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q-HEAT OPERATOR AND q-POISSON'S OPERATOR

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*Dedicated to Professor Khalifa Trimèche,
on the occasion of his 60th anniversary*

Abstract

In this paper we study the q-heat and q-Poisson's operators associated with the q-operator Δ_q (see[5]). We begin by summarizing some statements concerning the q-even translation operator $\mathcal{T}_{x,q}$, defined by Fitouhi and Bouzeffour in [5]. Then, we establish some basic properties of the q-heat semi-group such as boundedness and positivity. In the second part, we introduce the q-Poisson operator P^t, and address its main properties. We show in particular how these operators can be used to solve the initial and boundary value problems related to the q-heat and q-Laplace equation respectively.

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1. Introduction and preliminaries

1.1. Introduction

Let us recall the initial value problem for the classical heat equation associated with the second order derivative operator $\dfrac{\partial^2}{\partial x^2}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad x \in \mathbf{R}, t > 0, \ u(x,0) = f(x), x \in \mathbf{R}$$

which has a solution of the form

$$T^t f(x) = u(x,t) = (f * G(.,t))(x),$$

where $G(.,t)$ is the Gaussian kernel. The operator T^t is a bounded positive operator, and $\{T^t\}_{t\geq 0}$ form a semi-group. Our aim in this paper is to give the q-analogue of some well-known results associated to the heat and Poisson's operators as done in the classical case by Achour and Trimèche in [1] and by Stein in [14]. So we will turn our attention to the second order q-difference operator $\Delta_{q.x}$ defined by

$$\Delta_{q.x} f = (D_q^2 f)(q^{-1}x), \tag{1}$$

which has the q-cosine $\cos(\lambda x; q^2)$ and the q-sine $\sin(\lambda x; q^2)$ as eigenfunctions with eigenvalue $(-\lambda^2)$ (see [5]). We shall prove some facts about the q-even translation operator $\mathcal{T}_{x,q}$ such that the x-continuity of $\mathcal{T}_{x,q} f$ for f in appropriate spaces.

In a second part we study the q-heat equation

$$\Delta_{q.x}\, u(x,t) = D_{q^2.t}\, u(x,t). \tag{2}$$

We prove some properties of the q-Gaussian kernel $G(.,t;q^2)$ defined in [5], which enable us to establish some basic facts such as boundedness and positivity for the q-heat operator T^t, with methods similar to the ones used in [1, 4, 14]. Next, we construct the q-analogue of the Poisson operator P^t, we find the q-Poisson kernel, its expansion and its q-cosine Fourier transform. We give a q-integral representation of $u(x,t) = P^t f(x)$ and show that it is a solution of a q-difference equation analogous to the classical Laplace equation

$$(\Delta_{q.t} + \Delta_{q.x})\, u(x,t) = 0. \tag{3}$$

Finally, we give the q-analogue of some estimates given in [1, 4] for the function $u(x,t)$ and some of its q-derivatives.

1.2. Preliminaries

Let q be a positive real in $(0,1)$. We recall some notations and notions important in q-analysis (for more information the reader can consult [7, 9]):

The q-shifted factorials are defined for any $a \in \mathbf{C}$, by

$$(a;q)_n = \begin{cases} 1, & \text{if} \quad n = 0 \\ \prod_{k=0}^{n-1}(1 - aq^k), & \text{if} \quad n = 1, 2, \ldots, \infty \end{cases} \tag{4}$$

and we have

$$(a\,q^{-k};q)_k = (-1)^k q^{-k(k+1)/2} a^k (qa^{-1};q)_k, \quad k = 1,2,\dots. \tag{5}$$

The q-trigonometric functions (see [5, 10]), are defined by

$$\cos(x;q^2) = \sum_{n=0}^{+\infty} \frac{(-1)^n q^{n(n-1)}(1-q)^{2n}}{(q;q)_{2n}} x^{2n} = \sum_{n=0}^{+\infty} (-1)^n b_n(x;q^2), \tag{6}$$

$$\sin(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{1-q}{1-q^{2n+1}} b_n(x;q^2)\, x. \tag{7}$$

There are two q-analogues of the exponential function, given by

$$E(x;q^2) = (-(1-q^2)x;q^2)_\infty = \sum_{n=0}^{+\infty} \frac{(1-q^2)^n q^{n(n-1)}}{(q^2;q^2)_n} x^n, \quad x \in \mathbf{R} \tag{8}$$

$$e(x;q^2) = \frac{1}{((1-q^2)x;q^2)_\infty} = \sum_{n=0}^{+\infty} \frac{(1-q^2)^n}{(q^2;q^2)_n} x^n,$$

the last series converges for $|x| \le 1/(1-q^2)$; however, because of its product representation $e(x;q^2)$ has an analytic continuation to $\mathbf{C}\backslash\{\frac{q^{-2k}}{(1-q^2)}, k \in \mathbf{N}\}$. They satisfy the relation $e(x;q^2)\,E(-x;q^2) = 1$.

Let $\mathbf{C}_{q^2}\big[[x,y]\big]$ be the complex associative algebra with 1 of formal power series $\sum_{k,l=0}^{+\infty} c_{k,l}\, y^l x^k$, with arbitrary complex coefficients $c_{k,l}$ and where x and y are two q^2-commuting variables, i.e., $xy = q^2 yx$. Koornwinder [8] proves that the relation

$$e(y;q^2)\,e(x;q^2) = e(x+y;q^2), \tag{9}$$

holds in the algebra $\mathbf{C}_{q^2}\big[[x,y]\big]$.

The q-derivative $D_q f$ of a function f on $\mathbf{R}$, is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \ne 0 \tag{10}$$

and $D_q f(0) = f'(0)$, provided $f'(0)$ exists.

The q-Jackson integrals from 0 to a and from 0 to $\gamma.\infty$ with $a, \gamma \in \mathbf{R}$, are defined by

$$\int_0^a f(x) d_q x = (1-q) \sum_{n=0}^{+\infty} f(aq^n) a q^n, \quad \int_0^{\gamma.\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} f(\gamma q^n) \gamma q^n,$$

provided the sums converge absolutely. Then, we have the q-Chasles relation

$$\int_0^{+\infty} f(x)d_qx = \int_0^a f(x)d_qx + \int_a^{+\infty} f(x)d_qx, \quad a \in \mathbf{R}_{q,+} \tag{11}$$

where $\mathbf{R}_{q,+}$ will be lately defined by (13).

The q-integration by parts is given for suitable functions f and g by

$$\int_0^{+\infty} f(x)D_qg(x)d_qx = [f(x)g(x)]_0^{+\infty} - \int_0^{+\infty} D_q(f(q^{-1}x))\, g(x)\, d_qx. \tag{12}$$

Let us now consider the sets

$$\mathbf{R}_q = \{\pm q^k, k \in \mathbf{Z}\} \cup \{0\}, \quad \mathbf{R}_{q,+} = \{q^k, k \in \mathbf{Z}\}, \tag{13}$$

and recall that $S_{*.q}(\mathbf{R})$ is the space of even and indefinitely q-differentiable fast decreasing functions f, together with their q-derivatives, i.e., such that

$$\forall n, m \in \mathbf{N}, \quad P_{n,m,q}(f) = \sup_{\substack{x\in\mathbf{R} \\ 0\le k\le n}} |(1+x^2)^m D_q^k f(x)| < +\infty, \quad \text{(see [12])}$$

and $S_{*.q}(\mathbf{R}_q)$ is the space of skeletons $\widetilde{f}$ of f on $\mathbf{R}_q$, for f in $S_{*.q}(\mathbf{R})$. We equip $S_{*.q}(\mathbf{R})$ with the topology defined by the sequence of semi norms $P_{n,m,q}$, and the topology in $S_{*.q}(\mathbf{R}_q)$ is induced by the one of $S_{*.q}(\mathbf{R})$.

In the following we suppose that $\log(1-q)/\log(q) \in \mathbf{Z}$, and we recall some basic definitions useful for the remainder (see [5]).

We begin with the q-translation operator $\mathcal{T}_{x,q}$ defined for $f \in S_{*.q}(\mathbf{R}_q)$, by

$$\mathcal{T}_{x,q}f(y) = \int_0^{+\infty} f(t)d_q\mu(x;y)(t), \quad x, y \in \mathbf{R}_{q,+}, \tag{14}$$

with

$$d_q\mu(x;y) = \sum_{s=-\infty}^{+\infty} D(x,y;q^s)q^s\delta_{yq^s}, \quad \text{(see [5])}. \tag{15}$$

The q-convolution product of two suitable functions f and g is given by

$$(f *_q g)(x) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{+\infty} \mathcal{T}_{x,q}f(y)g(y)d_qy, \quad x \in \mathbf{R}_{q,+}. \tag{16}$$

Finally, the q-cosine Fourier transform [5] is defined as

$$\mathcal{F}_q(f)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{+\infty} f(x)\cos(\lambda x; q^2)\, d_qx, \quad \lambda \in \mathbf{R}_{q,+}, \tag{17}$$

and satisfies

$$\mathcal{F}_q(f *_q g)(\lambda) = \mathcal{F}_q(f)(\lambda)\, \mathcal{F}_q(g)(\lambda), \quad f, g \in S_{*.q}(\mathbf{R}_q). \tag{18}$$

We start on giving some interesting properties of the q-translation operator.

2. The q-Translation operator

Proposition 1. *If $f \in S_{*.q}(\boldsymbol{R}_q)$, then $\mathcal{T}_{x,q}f \in S_{*.q}(\boldsymbol{R}_q)$.*

P r o o f. We recall that the q-Fourier transform $\mathcal{F}_q$ is an isomorphism from $S_{*.q}(\mathbf{R}_q)$ into the same space (see [6]). So it suffices to show that

$$\mathcal{F}_q(\mathcal{T}_{x,q}f)(\lambda) = \cos(\lambda x; q^2)\mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbf{R}_{q,+} \quad \text{(see [5])}. \tag{19}$$

belongs to $S_{*.q}(\mathbf{R}_q)$, which is easy to prove. ∎

Proposition 2. *For $f \in L^1_q(\boldsymbol{R}_{q,+})$ and $x, y \in \boldsymbol{R}_q$ we have*

$$\mathcal{T}_{x,q}f(y) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{+\infty} \cos(\lambda x; q^2)\mathcal{F}_q(f)(\lambda)\cos(\lambda y; q^2)\, d_q\lambda. \tag{20}$$

P r o o f. It suffices to consider $f \in S_{*.q}(\mathbf{R}_q)$, then by the above proposition $\mathcal{T}_{x,q}f$ is in $S_{*.q}(R_q)$. The result follows from the formula (19) by using the q-Fourier inversion formula (see [5]). ∎

We denote by $\mathcal{C}_{0.q}(\mathbf{R}_q)$ the space of even functions f defined on $\mathbf{R}_q$ continuous at 0, such that for all $m \in \mathbf{N}$ and some $\epsilon > 0$, we have

$$|(D_q^m f)(\pm q^{-k})| = \mathcal{O}(q^{(1+\epsilon)k}), \quad k \longrightarrow +\infty. \tag{21}$$

Notice that $\sup_{k \in \mathbf{Z}} |f(q^k)| < +\infty$ for f in $\mathcal{C}_{0.q}(\mathbf{R}_q)$.

Proposition 3. *For $f \in \mathcal{C}_{0.q}(\boldsymbol{R}_q)$, the function $x \longrightarrow \mathcal{T}_{x,q}f$ is continuous at zero for the norm $\|.\|_{\infty.q}$ defined by*

$$\|f\|_{\infty.q} = \sup_{k \in \mathbf{Z}} |f(q^k)|. \tag{22}$$

P r o o f. Recall that $(\mathcal{T}_{x,q}f - f)(y) = \sum_{k=1}^{\infty} b_k(x; q^2)\Delta_q^k f(y)$ (see [5]).

By (21), there exist a strictly negative integer n_0 and a constant $M_1 > 0$, such that $\forall n \le n_0$, we have $|\Delta_q^k(f)(q^n)| < M_1, \quad \forall k \in \mathbf{N}$.

So $\exists a_1 > 0, / \forall x < a_1$, we have $\sup_{n \in (-\infty, n_0)} |\mathcal{T}_{x,q}f(q^n) - f(q^n)| < \epsilon$.

Using the inequality $|\mathcal{T}_{x,q}f(q^n) - f(q^n)| \le |\mathcal{T}_{x,q}f(q^n) - f(x)| + |f(x) - f(0)| + |f(0) - f(q^n)|$, the fact that the function f is continuous at 0, and $\lim_{n \longrightarrow +\infty} \mathcal{T}_{x,q}f(q^n) = f(x)$, we show that

$$[\forall \epsilon > 0, \exists\, a_2 > 0, n' > 0, / \forall x < a_2, \text{we have} \sup_{n \in (n', +\infty)} |\mathcal{T}_{x,q}f(q^n) - f(q^n)| < \epsilon].$$

Since the supremum over $[n_0, n']$ is attained, we deduce the result. ∎

REMARK 1. For $f \in \mathcal{C}_{0.q}(\mathbf{R}_q)$, we have

$$\|\mathcal{T}_{x,q} f\|_{\infty.q} = \sup_{k\in\mathbf{Z}} |\sum_{s=-\infty}^{+\infty} D(x, q^k; q^s) q^s f(q^{s+k})| \tag{23}$$

$$\leq \|f\|_{\infty.q} \sup_{k\in\mathbf{Z}} \|d_q\mu(x; q^k)\|_{var}. \tag{24}$$

But for any $x, y \in \mathbf{R}_{q,+}$ we have that

$$\|d_q\mu(x; y)\|_{var} \leq \frac{2(-q; q)_\infty^2}{(1-q)(q; q)_\infty} = K, \tag{25}$$

thus

$$\|\mathcal{T}_{x,q} f\|_{\infty.q} \leq K\|f\|_{\infty.q}. \tag{26}$$

For $1 \leq p < +\infty$, we denote by $L_q^p(\mathbf{R}_{q,+})$, the Banach space of functions $\widetilde{f}$ which are restrictions on $\mathbf{R}_{q,+}$ of functions f such that

$$\int_0^{+\infty} |f(x)|^p \, d_q x < +\infty, \tag{27}$$

this space is equipped with the $p.q$-norm $\|.\|$:

$$\|f\|_{p.q} = \left(\int_0^{+\infty} |f(x)|^p \, d_q x\right)^{1/p} \tag{28}$$

PROPOSITION 4. *For $f \in L_q^p(\mathbf{R}_{q,+})$, $1 \leq p < +\infty$, the function $x \longrightarrow \mathcal{T}_{x,q} f$ is continuous at zero for the norm $\|.\|_{p.q}$.*

P r o o f. It suffices to take the proof for $f \in S_{*.q}(\mathbf{R}_q)$ since the later space is dense in $L_q^p(\mathbf{R}_{q,+})$. By Proposition 1 we have that $\mathcal{T}_{x,q} f$ belongs to $S_{*.q}(\mathbf{R}_q)$, so, $\forall \epsilon > 0, \exists k_0 < 0$, such that $\sum_{k=-\infty}^{k_0} |\mathcal{T}_{x,q} f(q^k) - f(q^k)|^p q^k < \frac{\epsilon^p}{2(1-q)}$. So the proof follows in the same manner as Proposition 3. ∎

Finally, let us recall the following result concerning the positivity of the q-translation operator, proven by Fitouhi-Dhaouadi-El Kamel in [6].

THEOREM 1. *The operator $\mathcal{T}_{x,q}$ is positive if and only if q belongs to the (non empty) subset I_q of $(0, 1)$ defined by*

$$I_q = \{q \in (0, 1) \backslash \ {}_1\Phi_1(0; q; q, q) \geq 0\}. \tag{29}$$

3. q-Gaussian kernel and q-heat semi-group

3.1. The q-Gaussian kernel

We recall that the function $G(.,t;q^2), t>0$ is defined by

$$\mathcal{F}_q(G(.,t;q^2))(\lambda) = e(-\lambda^2 t;q^2) \tag{30}$$

and is given explicitly by (see [5])

$$G(x,t;q^2) = \frac{1}{A(t;q^2)} e(\frac{-x^2}{q(1+q)^2 t};q^2), \quad t>0, \tag{31}$$

where

$$A(t;q^2) = q^{-1/2}(1-q)^{1/2} \frac{(-\frac{(1-q)}{(1+q)t}, -\frac{(1+q)q^2 t}{(1-q)};q^2)_\infty}{(-\frac{(1-q)}{(1+q)qt}, -\frac{(1+q)q^3 t}{(1-q)};q^2)_\infty}. \tag{32}$$

Now we shall prove the following proposition.

PROPOSITION 5.

1. *$G(x,t;q^2)$ and $\mathcal{T}_{x,q}G(y,t;q^2)$ are positive for all q in I_q.*
2. *$G(.,t;q^2)$ belongs to $S_{*.q}(\boldsymbol{R}_q)$ and its $1.q$-norm is given by*

$$\|G(.,t;q^2)\|_{1.q} = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}}. \tag{33}$$

3. *If t_1, t_2 are two q^2-commuting variables, then*

$$(G(.,t_1;q^2)*_q G(.,t_2;q^2))(x) = G(x,t_2+t_1;q^2) \tag{34}$$

in the algebra $\boldsymbol{C}_{q^2}\big[[t_1,t_2]\big]$.

P r o o f.

1. Since $A(t;q^2)$ and $e(\frac{-x^2}{q(1+q)^2 t};q^2)$ are strictly positive, the same holds for $G(x,t;q^2)$. By Theorem 1 the q-translation function $\mathcal{T}_{x,q}G(y,t;q^2)$ inherits the same property.
2. We have

$$\|G(.,t;q^2)\|_{1.q} = \int_0^{+\infty} G(x,t;q^2)\, d_q x = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}} \mathcal{F}_q(G(.,t;q^2))(0).$$

By the formula (30) and the fact that $e(0;q^2)=1$, we obtain the result.

3. Using (18) and (30), we get

$$\begin{aligned}\mathcal{F}_q(G(.,t_1;q^2) *_q G(.,t_2;q^2))(\lambda) &= \mathcal{F}_q(G(.,t_1;q^2))(\lambda)\mathcal{F}_q(G(.,t_2;q^2))(\lambda)\\ = e(-\lambda^2 t_1;q^2)\, e(-\lambda^2 t_2;q^2) &= e(-\lambda^2(t_2+t_1);q^2)\\ &= \mathcal{F}_q(G(.,t_2+t_1;q^2))(\lambda) \qquad (35)\end{aligned}$$

in the algebra $\mathbf{C}_{q^2}\big[[t_1,t_2]\big]$, where in the thirst equality we have used the formula (9), since $G(.,t;q^2)$ is in $S_{*.q}(\mathbf{R}_q)$ then we can use the q-Fourier inversion formula, and get the result.

■

LEMMA 1. *For $a \in \boldsymbol{R}_{q,+}$ and $t > 0$, we have*

$$\lim_{t\longrightarrow 0}\int_a^{+\infty} G(y,t;q^2)d_q y = 0. \qquad (36)$$

P r o o f. If we replace $G(.,t;q^2)$ by its expansion (31) and use the q-Jackson integral definition [9], we obtain

$$\int_a^{+\infty} G(y,t;q^2)\, d_q y = \frac{(1-q)a}{A(t;q^2)(\frac{-(1-q)a^2}{q(1+q)t};q^2)_\infty}\sum_{k=-\infty}^{-1}\frac{q^k}{(-\frac{(1-q)a^2q^{2k}}{q(1+q)t};q^2)_{-k}}.$$

Replacing $A(t;q^2)$ by its expansion (32), we find, for t tending to zero:

$$\frac{1}{A(t;q^2)(-\frac{(1-q)a^2}{q(1+q)t};q^2)_\infty} \sim q^{1/2}(1-q)^{-1/2}\underbrace{\frac{(-\frac{(1-q)}{(1+q)qt};q^2)_\infty}{(-\frac{(1-q)}{(1+q)t};q^2)_\infty(-\frac{(1-q)a^2}{q(1+q)t};q^2)_\infty}}_{H(q,t)},$$

where $$H(q,t) = \lim_{j\longrightarrow+\infty}\prod_{k=0}^{j}\frac{(1+\frac{(1-q)}{(1+q)qt}q^{2k})}{(1+\frac{(1-q)}{(1+q)t}q^{2k})(1+\frac{(1-q)}{q(1+q)t}a^2q^{2k})}.$$

Then by a simple computation, we get that $H(q,t)$ is a bounded function of t, so it suffices to prove that $\lim_{t\longrightarrow 0}\sum_{k=-\infty}^{-1}\frac{q^k}{(-\frac{(1-q)a^2q^{2k}}{q(1+q)t};q^2)_{-k}} = 0.$

This is obtained by the change of variables $k' = -k$ and the use of the formula (5) with q^2 instead of q. ■

PROPOSITION 6. (q-Hölder's inequality) *Let p and p' be two conjugate reals with $p, p' > 1$. Then for $f \in L_q^p(\boldsymbol{R}_{q,+})$ and $g \in L_q^{p'}(\boldsymbol{R}_{q,+})$, we have*

$$\left| \int_0^{+\infty} f(x)\, g(x)\, d_q x \right| \leq \|f\|_{p.q} \|g\|_{p'.q}. \tag{37}$$

P r o o f. We replace the q-Jackson integral in the left-hand side by its expansion in a q-series form. Then, using the classical Hölder's inequality relative to sums, we get the result. ■

As a consequence, we obtain the following corollary.

COROLLARY 1. *If $f \in L_q^1(\boldsymbol{R}_{q,+})$ and $g \in L_q^p(\boldsymbol{R}_{q,+})$, then $f *_q g \in L_q^p(\boldsymbol{R}_{q,+})$, with*

$$\|f *_q g\|_{p.q} \leq \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \|f\|_{1.q} \|g\|_{p.q}. \tag{38}$$

P r o o f. In fact, we have

$$\|f *_q g\|_{p.q}^p = \int_0^{+\infty} \left| \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{+\infty} (\mathcal{T}_{x,q} f(y))^{\frac{1}{p}+\frac{1}{p'}} g(y) d_q y \right|^p d_q x.$$

By the q-Hölder's inequality and the fact that (see [5])

$$\|\mathcal{T}_{x,q} f\|_{1.q} \leq \|f\|_{1.q}, \tag{39}$$

we deduce the result. ■

3.2. The q-heat semi-group

For any $f \in L_q^p(\mathbf{R}_{q,+})$, $1 \leq p < +\infty$, we introduce the operator $T^t, t > 0$, by

$$T^t f(x) = (G(.,t;q^2) *_q f)(x). \tag{40}$$

REMARK 2. From the last corollary, we note that $T^t f \in L_q^p(\mathbf{R}_{q,+})$, and we have

$$\|T^t f\|_{p.q} \leq \|f\|_{p.q}. \tag{41}$$

Thus T^t is a contraction in $L_q^p(\mathbf{R}_{q,+})$, $1 \leq p < +\infty$.

PROPOSITION 7. *For $f \in \mathcal{C}_{0.q}(\boldsymbol{R}_q)$, we have*

$$\lim_{t \longrightarrow 0} \|T^t f - f\|_{\infty.q} = 0. \tag{42}$$

P r o o f. It follows from (16), (33) and (40), that

$$|T^t f(x) - f(x)| = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \left| \int_0^{+\infty} G(y,t;q^2)\big\{\mathcal{T}_{y,q} f(x) - f(x)\big\} d_q y \right|. \tag{43}$$

By the use of relation (11), and after simple computation, we obtain

$$\|T^t f - f\|_{\infty.q} \leq \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)}$$
$$\times \big(\sup_{y\in[0,a]} \|\mathcal{T}_{y,q} f - f\|_{\infty.q} \int_0^a G(y,t;q^2) d_q y + 2K\|f\|_{\infty.q} \int_a^{+\infty} G(y,t;q^2) d_q y\big),$$

where K is given in the formula (25). Now the result is a consequence from the last lemma and Proposition 3. ∎

Proposition 8. *If $f \in L^p_q(\boldsymbol{R}_{q,+})$, $1 \leq p < +\infty$, then the function $t \longrightarrow T^t f$ is continuous at zero for the norm $\|.\|_{p.q}$.*

P r o o f. Replacing $|T^t f(x) - f(x)|$ by its expansion (43) and using the definition of the $p.q$-norm to get $\|T^t f - f\|_{p.q}$.

Then after the use of the q-Hölder's inequality, the exchange of the q-integral signs and using the $1.q$-norm of $G(.,t;q^2)$ given by (33), we get

$$\|T^t f - f\|^p_{p.q} \leq \left(\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)}\right)^{p-p/p'} \int_0^{+\infty} \|\mathcal{T}_{y,q} f - f\|^p_{p.q} G(y,t;q^2) d_q y.$$

Now, we proceed as above by using the relation (11) and the fact that

$$\|\mathcal{T}_{y,q} f\|_{p.q} \leq K' \|f\|_{p.q}, \tag{44}$$

where K' is a constant. To complete the proof, we use the previous lemma and Proposition 4. ∎

Proposition 9. *Let $u(x,t) = T^t f(x)$, where $f \in L^1_q(\boldsymbol{R}_{q,+}), t > 0$ and $x \in \boldsymbol{R}_q$. Then $u(x,t)$ satisfies the following statements:*

1. *$u(x,t)$ has the q-integral representation*

$$u(x,t) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{+\infty} e(-\lambda^2 t; q^2) \mathcal{F}_q(f)(\lambda) \cos(\lambda x; q^2)\, d_q\lambda. \tag{45}$$

2. *i) The function $x \longrightarrow u(x,t)$ is an even infinitely q-derivable function from $\boldsymbol{R}_q$ into $\boldsymbol{R}$.*

   ii) The function $t \longrightarrow u(x,t)$ is infinitely q^2-derivable from $(0,+\infty)$ into $\boldsymbol{R}$.

P r o o f.

1. Using (18), (30) and (40), we get

$$\mathcal{F}_q(u(.,t))(\lambda) = e(-\lambda^2 t; q^2)\mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbf{R}_{q,+}. \tag{46}$$

Since $\mathcal{F}_q = \mathcal{F}_q^{-1}$, then by applying the q-Fourier inversion formula (see theorem 1 in [5]), we get that

$$u(x,t) = \mathcal{F}_q(e(-\lambda^2 t; q^2)\mathcal{F}_q(f))(x). \tag{47}$$

2. Applying $D_{q.x}$ and $[D_{q^2.t}; \text{for } t \geq t_0 > 0]$ to the last expansion (45) of $u(x,t)$, and since we have the following estimates (see [5])

$$\begin{aligned} |\mathcal{F}_q(f)(\lambda)| &\leq \frac{1}{[q(1-q)]^{1/2}(q;q)_\infty}\|f\|_{1.q}, \quad f \in L_q^1(\mathbf{R}_{q,+}), \\ |D_{q.x}^k \cos(\lambda x; q^2)| &\leq \frac{\lambda^k}{(q;q^2)_\infty^2}, \quad k \in \mathbf{N}, \end{aligned} \tag{48}$$

then, for all $t \geq t_0$, we have $|D_{q^2.t}u(x,t)| \leq c' \int_0^{+\infty} \lambda^2 e(-\lambda^2 t_0; q^2) d_q\lambda$, where c' is a constant depending on q. So to show $i)$ and $ii)$ we need to verify that for any $n \in \mathbf{N}$, we have

$$\int_0^{+\infty} e(-\lambda^2 t; q^2)\lambda^n \, d_q\lambda < +\infty. \tag{49}$$

To this end, we use the Ramanujan's identity [9]. ∎

THEOREM 2. (q-heat semi-group) *The operators T^t defined for $t > 0$ by the formula* (40), *satisfy the following properties:*

1. *Each T^t is a positive operator for all $q \in I_q$, a bounded operator on $L_q^p(\boldsymbol{R}_{q,+}), p \geq 1$, and $T^t 1 = 1$.*

2. *For every f in $L_q^1(\boldsymbol{R}_{q,+})$ the $\mathcal{F}_q(T^t f)$ is given by formula* (46) *and each T^t is a self-adjoint operator on $L_q^2(\boldsymbol{R}_{q,+})$.*

3. *If t_1 and t_2 are two q^2-commuting variables, then $T^{t_1}T^{t_2} = T^{t_2+t_1}$ in the algebra $C_{q^2}\big[[t_1, t_2]\big]$.*

4. *If $f \in L^p_q(R_{q,+}), p \geq 1$, then $\{T^t\}$ is strongly continuous, i.e., the function $t \longrightarrow T^t f$ is continuous from $[0, +\infty)$ into $L^p_q(\boldsymbol{R}_{q,+})$.*

5. *For $f \in \mathcal{C}_{0.q}(\boldsymbol{R}_q) \cap L^1_q(\boldsymbol{R}_{q,+})$, the functions*

   i) $x \longrightarrow u(x,t)$ is an even infinitely q-derivable function from $\boldsymbol{R}_q$ into $\boldsymbol{R}$,

   ii) $t \longrightarrow u(x,t)$ is an infinitely q^2-derivable from $[0, +\infty)$ into $\boldsymbol{R}$, and $u(x,t)$ is a solution of the q-system

$$\begin{cases} \Delta_{q.x} u(x,t) &= D_{q^2.t} u(x,t), \; x \in \boldsymbol{R}_q, t > 0, \\ u(x,0) &= f(x). \end{cases} \tag{50}$$

P r o o f.

1. The fact that T^t is a positive operator for all q in I_q follows from the positivity of $\mathcal{T}_{x,q}G(y,t;q^2)$ (see Proposition 5). The boundedness of the operator T^t follows from Remark 2, and it is easy to show that $T^t 1 = 1$.

2. For $f, g \in L^2_q(\mathbf{R}_{q,+})$, we have

   $|\langle T^t f, g\rangle| = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \left| \int_0^{+\infty} \{ \int_0^{+\infty} \mathcal{T}_{x,q} G(y,t;q^2) f(y) d_q y \} \overline{g(x)} d_q x \right|$.

   Applying the q-Hölder's inequality (37), we obtain $|\langle T^t f, g\rangle| < +\infty$, which enables us to exchange the q-integral signs and to get the result.

3. The definition (40) gives that $T^{t_1}T^{t_2} f = (G(.,t_1;q^2) *_q G(.,t_2;q^2)) *_q f$, $f \in L^p_q(\mathbf{R}_{q,+}), p \geq 1$, therefore the result follows by formula (34).

4. The continuity of the mapping $t \longrightarrow T^t f$ at 0 for the norm $\|.\|_{p.q}$ has been proved in Proposition 8. For the continuity at t_0, $t_0 > 0$, we have

$$\|T^t f - T^{t_0} f\|^p_{p.q} = \left(\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \right)^p \times \int_0^{+\infty} \left| \int_0^{+\infty} \mathcal{T}_{x,q}\big(G(y,t;q^2) - G(y,t_0;q^2)\big) f(y) d_q y \right|^p d_q x.$$

Applying the q-Hölder's inequality and after simple computations, we get $\|T^t f - T^{t_0} f\|_{p.q}^p < +\infty$. Thus, we can exchange the q-integral signs and the limit as $t \longrightarrow t_0$.
So to achieve the proof we need to show the t-continuity of $G(y,t;q^2)$ on $(0,+\infty)$. In fact, this is true on any interval $[\tau,+\infty), \tau > 0$. We replace $G(y,t;q^2)$ by its expression obtained by inverting formula (30). Then, for all $t \geq \tau$, we have, using (48) and (49):

$$\Big|\int_0^{+\infty} e(-\lambda^2 t;q^2)\cos(\lambda x;q^2)d_q\lambda\Big| \leq \frac{1}{(q;q^2)_\infty^2}\int_0^{+\infty} e(-\lambda^2\tau;q^2)d_q\lambda < \infty.$$

5. The result is a direct consequence from Propositions 7, 9. ∎

4. The q-Poisson's operator

Let x' and γ two q-commuting (formal) variables belonging for example to some non commutative algebra and such that x' possesses an inverse denoted by $(x')^{-1}$. We define the sets

$$\mathbf{R}_{q,x'} = \big\{ \pm q^k x', k \in \mathbf{Z}\big\} \cup \{0\}, \tag{51}$$

$$\mathbf{R}_{q,x',+} = \big\{q^k x', k \in \mathbf{Z}\big\}, \tag{52}$$

$\mathbf{R}_{q,(x')^{-1}}$ and $\mathbf{R}_{q,(x')^{-1},+}$ are defined similarly. These sets generalize the sets $\mathbf{R}_q$ and $\mathbf{R}_{q,+}$ obtained for $x' = 1$, x' real. Then we define the more general q-Jackson integral

$$\int_0^{x'.\infty} f(x)d_q x = (1-q)x' \sum_{k=-\infty}^{+\infty} f(q^k x')q^k \tag{53}$$

provided the sum in the right hand side converges absolutely.

Let $L_q^p(\mathbf{R}_{q,x',+})$, $1 \leq p < +\infty$, be the space of functions f such that

$$\|f\|_{p.q.x'} = \left(\int_0^{x'.\infty} |f(x)|^p d_q x\right)^{\frac{1}{p}} < +\infty. \tag{54}$$

For $f \in L_q^1(\mathbf{R}_{q,x',+})$, we define the x' q-cosine Fourier transform by

$$\mathcal{F}_{q.x'}(f)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{x'.\infty} f(x)\cos(\lambda x;q^2)d_q x, \quad \lambda \in \mathbf{R}_{q,(x')^{-1},+}. \tag{55}$$

Exactly as in [6], we can show the following proposition.

PROPOSITION 10. *The x' q-cosine Fourier transform $\mathcal{F}_{q.x'}$ is an isomorphism from $S_{*.q}(\boldsymbol{R}_{q,x'})$ into $S_{*.q}(\boldsymbol{R}_{q,(x')^{-1}})$, with inverse $\mathcal{F}^{-1}_{q.x'}$ given for $x \in \boldsymbol{R}_{q,x',+}$ by*

$$\mathcal{F}^{-1}_{q.x'}(f)(x) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{(x')^{-1}.\infty} f(\lambda)\cos(\lambda x; q^2)\, d_q\lambda. \tag{56}$$

Here $S_{*.q}(\boldsymbol{R}_{q,x'})$ is the space of even, indefinitely q-differentiable functions f on $\boldsymbol{R}_{q,x'}$, such that for any $n, m \in \mathbf{N}$, we have

$$P_{n,m,q}(f) = \sup_{\substack{x \in \boldsymbol{R}_{q,x'} \\ 0 \le k \le n}} |(1+x^2)^m\, D_q^k f(x)| < +\infty. \tag{57}$$

It holds then

$$\mathcal{F}_{q.x'}(G_{x'}(.,t;q^2))(\lambda) = e(-\lambda^2 t; q^2), \quad \lambda \in \mathbf{R}_{q,(x')^{-1},+}, \tag{58}$$

where

$$G_{x'}(x,t;q^2) = \frac{1}{A_{x'}(t;q^2)} e(-\frac{x^2}{q(1+q)^2 t}; q^2), \quad x \in \mathbf{R}_{q,x',+}, \tag{59}$$

and

$$A_{x'}(t;q^2) = x' A(\frac{t}{x'^2}; q^2). \tag{60}$$

For f and g in $L^1_q(\mathbf{R}_{q,x',+})$, we define their x' q-convolution by

$$(f *_{q.x'} g)(x) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{x'.\infty} \mathcal{T}_{x,q} f(y)\, g(y)\, d_q y, \quad x \in \mathbf{R}_{q,x',+}. \tag{61}$$

Then, we define the q-heat operator $T^t_{x'}$ on $L^p_q(\mathbf{R}_{q,x',+})$, $1 \le p < +\infty$, by

$$T^t_{x'} f(x) = (f *_{q.x'} G_{x'}(.,t;q^2))(x), \quad t > 0. \tag{62}$$

PROPOSITION 11.

1. *The q-norm of $G_{x'}(.,t;q^2)$ is given by*

$$\|G_{x'}(.,t;q^2)\|_{1.q.x'} = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}}. \tag{63}$$

2. *For $f \in L^p_q(\boldsymbol{R}_{q,x',+})$, $p \ge 1$, we have*

$$\|T^t_{x'} f\|_{p.q.x'} \le \|f\|_{p.q.x'}. \tag{64}$$

P r o o f. The proof is similar to that of Proposition 5 and Remark 2. ■

DEFINITION 1. For $x \in \mathbf{R}_{q,x',+}$ and $t > 0$, the q-Poisson operator is obtained from the q-heat semi-group by the formula

$$P^t f(x) = \frac{1}{c}\int_0^{\gamma.\infty} \frac{e(-u;q^2)}{A_{x'}(u;q^2)} T_{x'}^{\frac{t^2}{q(1+q)^2 u}} f(x) d_{q^2}u, \quad f \in L_q^p(\mathbf{R}_{q,x',+}), \tag{65}$$

with

$$c = \int_0^{\gamma.\infty} \frac{e(-u;q^2)}{A_{x'}(u;q^2)}\, d_{q^2}u. \tag{66}$$

Let us now introduce as previously the space $\mathcal{C}_{0.q}(\mathbf{R}_{q,x'})$ of even functions f continuous at 0, such that for all $m \in \mathbf{N}$, we have

$$|(D_q^m f)(\pm q^{-k}x')| = \mathcal{O}(q^{(1+\epsilon)k}), \quad k \longrightarrow +\infty, \tag{67}$$

for some $\epsilon > 0$.

The following proposition is a consequence from the properties of the q-heat semi-group.

PROPOSITION 12.

1. *The operator P^t is a positive operator for all $q \in I_q$ and is bounded on $L_q^p(\mathbf{R}_{q,x',+})$, $1 \leq p < +\infty$.*
2. *P^t is a self-adjoint operator on $L_q^2(\mathbf{R}_{q,x',+})$.*
3. *If $f \in \mathcal{C}_{0.q}(\mathbf{R}_{q,x'})$, then $P^t f$ converges uniformly to f as t tends to 0.*

P r o o f.

1. Using (54) and (65), we have for $f \in L_q^p(\mathbf{R}_{q,x',+})$:

$$\|P^t f\|_{p.q.x'}^p = \int_0^{x'.\infty} \left| \frac{1}{c}\int_0^{\gamma.\infty} [\frac{e(-u;q^2)}{A_{x'}(u;q^2)}]^{\frac{1}{p}+\frac{1}{p'}} T_{x'}^{\frac{t^2}{q(1+q)^2 u}} f(x) d_{q^2}u \right|^p d_q x.$$

Applying the q-Hölder's inequality to the second integral, we obtain

$$\|P^t f\|_{p.q.x'}^p < +\infty.$$

Now, since $x'\gamma = q\gamma x'$, then if we exchange the q-integral signs, we get a factor $\frac{1}{q}$ (see [8]). Taking account of the estimation (64), we obtain

$$\|P^t f\|_{p.q.x'}^p \leq \frac{1}{q}\|f\|_{p.q.x'}^p. \tag{68}$$

The positivity is deduced from the positivity of the q-heat operator.

2. By the q-Hölder's inequality, we have that for $f, g \in L_q^2(\mathbf{R}_{q,x',+})$ $|\langle P^t f, g\rangle| = |\int_0^{x'.\infty}(P^t f)(x)\overline{g(x)}\,d_q x| \le \|P^t f\|_{2.q.x'}\|g\|_{2.q.x'} < +\infty$. After two exchanges of the q-integral signs, taking account of the fact that $x'\gamma = q\gamma x'$ and using the self-adjointness of the q-heat operator, we deduce the result.

3. By the same arguments as in Proposition 7, we get that $T_{x'}^t f$ converges uniformly to f, as t tends to 0. Then the result follows from the expression (65) of $P^t f$.

■

THEOREM 3. *Let p_t be the function given for $x \in \mathbf{R}_{q,x',+}$ and $t > 0$ by*

$$p_t(x) = \frac{q}{c}\int_0^{\gamma.\infty} \frac{e(-u;q^2)}{A_{x'}(u;q^2)} G_{x'}(x, \frac{t^2}{q(1+q)^2 u}; q^2)\, d_{q^2}u, \tag{69}$$

where c is given by (66)*. Then we have the following statements:*

1. *The q-norm of p_t is given by*

$$\|p_t\|_{1.q.x'} = \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}}. \tag{70}$$

2. *The q-Poisson's operator has the form*

$$P^t f(x) = (p_t *_{q.x'} f)(x). \tag{71}$$

The p_t is therefore called the q-Poisson kernel.

P r o o f.

1. Using (69) and (54), we get $\|p_t\|_{1.q.x'} \le \left\|G_{x'}(., \frac{t^2}{q(1+q)^2 u}; q^2)\right\|_{1.q.x'} < \infty$. So we can exchange the q-integral signs, and after simple computations, we obtain the result.

2. Using (69) and definition (61), we get the expansion of $|(p_t *_{q.x'} f)(x)|$. Moreover, we can easily prove that

$$\|p_t *_{q.x'} f\|_{1.q.x'} \le \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)}\|p_t\|_{1.q.x'}\|f\|_{1.q.x'} < +\infty. \tag{72}$$

So the result follows by exchanging the q-integral signs in the expansion of $|(p_t *_{q.x'} f)(x)|$, and by using the expression (62) of the q-heat semi-group. ■

PROPOSITION 13. *For every $x \in \boldsymbol{R}_{q,x',+}$ and $t \in \boldsymbol{R}_{q,+}$ we have*

$$p_t(x) = K_{x',\gamma} \frac{t}{x^2+t^2}, \tag{73}$$

where

$$K_{x',\gamma} = \frac{q}{c} \frac{1}{A_{x'}(\gamma; q^2) A_{x'}(\frac{1}{q(1+q)^2\gamma}; q^2)}.$$

REMARK 3. Note that when $q \longrightarrow 1^-$, we obtain the classical Poisson kernel associated with the second order operator $\frac{\partial^2}{\partial x^2}$ (see [4]), namely

$$p_t(x) = \sqrt{\frac{2}{\pi}} \frac{t}{x^2+t^2}. \tag{74}$$

P r o o f. If we replace the q-Gaussian kernel by the formula (59) in the expansion of p_t we obtain: $p_t(x) = \frac{q}{c}\int_0^{\gamma.\infty} \frac{e(-u;q^2)}{A_{x'}(u;q^2)} \frac{e(-\frac{x^2}{t^2}u;q^2)}{A_{x'}(\frac{t^2}{q(1+q)^2u};q^2)} \, d_{q^2}u$,

since $({x'}^2\gamma).\gamma = q^2\gamma.({x'}^2\gamma)$, then by the q-addition formula (9), we get

$$e(-u;q^2)\, e(-\frac{x^2}{t^2}u;q^2) = e(-(\frac{x^2}{t^2}+1)u;q^2), \tag{75}$$

in the algebra $\mathbf{C}_{q^2}\big[[{x'}^2\gamma, \gamma]\big]$.

Now using the definition of the q-Jackson integral (see [8]) and the fact that for any $k \in \mathbf{Z}$ and any formal variable ω,

$$A_{x'}(\omega\, q^{2k}; q^2) = q^k A_{x'}(\omega; q^2),$$

we obtain

$$p_t(x) = \frac{q}{c} \frac{(1-q^2)}{A_{x'}(\gamma;q^2)A_{x'}(\frac{1}{q(1+q)^2\gamma};q^2)t} \sum_{k=-\infty}^{+\infty} \frac{q^{2k}\gamma}{\Big(-(1-q^2)(\frac{x^2}{t^2}+1)q^{2k}\gamma;q^2\Big)_\infty}.$$

The result follows then by use of the well-known Ramanujan identity [9]. ■

LEMMA 2. *Let*

$$c_0 = \int_0^{x'.\infty} \frac{1}{1+x^2}\, d_qx < +\infty. \tag{76}$$

Then the solution $g(\alpha;q)$ of the following q-problem

$$(S_1)\begin{cases} D^2_{q.\alpha}\, g(\alpha;q) &= g(\alpha q;q), \quad (E_1) \\ \lim\limits_{n\longrightarrow+\infty} g(\alpha q^n;q) &= c_0, \quad (s_1) \\ \lim\limits_{n\longrightarrow+\infty} g(\alpha q^{-n};q) &= 0, \quad (s_2) \end{cases} \tag{77}$$

is given by

$$g(\alpha;q) = c_0 E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\alpha), \tag{78}$$

where the function $E_q^{(\frac{1}{2})}$ *is defined by (see [2])*

$$E_q^{(\frac{1}{2})}(x) = \sum_{k=0}^{+\infty} \frac{q^{k^2/4}}{(q;q)_k} x^k. \tag{79}$$

P r o o f. We shall proceed by using the standard method given in [11].

Let $f_1(\alpha) = E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\alpha)$, and $f_2(\alpha) = E_q^{(\frac{1}{2})}(q^{-1/2}(1-q)\alpha)$. First we verify that f_1 and f_2 are solutions of the q-difference equation (E_1).

Using the properties of the q-exponential function $E_q^{(\frac{1}{2})}$ (see [2]), we show then that the q-Wronskian $W_q(f_1, f_2)$ is not identically zero. Thus any solution g of the q-difference equation (E_1) can be written in the form

$$g(\alpha;q) = p_1(\alpha)f_1(\alpha) + p_2(\alpha)f_2(\alpha), \tag{80}$$

where p_1 and p_2 are two q-periodic functions (see [11]). Moreover we have the following limits:

$$\lim_{n\longrightarrow+\infty} E_q^{(\frac{1}{2})}(q^{-1/2}(1-q)\alpha q^{-n}) = +\infty, \tag{81}$$

$$\lim_{n\longrightarrow+\infty} E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\alpha q^{-n}) = 0, \tag{82}$$

where the last statement (82) can be derived from the relation between the q-exponential function $E_q^{(\frac{1}{2})}$and the q-hypergeometric function ${}_1\Phi_1$ and its properties (see [3, 10]).

By using (81), (82) and the initial condition (s_2), we obtain that $g(\alpha;q) = p_1(\alpha)E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\alpha)$. So to finish the proof it suffices to use the other initial condition (s_1), and the limit $\lim_{n\longrightarrow+\infty} E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\alpha q^n) = 1$. ■

PROPOSITION 14. *The x' q-cosine Fourier transform of the q-Poisson kernel p_t is given, for $\lambda \in \boldsymbol{R}_{q,(x')^{-1},+}$ by*

$$\mathcal{F}_{q.x'}(p_t)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} c_0 K_{x',\gamma}\, E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\lambda t). \tag{83}$$

P r o o f. By using the definition (55) and replacing p_t by its expansion (73), we have

$$\mathcal{F}_{q.x'}(p_t)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} K_{x',\gamma} \int_0^{x'.\infty} \frac{t}{x^2+t^2} \cos(\lambda x; q^2)\, d_q x. \tag{84}$$

The change of variables $\widetilde{x} = \frac{x}{t}$ gives us

$$\mathcal{F}_{q.x'}(p_t)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} K_{x',\gamma} \int_0^{x'.\infty} \frac{1}{1+\widetilde{x}^2} \cos(\lambda t\widetilde{x}; q^2)\, d_q\widetilde{x}. \tag{85}$$

It is not difficult to show that the function

$$\lambda t \longrightarrow \int_0^{x'.\infty} \frac{1}{1+x^2} \cos(\lambda t x; q^2)\, d_q x,$$

verifies the q-system (S_1). So the result follows from Lemma 2. ∎

PROPOSITION 15. *For $f \in L_q^p(\boldsymbol{R}_{q,x',+})$, $p \geq 1, x \in \boldsymbol{R}_{q,x'}$ and $t \in \boldsymbol{R}_{q,+}$ we have*

1. *The function $P^t f(x)$ has the q-integral representation*

$$\begin{aligned} P^t f(x) &= \frac{(1+q^{-1})}{\Gamma_{q^2}^2(1/2)} c_0 K_{x',\gamma} \\ &\times \int_0^{(x')^{-1}.\infty} E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)\lambda t) \mathcal{F}_{q.x'}(f)(\lambda) \cos(\lambda x; q^2) d_q\lambda. \end{aligned} \tag{86}$$

2. *The function $P^t f(x)$ is an even function in x and satisfies the q-difference equation*

$$(\Delta_{q.x} + \Delta_{q.t}) P^t f(x) = 0. \tag{87}$$

P r o o f.

1. Applying the q-Fourier transform (55) to both sides of the formula (71), we get $\mathcal{F}_{q.x'}(P^t f)(\lambda) = \mathcal{F}_{q.x'}(p_t)(\lambda)\, \mathcal{F}_{q.x'}(f)(\lambda)$. So the result follows from (83) and the q-Fourier inversion formula (56).

2. Since the q-derivative operators $\Delta_{q.x}$ and $\Delta_{q.t}$ permute with the q-integral sign in (86), we can deduce the result.

∎

LEMMA 3. *For $t \in \boldsymbol{R}_{q,+}$, there exists a constant $M_2 > 0$, such that*

$$\|t D_{q.t}(p_t)\|_{1.q.x'} \leq M_2. \tag{88}$$

P r o o f. By the change of variables $\widetilde{u} = \frac{q(1+q)^2}{t^2}u$ in formula (69) and by use of the following relation (see [13])

$$\int_0^{A.\infty} \frac{1}{x^2} f(\frac{1}{x})\, d_q x = \int_0^{A^{-1}.\infty} f(x)\, d_q x, \quad A \in \mathbf{C}, \tag{89}$$

we obtain the expansion of $p_t(x)$, which we apply the q-derivative $D_{q.t}$ and after simple computation, we get

$$D_{q.t}(p_t(x)) = \frac{1}{c(1+q)^2(1-q)} \frac{1}{t} \int_0^{\frac{t^2}{q(1+q)^2}\gamma^{-1}.\infty} \frac{e(-\frac{t^2}{q(1+q)^2 u}; q^2)}{A_{x'}(\frac{t^2}{q(1+q)^2 u}; q^2)} \times \frac{t^2}{u^2} \left\{G_{x'}(x, u; q^2) - G_{x'}(x, q^2 u; q^2)\right\} d_{q^2} u.$$

On the other hand, we have $\|tD_{q.t}(p_t)\|_{1.q.x'} = \int_0^{x'.\infty} t|D_{q.t}(p_t)(x)|d_q x$. Since $\gamma^{-1}x' = qx'\gamma^{-1}$, then the exchange of the q-integral signs in the above formula produces a factor q (see [8]) and gives us

$$\|tD_{q.t}(p_t)\|_{1.q.x'} \le \frac{2\|G_{x'}(.,t;q^2)\|_{1.q.x'}}{q^{-1}(1+q)^2(1-q)c} \int_0^{\frac{t^2}{q(1+q)^2}\gamma^{-1}.\infty} \frac{t^2}{u^2} \frac{e(-\frac{t^2}{q(1+q)^2 u}; q^2)}{A_{x'}(\frac{t^2}{q(1+q)^2 u}; q^2)} d_{q^2} u.$$

Next, we use formula (89) and after simple computation we obtain the result with $M_2 = \frac{2q^2\,\Gamma_{q^2}(1/2)}{(1-q)(1+q^{-1})^{1/2}}$. ∎

LEMMA 4. *If $f \in S_{*.q}(\boldsymbol{R}_{q,x'})$, then the function*

$$u(x,t) = P^t f(x) \tag{90}$$

verifies for all $x \in \boldsymbol{R}_{q,x',+}$ and all $t \in \boldsymbol{R}_{q,+}$ the following estimation

$$|q(D_{q.x}u(x,t))(q^{-1}x)| \le \|\Delta_q f\|_{1.q.x'}. \tag{91}$$

P r o o f. By a q-integration by parts and taking account of the fact that the function $x \longrightarrow u(x,t)$ is even, we get:

$$|\int_0^x \Delta_{q.y} u(y,t) d_q y| = |q(D_{q.x}u(x,t))(q^{-1}x)| < +\infty. \tag{92}$$

Replacing $u(y,t)$ by (71), to get $|\int_0^x \Delta_{q.y} u(y,t) d_q y|$ which is finite by (92). Then by the exchange of the q-integral signs, we obtain

$$\int_0^x \Delta_{q.y} u(y,t) d_q y = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{x'.\infty} p_t(z) \{ \int_0^x \Delta_{q.y} \mathcal{T}_{y,q} f(z) d_q y \} d_q z.$$

Using the fact that $\Delta_{q.y}\mathcal{T}_{z,q}f(y) = \mathcal{T}_{z,q}(\Delta_{q.y}f(y))$, and since the q-Jackson integral is invariant under the q-translation, then we can deduce the result from (70). ∎

LEMMA 5. *For $f \in S_{*.q}(\boldsymbol{R}_{q,x'}), x \in \boldsymbol{R}_{q,x',+}, t \in \boldsymbol{R}_{q,+}$ and $n = 1, 2$, there exists a constant $K_2 > 0$, such that*

$$|D^n_{q.x}u(x,t)| + |D^n_{q.t}u(x,t)| \leq K_2. \tag{93}$$

P r o o f. Applying the q-derivative $D^n_{q.x}$, $n = 1, 2$ to the formula (86). So any q-derivative $D^n_{q.x}u(x,t)$ can be written in a q-series form, and by the majorization (48). We find that, for $n = 1, 2$, $|D^n_{q.x}u(x,t)|$ is bounded by

$$\widehat{K} \sum_{k=-\infty}^{+\infty} |E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)(x')^{-1}q^k t)\mathcal{F}_{q.x'}(f)((x')^{-1}q^k)((x')^{-1}q^k)^{n+1}|,$$

where $\quad \widehat{K} = (1-q)\dfrac{q^{-1}(1+q^{-1})}{(q;q^2)^2_\infty \Gamma^2_{q^2}(1/2)}|c_0 K_{x',\gamma}|.$

In same manner, for $n = 1, 2$, we have that $|D^n_{q.t}u(x,t)|$ is majorized by

$$\widehat{K} \sum_{k=-\infty}^{+\infty} |E_q^{(\frac{1}{2})}(-q^{\frac{n-1}{2}}(1-q)(x')^{-1}q^k t)\mathcal{F}_{q.x'}(f)((x')^{-1}q^k)((x')^{-1}q^k)^{n+1}|.$$

Therefore, it suffices to prove that

$$\sum_{k=-\infty}^{+\infty} |E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)(x')^{-1}q^k t)\mathcal{F}_{q.x'}(f)((x')^{-1}q^k)((x')^{-1}q^k)^{n+1}| \text{ is finite.}$$

To this end, we need to use the limits $\lim\limits_{k \longrightarrow -\infty} E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)(x')^{-1}q^k t) = 0$, and $\lim\limits_{k \longrightarrow +\infty} E_q^{(\frac{1}{2})}(-q^{-1/2}(1-q)(x')^{-1}q^k t) = 1$, and the fact that $\mathcal{F}_{q.x'}(f)$ belongs to $S_{*.q}(\mathbf{R}_{q,(x')^{-1}})$. ∎

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