0 \leq \delta < 1; j \geq n+p; n \in \mathbb{N}).$$

Then for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{1+\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| D_z^{1+\delta} p(z) \right|^{\mu} d\theta \qquad (\mu > 0).$$

Also, when $p = 1$, we have the following corollaries, see [8]:

Corollary 2.5. *Let* $f(z) \in \mathcal{A}_n$ *and*

$$p\,(z) = z + \sum_{s=1}^{m} b_{sj-s+1} z^{sj-s+1} \qquad (j \geqq n+1 \quad ; n \in \mathbb{N}; \quad m \geq 2) \qquad (2.4)$$

and supposed that

$$\sum_{k=n+1}^{\infty} (k-\lambda)_{\lambda+1}\,|a_k|$$
$$\leqq \sum_{s=1}^{m} \frac{\Gamma(sj-s+2)\Gamma(2-\lambda-\nu)\Gamma(n+2-\lambda-\delta)}{\Gamma(sj-s+2-\lambda-\nu)\Gamma(n+1-\lambda)\Gamma(2-\lambda-\delta)} \left| b_{sj-s+1} \right|$$

for $\lambda = 0$ *or 1* $(0 \leq \delta, \nu < 1)$ *and* $2 \leq \lambda \leq n(0 < \delta, \nu < 1)$. *Then for* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{\lambda+\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| \frac{\Gamma(2-\lambda-\nu)}{\Gamma(2-\lambda-\delta)} z^{\nu-\delta} D_z^{\lambda+\nu} p(z) \right|^{\mu} d\theta \qquad (\mu > 0).$$

Corollary 2.6. *Let* $f(z) \in \mathcal{A}_n$ *and* $p(z)$ *given by (2.4) and supposed that*

$$\sum_{k=n+1}^{\infty} k\,|a_k| \leqq \sum_{s=1}^{m} \frac{\Gamma(sj-s+2)\Gamma(2+\nu)\Gamma(n+2+\delta)}{\Gamma(sj-s+2+\nu)\Gamma(n+1)\Gamma(2+\delta)} \,|b_{sj-s+1}|$$

for δ *and* $\nu > 0$. *Then, for* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{-\delta} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| \frac{\Gamma(2+\nu)}{\Gamma(2+\delta)} z^{\delta-\nu} D_z^{-\nu} p(z) \right|^{\mu} d\theta \qquad (\mu > 0).$$

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