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EXISTENCE RESULTS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY AND APPLICATIONS TO CONTROL THEORY

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Abstract

In this paper we investigate the existence of solutions for fractional functional differential inclusions with infinite delay. In the last section we present an application of our main results in control theory.

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1. Introduction

This paper is concerned with the existence of solutions, for initial value problems (IVP for short), for fractional functional and neutral functional differential inclusions with infinite delay. In Section 3, we consider the initial value problem for fractional functional differential inclusions,

$$D^{\alpha}y(t) \in F(t, y_t), \ a.e. \ \ t \in J := [0, b], \quad 0 < \alpha < 1, \tag{1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2}$$

where D^{α} is the standard Riemman-Liouville fractional derivative, $F : J \times B \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact, convex values ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $\phi \in B, \phi(0) = 0$, and B is called a *phase space* that will be defined later. Section 4 is devoted to IVP's for fractional neutral functional differential inclusions,

$$D^{\alpha}[y(t) - g(t, y_t)] \in F(t, y_t),\ t \in J, \quad 0 < \alpha < 1, \tag{3}$$

$$y(t) = \phi(t),\ t \in (-\infty, 0], \tag{4}$$

where F, ϕ are as in problem (1)-(2), and $g : J \times B \to \mathbb{R}$, with $g(0, \phi) = 0$, is a given function.

In the literature devoted to equations with finite delay, the state space is usually the space of all continuous function on $[-r, 0],\ r > 0$ and $\alpha = 1$, endowed with the uniform norm topology; see the book of Hale and Lunel [29]. When the delay is infinite, the selection of the state B (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [28] (see also Kappel and Schappacher [34] and Schumacher [53]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [31]. For the case where ($\alpha = 1$), an extensive theory has been developed for the problems (1)-(2) and (3)-(4). We refer to Hale and Kato [28], Corduneanu and Lakshmikantham [12], Hino *et al* [31], Lakshmikantham *et al* [42] and Shin [54].

Differential equations of fractional order have recently proved valuable tools in the modeling of many physical phenomena [14, 25, 27, 43, 44]. There has been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [35], Miller and Ross [47], Podlubny [50], Samko *et al* [52], and the papers of Bai and Lu [6], Diethelm *et al* [14, 16, 17], El-Raheem [19], El-Sayed [20, 21, 22], El-Sayed and Ibrahim [23], Kilbas and Trujillo [36], Mainardi [43], Momani and Hadid [45], Momani *et al* [46], Nakhushev [48], Podlubny *et al* [51], and Yu and Gao [57, 58].

Very recently, some basic theory for initial value problems for fractional differential equations involving the Riemann-Liouville differential operator was discussed by Benchohra *et al* [8], Lakshmikantham [38], and Lakshmikantham and Vastala [39, 40, 41]. El-Sayed and Ibrahim [23] initiated the study of fractional multivalued differential inclusions. In the case where $\alpha \in (1, 2]$, existence results for fractional boundary value problem and relaxation theorem, were studied by Ouahab [49].

We shall present some existence results for the problems (1)-(2) and (3)-(4), and an application to control theory will considered. Our approach here is based on the nonlinear alternative of Leray-Schauder type for multivalued operators, see [18]. The results of this paper extend to the multivalued setting some very recently considered results by Benchohra *et al* [8] and others in the above cited literature.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 0 \leq t \leq b\},$$

and we let $L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \longrightarrow \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). The map G is *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Finally, we say that G is *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). We say that G has a *fixed point*, if there exists $x \in X$ such that $x \in G(x)$.

In what follows, $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_c(X) = \{Y \in \mathcal{P}(X) : Y$ convex$\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ closed$\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y$ bounded$\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ compact $\}$, $\mathcal{P}_{c,cp}(X) = \mathcal{P}_c(X) \cap \mathcal{P}_{cp}(X)$. A multi-valued map $G : J \longrightarrow \mathcal{P}_{cp}(R)$ is said to be *measurable*, if for each $x \in R$ the function $Y : J \longrightarrow R$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable.

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider the mapping $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow R^+ \cup \{\infty\}$, called *Hausdorff distance*, defined by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,\,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized complete metric space (see [37]). In the case where the multivalued maps have compact values, the Vietorice topology coincides with the Hausdorff topology.

For more details on the multi-valued maps, see the books of Aubin and Cellina [3], Aubin and Frankowska [4], Deimling [15], Górniewicz [26], and Hu and Papageorgiou [32].

Definition 2.1. The multivalued map $F : J \times B \longrightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory, if:

(i) $t \longmapsto F(t, y)$ is measurable for each $y \in B$;

(ii) $y \longmapsto F(t, y)$ is upper semi-continuous for almost all $t \in J$.

Definition 2.2. The fractional primitive of order $\alpha > 0$ of a function $h : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

provided the right side exists pointwise on $\mathbb{R}^+$, where Γ is the Gamma function.

When $h \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$, then $I^\alpha h \in C^0(\mathbb{R}_0^+)$ and moreover $I^\alpha h(0) = 0$.

Definition 2.3. The fractional derivative of order $\alpha > 0$ of a continuous function $h : (0, b] \to \mathbb{R}$ is given by

$$\frac{d^\alpha h(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds = \frac{d}{dt} I_0^{1-\alpha} h(t).$$

In this paper, we assume that the state space $(B, \|\cdot\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which was introduced at first by Hale and Kato in [28].

(A) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in B$, then for every $t \in [0, b]$ the following conditions hold:

(i) y_t is in B;

(ii) $\|y_t\|_B \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_B$,

where $H \ge 0$ is a constant, $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a B-valued continuous function on $[0, \infty)$.

(A-2) The space B is complete.

3. Fractional functional differential inclusions

The following space will be considered hereafter,

$$\Omega = \{y : (-\infty, b] : y \in B \cap C([0, b], \mathbb{R})\}.$$

Let us start by defining what we mean by a solution of problem.

Definition 3.1. A function $y \in \Omega$, for which its α-fractional derivative exists almost everywhere on $[0, b]$, is said to be a solution of (1)–(2), if there exists $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y_t)$ for a.e. $t \in J$, such that y satisfies the fractional differential equation $D^\alpha y(t) = v(t)$ a.e. on J, and condition (2).

For any $y \in C([-\infty, b], \mathbb{R})$, we define the set

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}) : \ v(t) \in F(t, y_t) \text{ for a.e. } \ t \in J\}.$$

This is known as the set of *selection functions*. For the existence results on the problem (1)-(2) we need the following auxiliary lemma.

Lemma 3.2. (see [13]) *Let $0 < \alpha < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds,$$

if and only if, y is a solution of the initial value problem for the fractional differential equation

$$D^\alpha y(t) = h(t), \ \ t \in (0, b],$$

$$y(0) = 0.$$

Essential for the main results of this section, we state a generalization of Gronwall's lemma for singular kernels ([30], Lemma 7.1.1).

LEMMA 3.3. *Let $v : [0, b] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b]$, and suppose there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha}ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \le w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^\alpha}ds,$$

for every $t \in [0, b]$.

THEOREM 3.4. *Assume the following hypotheses hold:*

(H1) $F : J \times B \longrightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map;*

(H2) there exist $p, q \in C(J, \mathbb{R})$ such that

$$\|F(t,u)\|_P = \sup\{|v| : v \in F(t,u)\} \le p(t) + q(t)\|u\|_B$$

for $t \in J$ and each $u \in B$;

(H3) there exists $l \in L^1(J, \mathbb{R})$, with $I^\alpha l < \infty$ such that

$$H_d(F(t,u), F(t,\overline{u})) \le l(t)\|u - \overline{u}\|_B \quad \text{for every } u, \overline{u} \in B.$$

Then the IVP (1)–(2) has at least one solution on $(-\infty, b]$.

P r o o f. Transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator $N : \Omega \longrightarrow \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{ h \in \Omega : h(t) = \left\{ \begin{array}{ll} \phi(t), & t \in (-\infty, 0] \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, & t \in [0,b],\ f \in S_{F,y} \end{array} \right. \right\}.$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \left\{ \begin{array}{ll} 0, & \text{if } t \in [0,b], \\ \phi(t), & \text{if } t \in (-\infty, 0]. \end{array} \right.$$

Then $x_0 = \phi$. For each $z \in C([0, b], \mathbb{R})$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0,b], \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t \leq b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq b$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad f \in S_{F, \bar{z}_s + x_s}. \tag{5}$$

Set

$$C_0 = \{z \in C([0,b], \mathbb{R}) : z_0 = 0\}.$$

Let $\| \cdot \|_0$ be the norm in C_0 defined by

$$\|z\|_0 = \|z_0\|_B + \sup\{|z(t)| : 0 \leq t \leq b\} = \sup\{|z(t)| : 0 \leq t \leq b\}, \ z \in C_0.$$

Let the operator $P : C_0 \to \mathcal{P}(C_0)$ be defined by

$$Pz = \left\{ h \in C_0 : \ h(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & t \in [0,b] \end{cases} \right\},$$

$$f \in S_{F, \bar{z}_s + x_s} = \{v \in L^1([0,b], \mathbb{R}) : v(t) \in F(t, \bar{z}_t + x_t)\}.$$

Obviously, that the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point.

Step 1: *$P(z)$ is convex for each $y \in C_0$.*

Indeed, if $h_1, \ h_2 \in P(z)$, then there exist $g_1, g_2 \in S_{F, \bar{z}_s + x_s}$ such that, for each $t \in J$, we have

$$h_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_i(s) ds, \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then for each $t \in J$ we have

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [dg_1(s) + (1-d)g_2(s)] ds.$$

Since $S_{F, \bar{z}_s + x_s}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in P(z).$$

Step 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $z \in \mathcal{B}_{\bar{q}} = \{y \in C_0 : \|z\|_0 \leq \bar{q}\}$, one has $\|P(z)\|_{\mathcal{P}} \leq \ell$.

Let $z \in \mathcal{B}_{\bar{q}}$ and $h \in P(z)$. Then there exists $f \in S_{F,\bar{z}_s+x_s}$ such that, for each $t \in J$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Then for each $t \in J$,

$$\begin{aligned} h(t) &\leq \frac{1}{\Gamma(\alpha} \int_0^b (b-s)^{\alpha-1} |f(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} [p(s) + q(s)\|\bar{z}_s + x_s\|_B]ds \\ &\leq \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)} q_* =: \ell\,, \end{aligned}$$

where

$$\|\bar{z}_s + x_s\|_B \leq \|\bar{z}_s\|_B + \|x_s\|_B \leq K_b \bar{q} + M_b \|\phi\|_B := q_*,$$

and

$$M_b = \sup\{|M(t)| : t \in [0,b]\}.$$

Step 3: *P maps bounded sets into equicontinuous sets of C_0.*

Let $t_1, t_2 \in [0,b],\ t_1 < t_2$ and let $\mathcal{B}_{\bar{q}}$ be a bounded set of C_0 as in Step 2. Let $z \in \mathcal{B}_{\bar{q}}$ and $h \in P(z)$. Then there exists $f \in S_{F,\bar{z}+x}$ such that, for each $t \in [0,b]$, we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} f(s)ds.$$

Then

$$\begin{aligned} |h(t_2) - h(t_1)| &= \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) f(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)ds \Big| \\ &\leq \frac{\|p\|_\infty + \|q\|_\infty q^*}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \end{aligned}$$

$$+ \frac{\|p\|_\infty + \|q\|_\infty q^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \leq \frac{\|p\|_\infty + \|q\|_\infty q^*}{\Gamma(\alpha+1)} [(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha]$$

$$+ \frac{\|p\|_\infty + \|q\|_\infty q^*}{\Gamma(\alpha+1)} (t_2-t_1)^\alpha \leq \frac{2[\|p\|_\infty + \|q\|_\infty q^*]}{\Gamma(\alpha+1)} (t_2-t_1)^\alpha.$$

As $t_1 \longrightarrow t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

As a consequence of Step 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $P : C_0 \longrightarrow \mathcal{P}_{cp}(C_0)$ is completely continuous.

Step 4: *P has a closed graph.*

Let $z_n \longrightarrow z_*$, $h_n \in P(z_n)$ and $h_n \longrightarrow h_*$. We will prove that $h_* \in P(z_*)$. Now $h_n \in P(z_n)$ implies there exists $f_n \in S_{F,z_n+x}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s)ds.$$

We must prove that there exists $f_* \in S_{F,\bar{z}_*+x}$ such that for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s)ds.$$

Since $F(t,\cdot)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exists $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$f_n(t) \in F(t, (\bar{z}_n)_t + x_t) \subset F(t, (\bar{z}_*)_t) + \varepsilon B(0,1), \text{ a.e. } t \in [0,b].$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $f_{n_m}(\cdot)$ such that

$$f_{n_m}(\cdot) \to f_*(\cdot) \text{ as } m \to \infty$$

and

$$f_*(t) \in F(t, (\bar{z}_*)_t + x_t), \text{ a.e. } t \in [0,b].$$

For every $w \in F(t, \bar{z}_t + x_t)$, we have

$$|f_{n_m}(t) - f_*(t)| \leq |f_{n_m}(t) - w| + |w - f_*(t)|.$$

Then

$$|f_{n_m}(t) - f_*(t)| \leq d(f_{n_m}(t), F(t, (\bar{z}_*)_t + x_t).$$

By an analogous relation, obtained by interchanging the roles of f_{n_m} and f_*, it follows that

$$|f_{n_m}(t) - f_*(t)| \leq H_d(F(t, (\bar{z}_n)_t + x_t), F(t, (\bar{z}_*)_t + x_t) \leq l(t)\|(\bar{z}_n)_t - (\bar{z}_*)_t\|_B.$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} |f_{n_m}(s) - f_*(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} l(s)ds \|z_{n_m} - z_*\|_0. \end{aligned}$$

Hence,

$$\|h_{n_m} - h_*\|_0 \quad \le \quad I^\alpha l\|z_{n_m} - z_*\|_0 \to 0 \text{ as } m \to \infty.$$

Step 5: A priori bounds. *We now show there exists an open set $U \subseteq C_0$ with $z \in \lambda P(z)$, for $\lambda \in (0,1)$ and $z \in \partial U$.*

Let $z \in C_0$ and $z \in \lambda P(z)$ for some $0 < \lambda < 1$. Thus, there exists $f \in S_{F,\bar{z}+x}$ such that for each $t \in [0,b]$, we have

$$z(t) = \lambda\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds\Big]. \tag{6}$$

This implies, by (H2),

$$|z(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s)\|\bar{z}_s + x_s\|_B ds + \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}. \tag{7}$$

But

$$\begin{aligned} \|\bar{z}_s + x_s\|_B &\le \|\bar{z}_s\|_B + \|x_s\|_B \\ &\le K(t)\sup\{|z(s)| : 0 \le s \le t\} + M(t)\|z_0\|_B \\ &\quad + K(t)\sup\{|x(s)| : 0 \le s \le t\} + M(t)\|x_0\|_B \\ &\le K_b \sup\{|z(s)| : 0 \le s \le t\} + M_b\|\phi\|_B. \end{aligned} \tag{8}$$

If we denote by $w(t)$ the right hand side of (8), then we have

$$\|\bar{z}_s + x_s\|_B \le w(t).$$

Therefore,

$$|z(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)ds + \frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}.$$

Using the above inequality in the definition of w, we have that

$$w(t) \le M_b\|\phi\|_B + K_b\frac{b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{K_b\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)ds.$$

Lemma 3.3 implies

$$|w(t)| \le M_b\|\phi\|_B + \frac{K_b b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + K(\alpha)\frac{K_b\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{R}ds,$$

where

$$\bar{R} = M_b\|\phi\|_B + \frac{K_b b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}.$$

Hence

$$\|w\|_\infty \leq \bar{R} + \frac{RK(\alpha)b^\alpha K_b}{\Gamma(\alpha+1)} := \widetilde{M}.$$

Then

$$\|z\|_\infty \quad \leq \quad \widetilde{M}\|I^\alpha q\|_\infty + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} := M^*.$$

Setting

$$U = \{z \in C_0 : \|z\|_0 < M^* + 1\},$$

$P : \overline{U} \to \mathcal{P}(C_0)$ is continuous and completely continuous. From the choice of U, there is no $z \in \partial U$ such that $z \in \lambda P(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [18], we deduce that P has a fixed point z in U. ∎

4. Neutral fractional functional differential inclusions

We start by defining what we mean by a solution of IVP (3)–(4).

DEFINITION 4.1. A function $y \in \Omega$, for which its α-fractional derivative exists almost everywhere on $[0,b]$, is said to be a solution of (3)–(4), if $y(t) = \phi(t),\ \ t \in (-\infty, 0]$, and there exists a function $v \in L^1(J,R)$ such that $v(t) \in F(t, y_t)$ a.e. $t \in J$ satisfies the equation

$$D^\alpha[y(t) - g(t, y_t)] = v(t), \quad a.e \ \ t \in [0,b], \quad 0 < \alpha < 1.$$

Our second existence result concerns the IVP (3)–(4) and is based also on the nonlinear alternative of Leray-Schauder.

THEOREM 4.2. *Assume (H1)-(H3), and that the following condition holds:*

(H4) the function g is continuous and completely continuous and for any bounded set $\mathcal{B}$ in $B \cap C([0,b],\mathbb{R})$, the set $\{t \to g(t,y_t) : y \in \mathcal{B}\}$ is equicontinuous in $C([0,b],\mathbb{R})$, and there exist constants $0 \leq d_1 < 1,\ d_2 \geq 0$ such that

$$|g(t,u)| \leq d_1\|u\|_B + d_2, \ \ t \in [0,b], \ u \in B.$$

Then the IVP (3)–(4) has at least one solution on $(-\infty, b]$.

P r o o f. Consider the operator $N_1 : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N_1(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0] \\ g(t, y_t) + & \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, & \text{if } t \in [0, b],\ g \in S_{F,y} \end{cases} \right\}.$$

In analogy to Theorem 3.4, we consider the operator $P_1 : C_0 \to \mathcal{P}(C_0)$ defined by

$$P_1(z) = \left\{ z \in C_0 : h(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ -g(t, \overline{z}_t + x_t) + & \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, & t \in [0, b], g \in S_{F,\bar{z}+x} \end{cases} \right\}.$$

We shall show that the operator P_1 is continuous and completely continuous.

Using (H4), it suffices to show that the operator $P_2 : C_0 \to \mathcal{P}(C_0)$ defined by

$$P_2(z) = \left\{ z \in C_0 : h(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, & t \in [0, b], g \in S_{F,\bar{z}+x} \end{cases} \right\}$$

is completely continuous. As in Theorem 3.4 we can show that P_2 is completely continuous. *We now show there exists an open set $U \subseteq C_0$ with $z \in \lambda P_1(z)$ for $\lambda \in (0, 1)$ and $z \in \partial U$.*

Let $z \in C_0$ and $z \in \lambda P_1(z)$ for some $0 < \lambda < 1$, $t \in (0, b]$,

$$z(t) = \lambda \left[g(t, \overline{z}_t + x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds \right].$$

Then

$$\begin{aligned} |z(t)| \quad &\leq \quad d_1 \|\bar{z}_t + x_t\|_B + d_2 + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) \|\bar{z}_s + x_s\|_B ds, \quad t \in (0, b]. \end{aligned} \tag{9}$$

Thus

$$\begin{aligned} w(t) \quad &\leq \quad \frac{1}{1 - K_b d_1} \Big[2K_b d_2 + \frac{K_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \\ &\quad + \frac{K_b \|q^*\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s) ds \Big], \quad t \in (0, b], \end{aligned}$$

and consequently, by Lemma 3.3,

$$\|w\|_\infty \quad \le R_1 + \frac{b^\alpha R_1 K_b \|q^*\|_\infty}{(1-K_b d_1)\Gamma(\alpha+1)} := L,$$

where

$$\|q^*\|_\infty = \frac{\|q\|_\infty}{1-K_b d_1} \text{ and } R_1 = \frac{1}{1-K_b d_1}\Big[2K_b d_2 + \frac{K_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)}\Big].$$

Then

$$\|z\|_\infty \le d_1\|\phi\|_{\mathcal{B}} + 2d_2 + Ld_1 + \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + L\|I^\alpha q\|_\infty := L^*.$$

Set

$$U_1 = \{z \in C_0 : \|z\|_0 < L^* + 1\}.$$

From the choice of U_1 there is no $z \in \partial U_1$ such that $z \in \lambda P_1(z)$ for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, [18], we deduce that P_1 has a fixed point z in U_1. Then N_1 has a fixed point, which is a solution of the IVP (3)–(4). ■

5. Example

As an application of the main results, we consider the fractional differential inclusions with infinite delay

$$D^\alpha y(t) \in F(t, y_t), \ \textit{a.e. } t \in J = [0,1], \quad 0 < \alpha \le 1, \tag{10}$$

$$y(t) = \phi(t), \ t \in (-\infty, 0], \ \phi(0) = 0. \tag{11}$$

Let γ be a positive real constant and

$$B_\gamma = \{y \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta} y(\theta), \text{ exists in } \mathbb{R}\}.$$

The norm of B_γ is given by

$$\|y\|_\gamma = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}|y(\theta)|.$$

Let $y : (-\infty, 1] \to \mathbb{R}$, such that $y_0 \in B_\gamma$. Then

$$\lim_{\theta\to-\infty} e^{\gamma\theta} y_t(\theta) = \lim_{\theta\to-\infty} e^{\gamma\theta} y(t+\theta)$$
$$= \lim_{\theta\to-\infty} e^{\gamma(\theta-t)} y(\theta) = e^{-\gamma t} \lim_{\theta\to-\infty} e^{\gamma\theta} y_0(\theta) < \infty.$$

Hence $y_t \in B_\gamma$. Finally we prove that

$$\|y_t\| \le K(t)\sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_\gamma,$$

where $K = M = 1$ and $H = 1$. If $\theta + t \leq 0$, we get

$$|y_t(\theta)| \leq \sup\{|y(s)| : -\infty < s \leq 0\}.$$

For $t + \theta \geq 0$, then we have

$$|y_t(\theta)| \leq \sup\{|y(s)| : 0 < s \leq t\}.$$

Thus for all $t + \theta \in \mathbb{R}$, we get

$$|y_t(\theta)| \leq \sup\{|y(s)| : -\infty < s \leq 0\} + \sup\{|y(s)| : 0 \leq s \leq t\}.$$

Then

$$\|y_t\|_\gamma \leq \|y_0\|_\gamma + \sup\{|y(s)| : 0 \leq s \leq t\}.$$

$(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ is a phase space.

Set

$$F(t, y_t) = \{v \in R : f_1(t, y_t) \leq v \leq f_2(t, y_t)\},$$

where $f_1, f_2 : [0,1] \times B_\gamma \to \mathbb{R}$. We assume that for each $t \in [0,1]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e, the set $\{y \in B_\gamma : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [0,1]$, $f_2(t, \cdot)$ is upper semi-continuous (i.e. the set $\{y \in B_\gamma : f_1(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p, q \in C([0,1], R)$ such that

$$\max(|f_1(t, y_t)|, \ |f_2(t, y_t)|) \leq p(t)\|y_t\|_\gamma + q(t), \quad t \in [0,1], \text{ and all } y_t \in B_\gamma.$$

It is clear that F is compact and convex valued, and is upper semi-continuous (see [15]). Since all the conditions of Theorem 3.4 are satisfied, problem (10)-(11) has at least one solution y on $[0,1]$.

6. Application to control theory

Many problems subject to controllability may be described by nonlinear differential equations of the form

$$x'(t) = f(t, x(t), u(t)), \quad 0 \leq t \leq b, \ u \in U, \tag{12}$$

$$x(0) = 0. \tag{13}$$

In this section we consider the fractional control problem

$$D^\alpha x(t) = f(t, x_t, u(t)), \quad 0 \leq t \leq b, \ u \in U, \ 0 < \alpha < 1, \tag{14}$$

$$x(t) = \phi(t),\ t \in (-\infty, 0],\ \ \phi(0) = 0, \tag{15}$$

with constrained control u. Here $f : J \times B \times \mathbb{R} \to \mathbb{R}$ is a single-valued function measurable in t and continuous in x, u. The time-varying set of constraints function $U : [0, b] \to \mathcal{P}_{cp}(\mathbb{R})$ is a measurable multi-valued function. By $u \in U$, we mean $u(t) \in U(t)$, for a.e. $t \in J$. Problem (14)-(15) is solved if there is a control function u for which the problem admits a solution. If we define the multi-function

$$F(t, x_t) = \{f(t, x_t, u),\ u \in U\}, \tag{16}$$

then the solution set of the controlled problem (14)-(15) coincides with the set of solutions of (1)-(2) with right-hand side given by (16).

The controllability of differential equations and inclusions have been investigated by many authors (see for instance [1, 2, 5, 7, 9, 10] and the references therein). There are many inclusions applications mainly in optimal control, economics, etc. Indeed, the first motivation for the study of differential inclusions arose from the development of some studies in control theory. For more information about the relation between the differential inclusions and control theory, see for instance [4, 24, 55, 56] and the references therein.

Hereafter, we apply the existence results obtained in Section 3 to study the fractional differential inclusion,

$$D^\alpha x(t) \in F(t, x_t), \quad 0 \leq t \leq b,\ \ 0 < \alpha < 1, \tag{17}$$

$$x(t) = \phi(t),\ t \in (-\infty, 0],\ \phi(0) = 0, \tag{18}$$

with F given by (16).

We will need the following auxiliary results in order to prove our main controllability theorem.

Lemma 6.1. (see [11]) *Let $G : J \to X$ be a set-valued map with closed nonempty images, where X is a separable Banach space. Then the following statements are equivalent:*

(a) G is measurable.

(b) There exist measurable selections $g_n(t) \in G(t)$ such that for every $t \in J$, we have

$$G(t) = \overline{\bigcup_{n \geq 1} \{g_n(t)\}}.$$

While this result characterizes measurability, the following lemma is a measurable selection result (Filippov's Theorem).

Lemma 6.2. (see [4], Th. 8.2.10) *Consider a complete $\sigma-$finite measurable space (Ω, A, μ) (A is a $\sigma-$algebra and μ is a positive measure). Let X, Y be two complete separable metric spaces. Let $F : X \to \mathcal{P}(Y)$ be a measurable set-valued map with closed nonempty values and $g : \Omega \times Y \to Y$ a Carathéodory map. Then for every measurable map $h : \Omega \to Y$ satisfying*

$$h(\omega) \in g(\omega, F(\omega)) \text{ for almost all } \omega \in \Omega,$$

there exists a measurable selection $f(\omega) \in F(\omega)$ such that

$$h(\omega) = g(\omega, f(\omega)) \text{ for almost all } \omega \in \Omega.$$

Next, we state our main existence result.

Theorem 6.3. *Assume that U and f satisfy the following hypotheses:*

$(H5)$ *$U : J \to \mathcal{P}_{cv,\,cp}(\mathbb{R})$ is a measurable multi-function.*

$(H6)$ *The function f is linear in the third argument, i.e. there exists a Carathéodory function $f_1 : J \times B \to \mathbb{R}$ such that for a.e. $t \in J$,*

$$f(t, x, u) = u + f_1(t, x),\ \forall\,(x, u) \in B \times \mathbb{R},$$

and there exists a positive number M such that

$$\sup\{|f_1(t, 0)| : t \in [0, b]\} \le M.$$

$(H7)$ *There exists $k \in L^1(J, (0, +\infty))$, with $I^\alpha k < \infty$ such that*

$$|f(t,x,u) - f(t,y,u)| \le k(t)\|x - y\|_B, \text{ for a.e. } t \in J,\ \forall\, x, y \in B \text{ and } \forall\, u \in U.$$

Then the problem (17)-(18) has at least one solution.

P r o o f. The proof is given in several claims.

Claim 1: Clearly, the map $t \to F(t, \cdot)$ is a measurable multifunction. From $(H5)$ and $(H6)$, we have that $F(\cdot, \cdot) \in \mathcal{P}_{cv}(\mathbb{R})$. Using the compactness of U and the continuity of f, we can easily show that $F(\cdot, \cdot) \in \mathcal{P}_{cp}(\mathbb{R})$; then $F(\cdot, \cdot) \in \mathcal{P}_{cp,\,cv}(\mathbb{R})$.

Claim 2: The selection set of F is not empty. Since U is a measurable multifunction, from Theorem 6.1, there exists a sequence of measurable selections $u_n \in U$ such that for a.e. $t \in [0, b]$,

$$U(t) = \overline{\{u_n(t);\ \ t \in [0, b],\ n \geq 1\}}.$$

Set $v_n(t) = f(t, x_t, u_n(t))$. Then v_n is measurable because f is a continuous function. It is clear that $\overline{\{v_n(t),\ n \geq 1\}} \subseteq F(t, x_t)$. Conversely, let $f(t, x_t, u) \in F(t, x_t)$ for some $u \in U$. Then there exists a subsequence $(u_{n_m})_{m\in N} \subset U$ such that u_{n_m} converges to u as $m \to \infty$. By continuity of f, we conclude that $f(t, x_t, u_{n_m})$ converges to $f(t, x_t, u)$, hence

$$F(t, x_t) = \overline{\{v_n(t),\ n \geq 1\}},$$

thus proving our claim.

Claim 3: From $(H7)$ we have

$$H_d(F(t, x), F(t, y)) \leq k(t)\|x - y\|_B \ \text{ for every } x, y \in B.$$

Therefore all conditions of Theorem 3.4 are fulfilled and then the IVP (17)-(18) has at least one solution.

Claim 4: The solutions of (17)-(18) and those of the problem (14)-(15) defined on J coincide. Let x be solution of the problem (14)-(15), then there exists a single-valued selection function

$$g \in S_{F,x} := \{g \in L^1([0, b], \mathbb{R}),\ g(t) \in F(t, x_t),\ \text{a.e. } t \in [0, b]\}$$

such that

$$D^\alpha x(t) = g(t),\ \text{ a.e. } \ t \in [0, b],\ \text{ and } x(t) = \phi(t),\ t \in (-\infty, 0].$$

We shall show that there exists $u \in U$ such that

$$g(t) = f(t, x_t, u(t)),\ \text{ a.e. } \ \text{ in } J. \tag{19}$$

Define the function $\Psi(t, u) = f(t, x_t, u)$. Then Ψ is measurable in t and continuous on u. Moreover for almost every $t \in [0, b]$, $g(t) \in \Psi(t, U(t)) := f(t, x_t, U(t))$. Hence from Theorem 6.2, we deduce the existence of some $u \in U$, verifying (19). Conversely, let x be a function satisfying the control problem, i.e. for some $u \in U$, we have

$$D^\alpha x(t) = f(t, x_t, u),\ \ t \in J, \quad x(t) = \phi(t), t \in (-\infty, 0].$$

Then x is solution of (14)-(15), and the proof of the theorem is completed. ■

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