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GENERALIZED SOBOLEV SPACES OF EXPONENTIAL TYPE ASSOCIATED WITH THE DUNKL OPERATORS

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Dedicated to Professor Khalifa Trimèche,
on the occasion of his 60th anniversary

Abstract

In this paper we study generalized Sobolev spaces H^s_G of exponential type associated with the Dunkl operators based on the space G of test functions for generalized hyperfunctions and investigate their properties. Moreover, we introduce a class of symbols of exponential type and their associated pseudodifferential operators related to the Dunkl operators, which act naturally on H^s_G.

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1. Introduction

The Sobolev space $H^s = H^s(\mathbb{R})$ associated with the Dunkl operators is defined by

$$H^s = \{u \in S'(\mathbb{R}) / \|u\|_s = \Big[\int_{\mathbb{R}} |\mathcal{F}_D(u)(\xi)|^2 (1+|\xi|^2)^s dm_\alpha(\xi)\Big]^{\frac{1}{2}} < \infty\}, \quad s \in \mathbb{R}. \tag{1.1}$$

In the present work we fix $\alpha \geq -\frac{1}{2}$ and introduce generalized Sobolev spaces of exponential type H^s_G [6] associated with the Dunkl operators replacing $(1+|\xi|^2)^s$ by an exponential weight function. Here and later on, G

will be the space of test functions for generalized hyperfunctions [5]. The generalized Sobolev spaces cited above consist of all u in $G^{'}$, the topological dual of G, such that $\mathcal{F}_D(u)$, the Dunkl transform [2] of u, is a function and satisfies

$$\|u\|_s = \Big[\int_{\mathbb{R}} |\mathcal{F}_D(u)(\xi)|^2 e^{2s|\xi|} dm_\alpha(\xi)\Big]^{\frac{1}{2}} < \infty,$$

where

$$dm_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx.$$

We recall here that the Fourier-Dunkl transform of a suitable function f is given by

$$\mathcal{F}_D(f)(\lambda) = \int_{\mathbb{R}} \psi^\alpha_{-\lambda}(x) f(x) dm_\alpha(x), \;\; \forall \lambda \in \mathbb{R}.$$

Moreover, $\mathcal{F}_D$ is a topological isomorphism from $S(\mathbb{R})$ onto it self. Its inverse is given by

$$\mathcal{F}_D^{-1}(h)(x) = \int_{\mathbb{R}} \psi^\alpha_{\lambda}(x) h(\lambda) dm_\alpha(\lambda), \;\; \forall x \in \mathbb{R}, \tag{1.2}$$

where the function ψ^α_λ can be expressed in terms of the normalized Bessel functions [7].

This paper is organized as follows: in Section 2, we collect some harmonic analysis properties associated with the Dunkl theory on the real line, see [4] and [2]. In Section 3, we introduce the generalized Sobolev spaces of exponential type associated with Dunkl operators and investigate their properties, in particular the structure theorem which asserts that for $s > 0$ and $u \in H^{-s}_G$, u takes the following form

$$u = \sum_{\beta \in \mathbb{N}} \frac{s^\beta}{i^\beta \beta!} {D_\alpha}^\beta g_\alpha, \qquad \beta \in \mathbb{N},$$

where g_β are square integrable functions. In Section 4, we introduce certain classes of pseudodifferential operators associated with the Dunkl operators whose symbols have suitable growth condition and which act naturally on the generalized Sobolev spaces cited above. In other words, for $r, l \in \mathbb{R}$, with $l > 0$ the space $S^{r,l}_{\exp}$ of symbols is defined so that its associated pseudodifferential operators maps H^s_G onto H^{r+s}_G. As an example, we give a differential operator of infinite order with variable coefficients of this class.

2. Preliminaries

In what follows we collect some harmonic analysis results related to Dunkl operators D_α, defined on the real line [4], given for an appropriate function f by

$$D_\alpha f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}, \qquad \alpha \geq -\frac{1}{2}.$$

It is well known that the following system

$$\begin{cases} D_\alpha f(x) &= i\lambda f(x), \quad \lambda \in \mathbb{C}, \\ f(0) &= 1, \end{cases}$$

admits a unique solution ψ_λ^α satisfying $|\psi_\lambda^\alpha| \leq 1$ and expressed in terms of the normalized spherical Bessel functions j_α and $j_{\alpha+1}$ as follows:

$$\psi_\lambda^\alpha(x) = j_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x), \quad \forall x \in \mathbb{R}.$$

The convolution product of a suitable pair of functions f and g is given by

$$f *_\alpha g(x) = \int_{\mathbb{R}} T_x^\alpha f(y) g(-y) dm_\alpha(y),$$

where T_x^α, $x \in \mathbb{R}$, are the translation operators associated with the Dunkl operators, given by

$$T_x^\alpha f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^\alpha(z),$$

$\mu_{x,y}^\alpha$ is signed measure given by

$$d\mu_{x,y}^\alpha(z) = \begin{cases} \mathcal{W}_\alpha(x,y,z) dm_\alpha(z), & \text{if} \quad xy \neq 0 \\ d\delta_x(z), & \text{if} \quad y = 0 \\ d\delta_y(z), & \text{if} \quad x = 0 \end{cases},$$

where δ_x denotes the Dirac measure at the point x, and

$$W_\alpha(x,y,z) = 2^{\alpha+1}\Gamma(\alpha+1)\rho(x,y,z)K_\alpha(|x|,|y|,|z|),$$

$K_\alpha(x,y,z)$ is given by:

$$K_\alpha(x,y,z) = \begin{cases} \dfrac{2^{1-2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \dfrac{[((x+y)^2 - z^2)(z^2 - (x-y)^2)]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}} \\ \qquad\qquad\qquad\qquad\qquad \text{if} \quad |x-y| < z < x+y, \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

while $\rho(x,y,z) = \frac{1}{2}[1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}]$, with $\sigma_{x,y,z} = \frac{x^2+y^2-z^2}{2xy}$, for all $x,\ y \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{R}$.

The translation operators given above satisfy the following formula [2]

$$\mathcal{F}_D(T_x^\alpha f)(\lambda) = \psi_\lambda^\alpha(x)\mathcal{F}_D(f)(\lambda) \quad \text{for all} \quad f \in L^1(\mathbb{R}, dm_\alpha). \tag{2.1}$$

3. Generalized Sobolev spaces of exponential type associated with the Dunkl operators

In this section we set some notations and collect some basic results about generalized Sobolev spaces of exponential type.

Definition 1. The space E of the Dunkl test functions is the set of all real C^∞ f, on $\mathbb{R}$, satisfying

$$|f|_{h,k} = \sup_{x\in\mathbb{R},\, \beta\in\mathbb{N}} \frac{|{D_\alpha}^\beta f(x)| \exp k|x|}{h^\beta \beta!} < \infty, \quad \forall h, k > 0,$$

and we denote

$$G = \{f \in E/ \quad \mathcal{F}_D(f) \in E\}.$$

The topology of G is defined by the above seminorms and we denote by $G^{'}$ the strong dual of the space G and call its elements as Dunkl ultrahyperfunctions [5].

Definition 2. We define the generalized Sobolev space H_G^s of exponential type associated with the Dunkl operators of order $s \in \mathbb{R}$ by

$$H_G^s = \{u \in G^{'}/\|u\|_s = \Big[\int_{\mathbb{R}} |\mathcal{F}_D(u)(\xi)|^2 e^{2s|\xi|} dm_\alpha(\xi)\Big]^{\frac{1}{2}} < \infty\}.$$

The following remark holds:

Remark 1.
1) G is contained in H_G^s for all $s \in \mathbb{R}$.
2) $H_G^0 = L^2(\mathbb{R}, dm_\alpha)$.
3) $H^s \subset H_G^s$ for $s < 0$ and $H_G^s \subset H^t \quad \forall s > 0$ and $\forall t \in \mathbb{R}$.
4) H_G^s has a Hilbert space structure with inner product given by

$$(u,v)_s = \int_{\mathbb{R}} e^{2s|\xi|} \mathcal{F}_D(u)(\xi)\overline{\mathcal{F}_D(v)(\xi)} dm_\alpha(\xi).$$

5) $H_G^t \subset H_G^s$ for $t > s$, with continuous inclusion.

Proposition 1. *Let $P(D_\alpha) = \sum_{\beta\in\mathbb{N}} a_\beta {D_\alpha}^\beta$ be a differential operator of infinite order such that there are constants $C > 0$ and $h > 0$ satisfying*

$$|a_\beta| \le C\frac{h^\beta}{\beta!}, \quad for\ all \quad \beta \in \mathbb{N}.$$

Then $P(D_\alpha)$ maps continuously H^s_G onto H^{s-h}_G.

P r o o f. Let $u \in H^s_G$. Then we have

$$\mathcal{F}_D(Pu)(\xi) = \sum_{\beta\in N} a_\beta \mathcal{F}_D({D_\alpha}^\beta u)(\xi) = \sum_{\beta\in N} a_\beta (i\xi)^\beta \mathcal{F}_D(u)(\xi).$$

Using the fact that

$$\Big|\sum_{\beta\in N} a_\beta (i\xi)^\beta\Big| \le C\sum_{\beta\in N}\frac{h^\beta}{\beta!}|\xi|^\beta = Ce^{h|\xi|},$$

we get

$$\|Pu\|_{H^s} \le C\|Pu\|_{H^{h+s}}$$

which gives the desired result. ∎

Example 1. Let $\Delta = {D_\alpha}^2$. Then the operator $\exp(\sqrt{1-\Delta})$ given by

$$\exp(\sqrt{1-\Delta})u = \int_{\mathbb{R}} \psi^\alpha_\xi(x) e^{\sqrt{1+|\xi|^2}} |\mathcal{F}_D(u)(\xi)| dm_\alpha(\xi),$$

is an isomorphism from H^s_G into H^{s-1}_G. Its inverse is given by $\exp(-\sqrt{1-\Delta})$.

P r o o f. For $u \in H^s_G$, one has

$$\begin{aligned} \|e^{(\sqrt{1-\Delta})}u\|_{s-1} &= \Big[\int_{\mathbb{R}} e^{2(s-1)|\xi|} \big|\mathcal{F}_D(e^{(\sqrt{1-\Delta})})u(\xi)\big|^2 dm_\alpha(\xi)\Big]^{\frac{1}{2}} \\ =\ & \Big[\int_{\mathbb{R}} e^{2(s-1)|\xi|} e^{2\sqrt{1+|\xi|^2}} |\mathcal{F}_D(u)(\xi)|^2 dm_\alpha(\xi)\Big]^{\frac{1}{2}} \\ \le\ & \Big[\sup_\xi e^{2\sqrt{1+|\xi|^2}-2|\xi|}\Big]^{\frac{1}{2}} \Big[\int_{\mathbb{R}} e^{2s|\xi|} |\mathcal{F}_D(u)(\xi)|^2 dm_\alpha(\xi)\Big]^{\frac{1}{2}} = e\|u\|_s. \end{aligned}$$

This completes the proof. ∎

Remark 2. Using standard argumentation and Riesz representation theorem [1], one can prove that for all real s, ${H_G}^{-s}$ can be identified with $(H^s_G)^*$ the dual of (H^s_G).

THEOREM 1. *Let $s > 0$, then all elements u of H_G^{-s} can be decomposed as an infinite sum of derivatives of square integrable functions g_β, that is:*

$$u = \sum_{\beta \in N} \frac{s^\beta}{i^\beta \beta!} {D_\alpha}^\beta g_\beta \,.$$

P r o o f. Taking into account that for all $u \in H_G^{-s}$, $e^{-s|\xi|}\mathcal{F}_D(u)(\xi)$ belongs to $L^2(\mathbb{R}, dm_\alpha)$ as well as the function g given by

$$\mathcal{F}_D(g)(\xi) = \frac{\mathcal{F}_D(u)(\xi)}{\displaystyle\sum_{\beta \in N} \frac{s^\beta}{\beta!}|\xi|^\beta},$$

one gets

$$\mathcal{F}_D(u)(\xi) = \sum_{\beta \in N} \frac{s^\beta}{\beta!} \xi^\beta \mathcal{F}_D(g_\beta)(\xi) = \sum_{\beta \in N} \frac{s^\beta}{i^\beta \beta!} \mathcal{F}_D({D_\alpha}^\beta(g_\beta))(\xi)$$

with $\mathcal{F}_D(g_\beta)(\xi) = \frac{|\xi|^\beta}{\xi^\beta}\mathcal{F}_D(g)(\xi)$. So, it follows

$$u(\xi) = \sum_{\beta \in N} \frac{s^\beta}{i^\beta \beta!} {D_\alpha}^\beta(g_\beta)(\xi),$$

which ends the proof. ∎

4. Pseudodifferential operators associated with Dunkl operators

The pseudodifferential operator $A(x, D_\alpha)$ associated with the symbol $a(x,\xi)$ is defined by

$$A(x, D_\alpha)(x) = \int_{\mathbb{R}} \psi_x^\alpha(\xi) a(x,\xi) \mathcal{F}_D(u)(\xi) dm_\alpha(\xi), \; u \in G,$$

where $a(x, \xi)$ belongs to the class $S_{\exp}^{r,l}$ defined below.

DEFINITION 3. ([3]) Let $r, l \in \mathbb{R}$ be real numbers with $l > 0$. We say that the function $a(x,\xi) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ belongs to $S_{\exp}^{r,l}$ if and only if $a(x,\xi) \in C^\infty(\mathbb{R} \times \mathbb{R})$ and for each $L > 0$ and γ, $\beta \in \mathbb{N}$, there exists a positive constant $C = C_{r,l,\gamma}$ such that

$$|{D_{\alpha,\xi}}^\gamma {D_{\alpha,x}}^\beta a(x,\xi)| \le C L^{|\beta|} |\beta|! e^{r|\xi| - l|x|}.$$

To obtain some deep and interesting results we need the following alternative form of $A(x, D_\alpha)$ given by the following lemma.

LEMMA 1. *For any symbol $a(x,\xi) \in S^{r,l}_{exp}$, the pseudodifferential operator $A(x,D_\alpha)$ admits the following representation:*

$$(A(x,D_\alpha)u)(x)=\int_{\mathbb{R}}\psi^\alpha_x(\xi)\int_{\mathbb{R}}\mathcal{F}_D(T_\eta a(.,\eta))(\xi)\mathcal{F}_D(u)(\eta)dm_\alpha(\eta)dm_\alpha(\xi),\ \forall u \in G,$$

where all the involved integrals are absolutely convergent.

P r o o f. We prove the theorem above in two steps.

First step. We shall prove that for all symbols $a(x,\xi) \in S^{r,l}_{\exp}$ and for a suitable real $\tau > 0$, we have

$$|\mathcal{F}_D(a(.,\xi))(y)| \ \leq\ Ce^{r|\xi|-\tau|y|}. \tag{4.1}$$

Indeed,

$$\begin{aligned} e^{\tau|y|}|\mathcal{F}_D(a(y,\xi))| &= \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}|y|^k|\mathcal{F}_D(a(.,\xi))(y)| \\ &= \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}|\mathcal{F}_D({D_{\alpha,y}}^k a(.,\xi))(y)| \leq \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}\int_{\mathbb{R}}|{D_{\alpha,y}}^k a(.,\xi)(y)|dm_\alpha(y) \\ &\leq \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}L^{|k|}k!e^{r|\xi|}\int_{\mathbb{R}}e^{-l|y|}dm_\alpha(y) = \sum_{k=0}^{+\infty}(\tau L)^k e^{r|\xi|}\int_{\mathbb{R}}e^{-l|y|}dm_\alpha(y). \end{aligned}$$

Choosing $\tau < L^{-1}$, we get

$$e^{\tau|y|}|\mathcal{F}_D(a(.,\xi))(y)| \leq Ce^{r|\xi|}, \tag{4.2}$$

the desired inequality, for a constant C depending on r, l and L.

Second step. Using the fact that $|\psi^\alpha_\lambda| \leq 1$ and relation (2.1) together with the estimate (4.1), we get

$$|\mathcal{F}_D(T_\eta a(.,\eta))(\xi)\mathcal{F}_D(u)(\eta)| \leq Ce^{r|\eta|-\tau|\xi|-\lambda|\eta|}.$$

Let now $g(\xi) = \int_{\mathbb{R}}\mathcal{F}_D(T_\eta a(.,\eta))(\xi)\mathcal{F}_D(u)(\eta)dm_\alpha(\eta)$, we shall prove that g belongs to $L^1(dm_\alpha(\xi))$. Hence,

$$\begin{aligned} |g(\xi)| &\leq \int_{\mathbb{R}}|\mathcal{F}_D(T_\eta a(.,\eta))(\xi)\mathcal{F}_D(u)(\eta)|dm_\alpha(\eta) \\ &\leq C\int_{\mathbb{R}}e^{-\tau|\xi|}e^{(r-\lambda)|\eta|}dm_\alpha(\eta) = Ce^{-\tau|\xi|}\int_{\mathbb{R}}e^{(r-\lambda)|\eta|}dm_\alpha(\eta). \end{aligned}$$

Then for $\lambda > r$, the desired follows by applying the inverse Dunkl transform (1.2). ∎

Theorem 2. *Let* $a(x,\xi) \in S^{r,l}_{exp}$. *Then the associated pseudodifferential operator* $A(x,D_\alpha)$ *maps continuously* H_G^{s+r} *to* H_G^s *for all* $s>0$. *Moreover we have for all* $u \in G$,

$$\|A(x,D_\alpha)u\|_s \leq C_s \|u\|_{r+s}.$$

P r o o f. Putting

$$U_s(\xi) = e^{s|\xi|} \int_{\mathbb{R}} \mathcal{F}_D(T_\eta a(.,\eta))(\xi)\mathcal{F}_D(u)(\eta)dm_\alpha(\eta)$$

and using (4.1) together with (2.1), we obtain

$$|U_s(\xi)| = Ce^{(s-\tau)|\xi|} \int_{\mathbb{R}} e^{(r+s)|\eta|} e^{-s|\eta|} |\mathcal{F}_D(u)(\eta)| dm_\alpha(\eta).$$

Applying Hölder's inequality, it follows

$$|U_s(\xi)| \leq Ce^{(s-\tau)|\xi|} \|u\|_{r+s}.$$

And hence, for a suitable τ we get

$$\begin{aligned} \|A(x,D_\alpha)u\|_s = \Big[\int_{\mathbb{R}} |U_s(\xi)|^2 dm_\alpha(\xi)\Big]^{\frac{1}{2}} &\leq C\|u\|_{r+s} \Big[\int_{\mathbb{R}} e^{2(s-\tau)|\xi|} dm_\alpha(\xi)\Big]^{\frac{1}{2}} \\ &\leq C_s \|u\|_{r+s}. \end{aligned}$$

■

Corollary 1. *Let* $h, s > 0$ *and* $a_\gamma(x)$ *be a sequence of symbols satisfying*

$$|a_\gamma(x)| < C\frac{h^\gamma}{\gamma!} e^{-l|x|}, \quad l > 0, \tag{4.3}$$

then the pseudodifferential operator

$$P(x,D_\alpha) = \sum_{\gamma \in \mathbb{N}} a_\gamma(x) D_\alpha^\gamma$$

maps continuously H_G^{s+h} *to* H_G^s *for all* $s \in \mathbb{R}$. *Moreover, we have for all* $u \in G$,

$$\|P(x,D_\alpha)u\|_s \leq C_s \|u\|_{s+h}.$$

P r o o f. Using condition (4.3) above, we obtain

$$\begin{aligned}
\|P(x,D_\alpha)u\|_s^2 &= \int_{\mathbb{R}} e^{2s|\xi|}|\mathcal{F}_D(P(x,D_\alpha)u)(\xi)|^2 dm_\alpha(\xi) \\
&\leq C\int_{\mathbb{R}} e^{2s|\xi|}\Big(\sum_{\gamma\in\mathbb{N}} \frac{h^\gamma}{\gamma!}e^{-l|x|}|\xi|^\gamma\Big)^2 |\mathcal{F}_D u(\xi)|^2 dm_\alpha(\xi) \\
&= Ce^{-2l|x|}\int_{\mathbb{R}} e^{2(s+h)|\xi|}|\mathcal{F}_D u(\xi)|^2 dm_\alpha(\xi) \leq C\|u\|_{s+h}.
\end{aligned}$$

This completes the proof. ■

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