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AN ANALOGUE OF BEURLING-HÖRMANDER'S THEOREM FOR THE DUNKL-BESSEL TRANSFORM

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Dedicated to Professor Khalifa Trimèche, on the occasion of his 60th anniversary

Abstract

We establish an analogue of Beurling-Hörmander's theorem for the Dunkl-Bessel transform $\mathcal{F}_{D,B}$ on $I\!R_+^{d+1}$. We deduce an analogue of Gelfand-Shilov, Hardy, Cowling-Price and Morgan theorems on $I\!R_+^{d+1}$ by using the heat kernel associated to the Dunkl-Bessel-Laplace operator.

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1. Introduction

There are many theorems known which state that a function and its classical Fourier transform on $I\!R$ cannot simultaneously be very small at infinity. This principle has several version which were proved by G.H. Hardy [6], G.W. Morgan [11], M.G. Cowling and J.F. Price [4], A. Beurling [1].

The Beurling theorem for the classical Fourier transform on $I\!R$ which was proved by L. Hörmander [7], says that for any non trivial function f in $L^2(I\!R)$, the function $f(x)\mathcal{F}(f)(y)$ is never integrable on $I\!R^2$ with respect to the measure $e^{|xy|}dxdy$. A far reaching generalization of this result has been recently proved in [2]. In this paper the author proves that a square integrable function f on $\mathbb{R}^d$ satisfying for an integer N:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)||\mathcal{F}(f)(y)|}{(1+||x||+||y||)^N} e^{||x||||y||} dxdy < +\infty$$

has the form $f(x) = P(x)e^{-\beta||x||^2}$, where P is a polynomial of degree strictly lower than $\frac{N-d}{2}$ and $\beta > 0$.

This version has been studied in other situations by many authors in particular L. Bouattour and K. Trimèche [3], L. Kamoun and K. Trimèche [8] and K. Trimèche [13]. There, an analogue of Beurling-Hörmander's theorem has been proved, for the Chébli-Trimèche transform, a Fourier transform associated with partial differential operators and the Dunkl transform.

In this paper we study an analogue of Beurling-Hörmander's theorem for the Dunkl-Bessel transform on $\mathbb{R}^{d+1}_+$.

The contents of the paper is as follows: In Section 2 we recall the Dunkl operators and the Dunkl kernel. We introduce in the third section the Dunkl-Bessel-Laplace operator and define the Dunkl-Bessel transform, the Dunkl-Bessel intertwining operator and its dual, and give their properties. Section 4 is devoted to the heat functions $W^{k,\beta}_{s,p}$ related to the Dunkl-Bessel Laplace operator. These functions are used in the statement of the main result. In Section 5 we give an analogue of Beurling-Hörmander's theorem for the Dunkl-Bessel transform. In the last section, an analogue of Hardy and Morgan theorems is obtained for the Dunkl-Bessel transform. For other proofs of these theorems (see [9], [10]).

2. Dunkl operators and Dunkl kernel

In this section we collect some notations on Dunkl operators and the Dunkl kernel (see [5]).

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha. \tag{1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $R \cap \mathbb{R}^d.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflection

$\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R . All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R \,/ \langle\alpha,\beta\rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \longrightarrow \mathbb{C}$ on a root system R is called a multiplicity function, if it is invariant under the action of the associated reflection group W.

Moreover, let ω_k denote the weight function

$$\forall\, x \in \mathbb{R}^d,\ \omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}. \tag{2}$$

The Dunkl operators $T_j,\ j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given for a function of class C^1 by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle} \tag{3}$$

In the case $k = 0$, the $T_j,\ j = 1, ..., d$, reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$.

We define the Dunkl-Laplace operator on $\mathbb{R}^d$ by

$$\triangle_k f(x) = \sum_{j=1}^{d} T_j^2 f(x) = \triangle_d f(x) + 2 \sum_{\alpha\in R^+} k(\alpha)\Big[\frac{\langle\nabla f(x), \alpha\rangle}{\langle\alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle^2}\Big]. \tag{4}$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x, y) &= y_j u(x, y), \quad j = 1, ..., d, \\ u(0, y) &= 1, \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted $K(x, y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The function $K(x, z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{5}$$

where μ_x is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(o, ||x||)$ of center o and radius $||x||$.

3. Harmonic analysis associated with the Dunkl-Bessel-Laplace operator

In this section we collect some notations and results on the Dunkl-Bessel Laplace operator, the Dunkl-Bessel intertwining operator and its dual, and the Dunkl-Bessel transform (see [10]).

Notations. We denote by
-$\mathbb{R}_+^{d+1} = \mathbb{R}^d \times [0, +\infty[$.
- $x = (x_1, ..., x_d, x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}_+^{d+1}$.
- $C_*(\mathbb{R}^{d+1})(resp.\ C_{*,c}(\mathbb{R}^{d+1}))$ the space of continuous functions on $\mathbb{R}^{d+1}$ (resp. with compact support), even with respect to the last variable.
- $C_*^p(\mathbb{R}^{d+1})(resp.\ C_{*,c}^p(\mathbb{R}^{d+1}))$ the space of functions of class C^p on $\mathbb{R}^{d+1}$, (resp. with compact support), even with respect to the last variable .
-$\mathcal{E}_*(\mathbb{R}^{d+1})$ (resp. $D_*(\mathbb{R}^{d+1})$) the space of C^∞-functions on $\mathbb{R}^{d+1}$ (resp. with compact support), even with respect to the last variable.

We provide these spaces with the classical topology.

3.1. The Dunkl-Bessel-Laplace operator and the Dunkl-Bessel intertwining operator

We consider the Dunkl-Bessel-Laplace operator $\triangle_{k,\beta}$ defined by $\forall\, x = (x', x_{d+1}) \in \mathbb{R}^d \times]0, +\infty[$,

$$\triangle_{k,\beta} f(x) = \triangle_{k,x'} f(x', x_{d+1}) + \mathcal{L}_{\beta,x_{d+1}} f(x', x_{d+1}),\ f \in C_*^2(\mathbb{R}^{d+1}), \qquad (6)$$

where $\triangle_k$ is the Dunkl-Laplace operator on $\mathbb{R}^d$, and $\mathcal{L}_\beta$ the Bessel operator on $]0, +\infty[$ given by

$$\mathcal{L}_\beta = \frac{d^2}{dx_{d+1}^2} + \frac{2\beta + 1}{x_{d+1}} \frac{d}{dx_{d+1}}, \quad \beta > -\frac{1}{2}.$$

For all $x \in \mathbb{R}_+^{d+1}$, we define the measure $\zeta_x^{k,\beta}$ on $\mathbb{R}^d \times]0, +\infty[$ by

$$d\zeta_x^{k,\beta}(y) = \frac{2\Gamma(\beta + 1)}{\sqrt{\pi}\Gamma(\beta + \frac{1}{2})} x_{d+1}^{-2\beta} (x_{d+1}^2 - y_{d+1}^2)^{\beta - \frac{1}{2}} 1_{]0, x_{d+1}[}(y_{d+1}) d\mu_{x'}(y') dy_{d+1}, \qquad (7)$$

where $\mu_{x'}$ is the measure given by (5) and $1_{]0,x_{d+1}[}$ is the characteristic function of the interval $]0, x_{d+1}[$.

The Dunkl-Bessel intertwining operator is the operator $\mathcal{R}_{k,\beta}$ defined on $C_*(I\!R^{d+1})$ by

$$\forall\, x \in I\!R_+^{d+1},\ \mathcal{R}_{k,\beta} f(x) = \int\limits_{I\!R_+^{d+1}} f(y) d\zeta_x^{k,\beta}(y). \tag{8}$$

3.2. The dual of the Dunkl-Bessel intertwining operator

The dual of the Dunkl-Bessel intertwining operator $\mathcal{R}_{k,\beta}$ is the operator ${}^t\mathcal{R}_{k,\beta}$ defined on $D_*(I\!R^{d+1})$ by: $\forall y = (y', y_{d+1}) \in I\!R^d \times [0, \infty[$,

$${}^t\mathcal{R}_{k,\beta}(f)(y', y_{d+1}) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})} \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta-\frac{1}{2}}\ {}^tV_k f(y', s) s ds, \tag{9}$$

where ${}^tV_k$ is the dual Dunkl intertwining operator defined by K. Trimèche in [12] by

$$\forall\, y \in I\!R^d,\ {}^tV_k(f)(y) = \int_{I\!R^d} f(x) d\nu_y(x), \tag{10}$$

where ν_y is a positive measure on $I\!R^d$ with support in the set $\{x \in I\!R^d, ||x|| \geq ||y||\}$.

For all $y \in I\!R_+^{d+1}$, we define the measure $\varrho_y^{k,\beta}$ on $I\!R^d \times]0, +\infty[$, by

$$d\varrho_y^{k,\beta}(x) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})} (x_{d+1}^2 - y_{d+1}^2)^{\beta-\frac{1}{2}} x_{d+1} 1_{]y_{d+1}, +\infty[}(x_{d+1}) d\nu_{y'}(x') dx_{d+1}, \tag{11}$$

From (9) the operator ${}^t\mathcal{R}_{k,\beta}$ can also be written in the form

$$\forall\, y \in I\!R_+^{d+1},\ {}^t\mathcal{R}_{k,\beta}(f)(y) = \int\limits_{I\!R_+^{d+1}} f(x) d\varrho_y^{k,\beta}(x). \tag{12}$$

Notation. We denote by $L_{k,\beta}^p(I\!R_+^{d+1})$ the space of measurable functions on $I\!R_+^{d+1}$ such that

$$||f||_{k,\beta,p} = \Big(\int\limits_{I\!R_+^{d+1}} |f(x)|^p d\mu_{k,\beta}(x)\, dx \Big)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$||f||_{k,\beta,\infty} = ess\ sup_{x \in I\!R^d} |f(x)| < +\infty,$$

where $\mu_{k,\beta}$ is the measure on $\mathbb{R}_+^{d+1}$ given by

$$d\mu_{k,\beta}(x', x_{d+1}) = \omega_k(x') x_{d+1}^{2\beta+1} dx' dx_{d+1}.$$

THEOREM 3.1. *Let $(\varrho_y^{k,\beta})_{y\in\mathbb{R}_+^{d+1}}$ be the family of measures defined by (11) and f in $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$. Then for almost all y (with respect to the Lebesgue measure on $\mathbb{R}_+^{d+1}$), f is $\varrho_y^{k,\beta}$-integrable, the function*

$$y \mapsto \int_{\mathbb{R}_+^{d+1}} f(y) \varrho_y^{k,\beta}(x) dy,$$

which will be denoted also by ${}^t\mathcal{R}_{k,\beta}(f)$, is defined almost every where on $\mathbb{R}_+^{d+1}$, and for all bounded function g in $C_*(\mathbb{R}^{d+1})$ we have the formula

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_{k,\beta}(f)(y) g(y) dy = \int_{\mathbb{R}_+^{d+1}} f(x)\, \mathcal{R}_{k,\beta}(g)(x) d\mu_{k,\beta}(x). \qquad (13)$$

REMARK 3.2. Let f be in $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$. By taking $g \equiv 1$ in the relation (13) we deduce that

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_{k,\beta}(f)(y) dy = \int_{\mathbb{R}_+^{d+1}} f(x)\, d\mu_{k,\beta}(x). \qquad (14)$$

3.3. The Dunkl-Bessel transform

DEFINITION 3.3. The Dunkl-Bessel transform is given for f in $D_*(\mathbb{R}^{d+1})$ by

$$\forall\, y = (y', y_{d+1}) \in \mathbb{R}_+^{d+1},\ \mathcal{F}_{D,B}(f)(y', y_{d+1}) = \int_{\mathbb{R}_+^{d+1}} f(x', x_{d+1}) \Lambda(x, y) d\mu_{k,\beta}(x), \qquad (15)$$

where Λ is given by

$$\Lambda(x, z) = K(-ix', z') j_\beta(x_{d+1} z_{d+1}), \quad (x, z) \in \mathbb{R}_+^{d+1} \times \mathbb{C}^{d+1}. \qquad (16)$$

From Theorem 3.1 we deduce the following proposition.

PROPOSITION 3.4. *For all f in $L^1_{k,\beta}(I\!R^{d+1}_+)$, we have*

$$\mathcal{F}_{D,B}(f)(y) = \mathcal{F}_o \circ {}^t\mathcal{R}_{k,\beta}(f)(y), \quad y \in I\!R^{d+1}_+, \tag{17}$$

where $\mathcal{F}_o$ is the transform defined by: $\forall\, y = (y', y_{d+1}) \in I\!R^d \times [0, +\infty[$, $f \in C_{*,c}(I\!R^{d+1})$

$$\mathcal{F}_o(f)(y', y_{d+1}) = \int_{I\!R^{d+1}_+} f(x', x_{d+1}) e^{-i\langle y', x'\rangle} \cos(x_{d+1} y_{d+1}) dx' dx_{d+1}.$$

4. Heat functions related to the Dunkl-Bessel Laplacian $\triangle_{k,\beta}$

For $r > 0$, $p \in I\!N$ and $s \in I\!N^d$, we define the heat functions $W^{k,\beta}_{s,p}(r,.)$ related to the Dunkl-Bessel Laplacian $\triangle_{k,\beta}$ by

$$\forall y \in I\!R^{d+1}_+, \quad W^{k,\beta}_{s,p}(r, y) \tag{18}$$

$$= \frac{i^{|s|}(-1)^p c_k^2}{4^{\gamma+\beta+d}(\Gamma(\beta+1))^2} \int_{I\!R^{d+1}_+} x_1^{s_1} ... x_d^{s_d} x_{d+1}^{2p}\ e^{-r||x||^2} \Lambda(x, y) d\mu_{k,\beta}(x).$$

These functions satisfy the following properties:

i) $W^{k,\beta}_{0,0}(r, x) = E^{k,\beta}_r(x)$ the Gaussian kernel associated to the Dunkl-Bessel Laplacian, defined by

$$\forall\, x \in I\!R^{d+1}_+,\ E^{k,\beta}_r(x) = \frac{c_k^2}{4^{\gamma+\beta+d}(\Gamma(\beta+1))^2} \int_{I\!R^{d+1}_+} e^{-r||x||^2} \Lambda(x, y) d\mu_{k,\beta}(x). \tag{19}$$

ii) $W^{k,\beta}_{s,p}(r,.)$ is a C^∞-function on $I\!R^{d+1}$, even with respect to the last variable and we have

$$W^{k,\beta}_{s,p}(r, x) = T^s_{x'} \mathcal{L}^p_{\beta, x_{d+1}} E^{k,\beta}_r(x),\ x \in I\!R^{d+1}_+,$$

where T^s is the operator $T^s = T_1^{s_1} \circ T_2^{s_2} \circ ... T_d^{s_d}$, with T_j, $j = 1, 2, ..., d$, the Dunkl operators.

iii) For all $r > 0$, the kernel $E^{k,\beta}_r$ solves the generalized heat equation

$$\frac{\partial}{\partial r} E^{k,\beta}_r(x) - \triangle_{k,\beta} E^{k,\beta}_r(x) = 0,\ x \in I\!R^d \times]0, +\infty[.$$

iv) For $p \in I\!N$, $s \in I\!N^d$ we have

$$\forall\, y \in I\!R_+^{d+1},\ \mathcal{F}_{D,B}(W_{s,p}^{k,\beta}(r,.))(y) = i^{|s|}(-1)^p y_1^{s_1}...y_d^{s_d} y_{d+1}^{2p}\ e^{-r||y||^2}. \quad (20)$$

Notation. We denote by $\mathcal{P}_m^{d+1}$ the set of homogeneous polynomials on $I\!R^{d+1}$ of degree m even with respect to the last variable.

We state now the following proposition given in [10].

PROPOSITION 4.1. *Let ψ be in $\mathcal{P}_m^{d+1}$, for all $\delta > 0$, there exists a polynomial $Q \in \mathcal{P}_m^{d+1}$ such that*

$$\forall\, y \in I\!R_+^{d+1},\ \mathcal{F}_{D,B}(\psi\, e^{-\delta||x||^2})(y) = Q(y) e^{-\frac{1}{4\delta}||y||^2}.$$

5. Beurling-Hörmander's theorem for the Dunkl-Bessel transform

We need the following lemmas for the proof of the main theorem of this section.

LEMMA 5.1. *Let $N \geq 0$. We consider f in $L_{k,\beta}^2(I\!R_+^{d+1})$ satisfying*

$$\int_{I\!R_+^{d+1}} \int_{I\!R_+^{d+1}} \frac{|f(x)||\mathcal{F}_{D,B}(f)(y)|}{(1 + ||x|| + ||y||)^N} e^{||x||||y||} d\mu_{k,\beta}(x) dy < +\infty. \quad (21)$$

Then $f \in L_{k,\beta}^1(I\!R_+^{d+1})$.

P r o o f. From the relation (21) and Fubini's theorem we have for almost every $y \in I\!R_+^{d+1}$:

$$\frac{|\mathcal{F}_{D,B}(f)(y)|}{(1 + ||y||)^N} \int_{I\!R_+^{d+1}} \frac{|f(x)|}{(1 + ||x||)^N} e^{||x||||y||} d\mu_{k,\beta}(x) < +\infty.$$

As f is not negligible, there exists $y_0 \in I\!R_+^{d+1}$, $y_0 \neq 0$ such that $\mathcal{F}_{D,B}(f)(y_0) \neq 0$.

Thus

$$\int_{I\!R_+^{d+1}} \frac{|f(x)|}{(1 + ||x||)^N} e^{||x||||y_0||} d\mu_{k,\beta}(x) < +\infty. \quad (22)$$

As the function $\dfrac{e^{||x||||y_0||}}{(1+||x||)^N}$ is greater than 1 for large $||x||$, then

$$\int_{\mathbb{R}_+^{d+1}} |f(x)| d\mu_{k,\beta}(x) < +\infty.$$

■

THEOREM 5.2. *Let $N \in \mathbb{N}$ and f in $L^2_{k,\beta}(\mathbb{R}_+^{d+1})$ satisfying (21). Then:*
• *If $N \geq d+2$ we have*

$$f(y) = \sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(r,y), \; y \in \mathbb{R}_+^{d+1},$$

where $r > 0$, $a_{s,p}^{k,\beta} \in \mathbb{C}$ and $W_{s,p}^{k,\beta}(r,.)$ given by the relation (18).
• *Else $f(y) = 0$ a.e. $y \in \mathbb{R}_+^{d+1}$.*

P r o o f. From Lemma 5.1 and Theorem 3.1, the function f belongs to $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$ and the function ${}^tR_{k,\beta}(f)$ is defined almost everywhere on $\mathbb{R}_+^{d+1}$. We shall prove that we have

$$\int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \frac{e^{||x||||y||} |{}^tR_{k,\beta} f(x)||\mathcal{F}_0({}^tR_{k,\beta})(y)|}{(1+||x||+||y||)^N} dydx < +\infty. \tag{23}$$

Take y_0 as in Lemma 5.1. We write the above integral as a sum of the following integrals

$$I = \int_{\mathbb{R}_+^{d+1}} \int_{||y|| \leq ||y_0||} \frac{e^{||x||||y||}}{(1+||x||+||y||)^N} |{}^tR_{k,\beta} f(x)||\mathcal{F}_0({}^tR_{k,\beta}(f))(y)| dydx$$

and

$$J = \int_{\mathbb{R}_+^{d+1}} \int_{||y|| \geq ||y_0||} \frac{e^{||x||||y||}}{(1+||x||+||y||)^N} |{}^tR_{k,\beta} f(x)||\mathcal{F}_0({}^tR_{k,\beta}(f))(y)| dydx.$$

We will prove that I and J are finite, which implies (23).

• As the functions $|\mathcal{F}_{D,B}(f)(y)|$ is continuous in the compact $\{y \in \mathbb{R}_+^{d+1} / ||y|| \leq ||y_0||\}$, so we get

$$I \leq const \int_{\mathbb{R}_+^{d+1}} \frac{e^{||x||||y_0||} | \, {}^tR_{k,\beta} f(x)|}{(1+||x||)^N} dx.$$

Writing the integral of the second member as $I_1 + I_2$ with

$$I_1 = \int_{||x|| \leq \frac{N}{||y_0||}} \frac{e^{||x|| ||y_0||} \, |{}^tR_{k,\beta} f(x)|}{(1 + ||x||)^N} dx,$$

and

$$I_2 = \int_{||x|| \geq \frac{N}{||y_0||}} \frac{e^{||x|| ||y_0||} \, |{}^tR_{k,\beta} f(x)|}{(1 + ||x||)^N} dx.$$

Therefore, we have the following results:

– As the function $x \mapsto \frac{e^{||x|| ||y_0||}}{(1+||x||)^N}$ is continuous in the compact $\{x \in \mathbb{R}_+^{d+1} / ||x|| \leq \frac{N}{||y_0||}\}$, and f is in $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$ we deduce by using Fubini-Tonelli's theorem, and the relations (12),(11) that ${}^tR_{k,\beta}(|f|)$ belongs to $L^1_{k,\beta}(\mathbb{R}_+^{d+1})$. Hence I_1 is finite.

– On the other hand, for $t > \frac{N}{||y_0||}$, the function $t \mapsto \frac{e^{t||y_0||}}{(1+t)^N}$ is increasing, so we obtain by using Fubini-Tonelli's theorem, and (12),(11) and (14), that

$$I_2 \leq \int_{\mathbb{R}_+^{d+1}} \frac{e^{||\xi|| ||y_0||}}{(1 + ||\xi||)^N} |f(\xi)| d\mu_{k,\beta}(\xi).$$

The inequality (22) assert that I_2 is finite. This proves that I is finite.

- We suppose $||y_0|| \leq N$. Let $J = J_1 + J_2 + J_3$, with

$$J_1 = \int_{||x|| \leq \frac{N}{||y_0||}} \int_{||y_0|| \leq ||y|| \leq N} \frac{e^{||x|| ||y||}}{(1 + ||x|| + ||y||)^N} |{}^tR_{k,\beta}(f)(x)| |\mathcal{F}_{D,B}(f)(y)| dy dx,$$

$$J_2 = \int_{||x|| \geq \frac{N}{||y_0||}} \int_{||y_0|| \leq ||y|| \leq N} \frac{e^{||x|| ||y||}}{(1 + ||x|| + ||y||)^N} |{}^tR_{k,\beta}(f)(x)| |\mathcal{F}_{D,B}(f)(y)| dy dx,$$

$$J_3 = \int_{\mathbb{R}_+^{d+1}} \int_{||y|| \geq N} \frac{e^{||x|| ||y||}}{(1 + ||x|| + ||y||)^N} |{}^tR_{k,\beta}(f)(x)| |\mathcal{F}_{D,B}(f)(y)| dy dx.$$

– As the function $(x, y) \mapsto \dfrac{e^{||x|| ||y||}}{(1 + ||x|| + ||y||)^N} |\mathcal{F}_{D,B}(f)(y)|$ is bounded in the compact $\{x \in \mathbb{R}_+^{d+1} / \ ||x|| \leq \frac{N}{||y_0||}\} \times \{\xi \in \mathbb{R}_+^{d+1} / \ ||y_0|| \leq ||\xi|| \leq N\}$ and ${}^tR_{k,\beta}(|f|)(x)$ is Lebesgue-integrable on $\mathbb{R}_+^{d+1}$, then J_1 is finite.

– Let $\lambda > 0$. As the function $t \mapsto \frac{e^{\lambda t}}{(1+t+\lambda)^N}$ is increasing for $t > \frac{N}{\lambda}$. Thus, for all $(x, y) \in C(\xi, y_0, N)$ we have the inequality

$$\frac{e^{||x||||y||}}{(1 + ||x|| + ||y||)^N} \leq \frac{e^{||\xi||||y||}}{(1 + ||\xi|| + ||y||)^N},$$

with

$$C(\xi, y_0, N) = \{(x, y) \in I\!R_+^{d+1} \times I\!R_+^{d+1} / \frac{N}{||y||} \leq ||x|| \leq ||\xi|| \text{ and } ||y_0|| \leq ||y|| \leq N\}.$$

Therefore, from Fubini-Tonelli's theorem and the relations (12),(11), we get

$$J_2 \leq \int_{I\!R_+^{d+1}} \int_{I\!R_+^{d+1}} |f(\xi)||\mathcal{F}_{D,B}(f)(y)| \frac{e^{||\xi||||y||}}{(1 + ||\xi|| + ||y||)^N} dy d\mu_{k,\beta}(\xi).$$

Taking account of the condition (21), we deduce that J_2 is finite.

– For $||y|| > N$, the function $t \mapsto \frac{e^{t||y||}}{(1+t+||y||)^N}$ is increasing. We deduce, by using Fubini-Tonelli's theorem and the relations (12),(11),(21), that

$$J_3 \leq \int_{I\!R_+^{d+1}} \int_{||y|| \geq N} |f(\xi)||\mathcal{F}_{D,B}(f)(y)| \frac{e^{||\xi||||y||}}{(1 + ||\xi|| + ||y||)^N} dy d\mu_{k,\beta}(\xi) < +\infty.$$

This implies that J is finite.

Finally for $||y_0|| > N$, we have $J \leq J_3 < \infty$. This completes the proof of the relation (23).

According to Corollary 3.1, ii) of [2], we conclude that

$$\forall\, x \in I\!R_+^{d+1}, \quad {}^tR_{k,\beta} f(x) = P(x) e^{-\delta ||x||^2}$$

with $\delta > 0$ and P a polynomial of degree strictly lower than $\frac{N-d-1}{2}$.

Using this relation and (18), we deduce that

$$\forall\, y \in I\!R_+^{d+1}, \quad \mathcal{F}_{D,B}(f)(y) = \mathcal{F}_0 \circ {}^tR_{k,\beta}(f)(y) = \mathcal{F}_0(P(x) e^{-\delta ||x||^2})(y). \quad (24)$$

But

$$\forall\, y \in I\!R_+^{d+1}, \quad \mathcal{F}_0(P(x) e^{-\delta ||x||^2})(y) = Q(y) e^{-\frac{||y||^2}{4\delta}}, \quad (25)$$

with Q a polynomial of degree strictly lower than $\frac{N-d-1}{2}$.

Thus from (20) we obtain

$$\forall\, y \in I\!R_+^{d+1}, \quad \mathcal{F}_{D,B}(f)(y) = \mathcal{F}_{D,B}\Big(\sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}\big(\frac{1}{4\delta},.\big)\Big)(y).$$

The injectivity of the transform $\mathcal{F}_{D,B}$ implies

$$f(x) = \sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}\big(\frac{1}{4\delta},.\big)(x) \;\; \text{a.e.},$$

and the theorem is proved. ∎

6. Applications

In this section we give analogues of the Gelfand-Shilov, Hardy, Cowling-Price and Morgan theorems for the Dunkl-Bessel transform $\mathcal{F}_{D,B}$.

THEOREM 6.1. (Gelfand-Shilov type) *Let $N \in I\!N$ and assume that f in $L^2_{k,\beta}(I\!R_+^{d+1})$ is such that*

$$\int_{I\!R_+^{d+1}} \frac{|f(x)| e^{\frac{(2a)^p}{p}||x||^p}}{(1+||x||)^N} d\mu_{k,\beta}(x) < +\infty, \tag{26}$$

$$\int_{I\!R_+^{d+1}} \frac{|\mathcal{F}_{D,B}(f)(y)| e^{\frac{(2b)^q}{q}||y||^q}}{(1+||y||)^N} dy < +\infty, \tag{27}$$

where $1 < p, q < +\infty$, $\frac{1}{p}+\frac{1}{q} = 1$, $a > 0$, $b > 0$ and $ab \geq \frac{1}{4}$. Then:

1) If $ab > \frac{1}{4}$, we have $f(x) = 0$ a.e.

2) We suppose that $ab = \frac{1}{4}$.

i) If $N < \frac{d}{2}+1$, $1 < p, q < +\infty$, we have $f(x) = 0$, a.e. $x \in I\!R^d$.

ii) If $N \geq \frac{d}{2}+1$.

• For the cases: $2 \leq q < +\infty$, $1 < p < +\infty$,
$1 < q < 2$, $2 < p < +\infty$,
$q = 2$, $p = 2$,

we have $f(x) = 0$, a.e. $x \in I\!R^d$.

• For the case: $1 < q < 2$, $1 < p < 2$

we have

$$f(x) = \sum_{|s|+p<\frac{2N-d-1}{2}} a_{s,p}^{k,\beta} W_{s,p}^{k,\beta}(r,x), \;\; a.e.\, x \in I\!R_+^{d+1}, \tag{28}$$

where $r > 0$ *and* $a^{k,\beta}_{s,p} \in \mathbb{C}$.

- *For the case* $q = 2$, $1 < p < 2$

  - *If* $0 < r \leq 2b^2$

we have $f(x) = 0$ a.e. $x \in \mathbb{R}^{d+1}_+$.

  - *If* $r > 2b^2$

the function f *is given by the relation (28).*

- *For the case* $p = 2$, $1 < q < 2$

  - *If* $r \geq 2b^2$

we have $f(x) = 0$ a.e. $x \in \mathbb{R}^{d+1}_+$.

  - *If* $0 < r < 2b^2$

the function f *is given by the relation (28).*

P r o o f. Using the inequality

$$4ab||x||||y|| \leq \frac{(2a)^p}{p}||x||^p + \frac{(2b)^q}{q}||y||^q,$$

we get

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)||\mathcal{F}_{D,B}(f)(y)|}{(1+||x||+||y||)^{2N}} e^{4ab||x||||y||} d\mu_{k,\beta}(x)dy \leq$$

$$\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|e^{\frac{(2a)^p}{p}||x||^p}}{(1+||x||)^N} d\mu_{k,\beta}(x) \int_{\mathbb{R}^{d+1}_+} \frac{|\mathcal{F}_{D,B}(f)(y)|e^{\frac{(2b)^q}{q}||y||^q}}{(1+||y||)^N} dy < +\infty. \quad (29)$$

As $ab \geq \frac{1}{4}$, then from (29) we deduce that the condition (22) is satisfied. By using the proof of Theorem 5.2, we obtain, $\forall\, x \in \mathbb{R}^{d+1}_+$,

$${}^tR_{k,\beta}(f)(x) = P(x)e^{-\frac{||x||^2}{4r}};\ \forall\, y \in \mathbb{R}^{d+1}_+,\ \mathcal{F}_{D,B}(f)(y) = Q(y)e^{-r||y||^2}, \quad (30)$$

where r is a positive constant and P,Q are polynomials of the same degree which is strictly lower than $\frac{2N-d-1}{2}$.

1) From (29) and the proof of (23) we deduce that

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \frac{|{}^tR_{k,\beta}(f)(x)||\mathcal{F}_o({}^tR_{k,\beta}(f))(y)|}{(1+||x||+||y||)^{2N}} e^{4ab||x||||y||} dxdy < +\infty. \quad (31)$$

By replacing in (31) the functions ${}^tR_{k,\beta}(f)(x)$ and $\mathcal{F}_{D,B}(f)(y)$ by their expression given in (30), we get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|P(x)||Q(y)|}{(1+||x||+||y||)^{2N}} e^{-(\sqrt{r}||y||-\frac{1}{2\sqrt{r}}||x||)^2} e^{(4ab-1)||x||||y||} dxdy < +\infty. \quad (32)$$

As $ab > \frac{1}{4}$, there exists $\varepsilon > 0$ such that $4ab-1-\varepsilon > 0$. If P is non null, Q is also non null and we have

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+}\frac{|P(x)||Q(y)|}{(1+||x||+||y||)^{2N}}e^{-(\sqrt{r}||y||-\frac{1}{2\sqrt{r}}||x||)^2}e^{(4ab-1)||x||||y||}dxdy$$
$$\geq C\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+}e^{-(\sqrt{r}||y||-\frac{1}{2\sqrt{r}}||x||)^2}e^{(4ab-1-\varepsilon)||x||||y||}dxdy,$$

where C is a positive constant. But the function

$$e^{-(\sqrt{r}||y||-\frac{1}{2\sqrt{r}}||x||)^2}e^{(4ab-1-\varepsilon)||x||||y||}$$

is not integrable, (32) does not hold. Hence $f(x) = 0$ a.e.

2)

i) We deduce the result from (29) and Theorem 5.2.

ii) By using (29) the relations (26),(27) can also be written in the form

$$\int_{\mathbb{R}^d}\frac{|\mathcal{F}_D(f)(y)|e^{\frac{(2b)^q}{q}||y||^q}}{(1+||y||)^N}dy = \int_{\mathbb{R}^d}\frac{|Q(y)|e^{-r||y||^2}e^{\frac{(2b)^q}{q}||y||^q}}{(1+||y||)^N}dy.$$

and

$$\int_{\mathbb{R}^d}\frac{|f(x)|e^{\frac{(2a)^p}{p}||x||^p}}{(1+||x||)^N}\omega_k(x)dx = \int_{\mathbb{R}^d}\frac{|P(x)|e^{-\frac{||x||^2}{4r}}e^{\frac{(2a)^p}{p}||x||^p}}{(1+||x||)^N}\omega_k(x)dx.$$

We obtain ii) from Theorem 5.2 and by studying the convergence of these integrals as we have made it in the 1).

■

Theorem 6.2. (Hardy type) *Let $N \in \mathbb{N}$. Assume that f in $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ is such that*

$$|f(x)| \leq Me^{-\frac{1}{4a}||x||^2}\text{a.e.}$$
$$\text{and } \forall y \in \mathbb{R}^{d+1}_+,\ |\mathcal{F}_{D,B}(f)(y)| \leq M(1+|y_j|)^N e^{-b|y_j|^2},\ j = 1, ..., d+1, \tag{33}$$

for some constants $a > 0$, $b > 0$ and $M > 0$. Then,

i) If $ab > \frac{1}{4}$, then $f = 0$ a.e.

ii) If $ab = \frac{1}{4}$, the function f is of the form $f(x) = \sum_{|s|+p\leq N} a^{k,\beta}_{s,p}W^{k,\beta}_{s,p}(\frac{1}{4a}, x)$ a.e. where $a^{k,\beta}_{s,p} \in \mathbb{C}$.

iii) If $ab < \frac{1}{4}$, there are infinity many nonzero functions f satisfying the conditions (33).

P r o o f. The first condition of (33) implies that $f \in L^1_{k,\beta}(I\!R^{d+1}_+)$. So by Theorem 3.1, the function ${}^tR_{k,\beta}(f)$ is defined almost everywhere. By using the relation (17) we deduce that for all $x \in I\!R^{d+1}_+$,

$$|{}^tR_{k,\beta}(f)(x)| \leq M_0 e^{-a||x||^2},$$

where M_0 is a positive constant.

So,

$$|{}^tR_{k,\beta}(f)(x)| \leq M_0(1+|x_j|)^N e^{-a|x_j|^2},\ j=1,...,d+1. \tag{34}$$

On the other hand from (17) and (33) we have for all $y \in I\!R^{d+1}_+$,

$$|\mathcal{F}_o({}^tR_{k,\beta})(f)(y)| \leq M(1+|y_j|)^N e^{-b|y_j|^2},\ j=1,...,d+1. \tag{35}$$

The relations (34) and (35) show that the conditions of Proposition 3.4 of [2], p.36, are satisfied by the function ${}^tR_{k,\beta}(f)$. Thus we get:

i) If $ab > \frac{1}{4}$, ${}^tR_{k,\beta}(f) = 0$ a.e.

Using (17) we deduce

$$\forall\, y \in I\!R^{d+1}_+,\ \mathcal{F}_{D,B}(f)(y) = \mathcal{F}_o \circ ({}^tR_{k,\beta})(f)(y) = 0.$$

Then from Theorem 2.3.1 of [10] we have $f = 0$ a.e.

ii) If $ab = \frac{1}{4}$, then ${}^tR_{k,\beta}(f)(x) = P(x)e^{-a||x||^2}$, where P is a polynomial of degree strictly lower than N. The same proof as of the end of Theorem 5.2 shows that

$$f(x) = \sum_{|s|+p\leq N} a^{k,\beta}_{s,p} W^{k,\beta}_{s,p}(\frac{1}{4a}, x)\ a.e.$$

iii) If $ab < \frac{1}{4}$, let $t \in]a, \frac{1}{4b}[$ and $f(x) = ce^{-t||x||^2}$ for some real constant c, these functions satisfy the conditions (33). ∎

Theorem 6.3. (Cowling-Price type) *Let $N \in I\!N$. Assume that f in $L^2_{k,\beta}(I\!R^{d+1}_+)$ is such that*

$$\int_{I\!R^{d+1}_+} e^{a||x||^2}|f(x)|d\mu_{k,\beta}(x) < +\infty \ \text{ and } \int_{I\!R^{d+1}_+} \frac{e^{b||y||^2}}{(1+||y||)^N}|\mathcal{F}_{D,B}(f)|dy < +\infty \tag{36}$$

for some constants $a > 0, b > 0$. Then,

i) If $ab > \frac{1}{4}$, we have $f = 0$ a.e.

ii) If $ab = \frac{1}{4}$, then when $N \geq d+2$ we have

$$f(x)=\sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta}W_{s,p}^{k,\beta}(\frac{1}{4a},x) \text{ a.e.},$$

where $a_{\mu,p}^{k,\beta}\in\mathbb{C}$.

iii) If $ab<\frac{1}{4}$, *there are infinity many nonzero functions* f *satisfying the conditions (36).*

P r o o f. From the first condition of (36) we deduce that $f\in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. So by Theorem 3.1, the function ${}^tR_{k,\beta}(f)$ is defined almost everywhere. By using the relations (12), (14) and (36) we have:

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+}\frac{|{}^tR_{k,\beta}(f)(x)|e^{a||x||^2}}{(1+||x||)^N}dx &\leq \int_{\mathbb{R}^{d+1}_+}{}^tR_{k,\beta}(e^{a||x||^2}|f|)(x)dx,\\ &\leq \int_{\mathbb{R}^{d+1}_+}e^{a||y||^2}|f(y)|d\mu_{k,\beta}(y)<+\infty.\end{aligned}$$

So,

$$\int_{\mathbb{R}^{d+1}_+}\frac{|{}^tR_{k,\beta}(f)(x)|e^{a||x||^2}}{(1+||x||)^N}dx<+\infty. \tag{37}$$

On the other hand from (17) and (36) we have:

$$\int_{\mathbb{R}^{d+1}_+}\frac{e^{b||y||^2}}{(1+||y||)^N}|\mathcal{F}_{D,B}(f)|dy=\int_{\mathbb{R}^{d+1}_+}\frac{e^{b||y||^2}}{(1+||y||)^N}|\mathcal{F}_o({}^tR_{k,\beta})(f)(y)|dy<+\infty. \tag{38}$$

The relations (37) and (38) are the conditions of Proposition 3.2 of [2] p.35, which are satisfied by the function ${}^tR_{k,\beta}(f)$. Thus we get:

i) If $ab>\frac{1}{4}$, ${}^tR_{k,\beta}(f)=0$ a.e.

Using the same proof as of Theorem 6.2, we deduce $f(y)=0$. a.e. $y\in\mathbb{R}^{d+1}_+$.

ii) If $ab=\frac{1}{4}$, then ${}^tR_{k,\beta}(f)(x)=P(x)e^{-a||x||^2}$ where P is a polynomial of degree strictly lower than $\frac{N-d-1}{2}$. The same proof as of the end of Theorem 5.2 shows that

$$f(x)=\sum_{|s|+p<\frac{N-d-1}{2}} a_{s,p}^{k,\beta}W_{s,p}^{k,\beta}(\frac{1}{4a},x) \text{ a.e.}$$

iii) If $ab<\frac{1}{4}$, let $t\in]a,\frac{1}{4b}[$ and $f(x)=ce^{-t||x||^2}$ for some real constant c, these functions satisfy the conditions (36). This completes the proof. ■

Theorem 6.4. (Morgan type) *Let* $1 < p < 2$ *and* q *be the conjugate exponent of* p. *Assume that* f *in* $L^2_{k,\beta}(\mathbb{R}^{d+1}_+)$ *satisfies*

$$\int_{\mathbb{R}^{d+1}_+} e^{\frac{a^p}{p}||x||^p}|f(x)|d\mu_{k,\beta}(x) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^{d+1}_+} e^{\frac{b^q}{q}||y||^q}|\mathcal{F}_{D,B}(f)(y)|dy < +\infty, \tag{39}$$

for some constants $a > 0, b > 0$.

Then if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$, *we have* $f = 0$ *a.e.*

P r o o f. The first condition of (39) implies that $f \in L^1_{k,\beta}(\mathbb{R}^{d+1}_+)$. So by Theorem 3.1 , the function ${}^tR_{k,\beta}(f)$ is defined almost everywhere. By using the relations (12) and (39) we deduce that:

$$\int_{\mathbb{R}^{d+1}_+} |{}^tR_{k,\beta}(f)(x)|e^{\frac{a^p}{p}||x||^p}dx \leq \int_{\mathbb{R}^{d+1}_+} e^{\frac{a^p}{p}||y||^p}|f(y)|d\mu_{k,\beta}(y) < +\infty.$$

So,

$$\int_{\mathbb{R}^{d+1}_+} |{}^tR_{k,\beta}(f)(x)|e^{\frac{a^p}{p}||x||^p}dx < +\infty. \tag{40}$$

On the other hand, from (17) and (39) we have:

$$\int_{\mathbb{R}^{d+1}_+} e^{\frac{b^q}{q}||y||^q}|\mathcal{F}_{D,B}(f)(y)|dy = \int_{\mathbb{R}^{d+1}_+} e^{\frac{b^q}{q}||y||^q}|\mathcal{F}_o({}^tR_{k,\beta})(f)(y)|dy < +\infty. \tag{41}$$

The relations (40) and (41) are the conditions of Theorem 1.4, p.26 of [2], which are satisfied by the function ${}^tR_{k,\beta}(f)$. Thus we deduce that if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$ we have ${}^tR_{k,\beta}(f) = 0$ a.e.

Using the same proof as of Theorem 6.2 we obtain $f(y) = 0$. a.e. $y \in \mathbb{R}^{d+1}_+$. ∎

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