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CAPUTO DERIVATIVES IN VISCOELASTICITY: A NON-LINEAR FINITE-DEFORMATION THEORY FOR TISSUE

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Dedicated to Professor Michele Caputo
on the occasion of his 80th birthday

Abstract

The popular elastic law of Fung that describes the non-linear stress-strain behavior of soft biological tissues is extended into a viscoelastic material model that incorporates fractional derivatives in the sense of Caputo. This one-dimensional material model is then transformed into a three-dimensional constitutive model that is suitable for general analysis. The model is derived in a configuration that differs from the current, or spatial, configuration by a rigid-body rotation; it being the polar configuration. Mappings for the fractional-order operators of integration and differentiation between the polar and spatial configurations are presented as a theorem. These mappings are used in the construction of the proposed viscoelastic model.

Mathematics Subject Classification: 26A33, 74B20, 74D10, 74L15
Key Words and Phrases: hyper-elasticity, hypo-elasticity, viscoelasticity, soft biological tissue, three-dimensional material model, Caputo derivative, polar configuration, fractional polar derivative, fractional polar integral

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[a] This work began while ADF was still an employee of the Bio Science & Technology Branch at NASA's Glenn Research Center in Cleveland, Ohio.

1. Definitions

Liouville [37, §5, Eq. A] and Riemann[1] defined fractional-order integration as an analytic continuation of Cauchy's n-fold integral by replacing its factorial with the Gamma function, noting the identity $\Gamma(n) = (n-1)!$, $n \in I\!N$, thereby introducing

$$J^\alpha \mathbf{y}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \big(x - x'\big)^{\alpha-1} \mathbf{y}(x')\,\mathrm{d}x', \tag{1}$$

where $\alpha, x \in I\!R_+$, with J^α being called the *Riemann-Liouville integral operator of order* α, which obeys $J^0\mathbf{y}(x) = \mathbf{y}(x)$.

From this single definition for fractional integration, one can construct several definitions for fractional differentiation (cf., e.g., with Podlubny [41] and Samko et al. [44]). The special operator $D_\star^\alpha$ chosen for use herein requires the dependent variables, e.g., $\mathbf{y}$ and $\mathbf{z}$ below, to be continuous and $\lceil\alpha\rceil$-times differentiable in the independent variable x, and is defined by

$$D_\star^\alpha \mathbf{y}(x) = J^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil} \mathbf{y}(x), \tag{2}$$

with properties

$$D_\star^\alpha J^\alpha \mathbf{y}(x) = \mathbf{y}(x), \tag{3}$$

$$J^\alpha D_\star^\alpha \mathbf{y}(x) = \mathbf{y}(x) - \sum_{n=0}^{\lfloor\alpha\rfloor} \frac{x^n}{n!} D^n \mathbf{y}(0^+), \tag{4}$$

$$D_\star^\alpha (a\mathbf{y} + b\mathbf{z})(x) = a D_\star^\alpha \mathbf{y}(x) + b D_\star^\alpha \mathbf{z}(x), \tag{5}$$

and

$$D_\star^\alpha c = 0 \text{ for any constant } c, \tag{6}$$

wherein $a, b, c \in I\!R$; $\alpha \in I\!R_+$; and where D^n, $n \in I\!N$, denotes the classic differential operator. Whether or not we have the relation

$$D_\star^\alpha D_\star^\beta \mathbf{y}(x) = D_\star^\beta D_\star^\alpha \mathbf{y}(x) = D_\star^{\alpha+\beta} \mathbf{y}(x) \tag{7}$$

for $\alpha, \beta \in I\!R_+$ cannot be said directly. The answer to this question depends on the exact nature of the function $\mathbf{y}$, and on the values of α and β. Given that $x > 0$, we also have

$$\lim_{\alpha \to n-} D_\star^\alpha \mathbf{y}(x) = D^n \mathbf{y}(x) \tag{8}$$

[1] Riemann's pioneering work in the field of fractional calculus was done during his student years, but published posthumous—forty-four years after Liouville first published in the field [43].

in a pointwise sense, where $\alpha \to n-$ means α approaches n from below; however, this relation does not hold at $x = 0$ because, from its definition in Eq. (2), one always has

$$D_\star^\alpha \mathbf{y}(0) = \mathbf{0}, \tag{9}$$

whenever α is not an integer and $\mathbf{y}$ has $\lceil \alpha \rceil$ continuous derivatives.

It has become accepted practice to call $D_\star^\alpha$ the *Caputo*[2] *differential operator of order* α, since Caputo [8] was amongst the first to use this operator in applications, and to study some of its properties.

Remark 1. The chain rule and the Leibniz product rule, as they pertain to the Caputo derivative, do not reduce to simple forms like their classic analogs from integer calculus. For example, for the case where $0 < \alpha < 1$, the Leibniz product rule takes on the form

$$\begin{aligned} D_\star^\alpha(\mathbf{y} \times \mathbf{z})(x) &= \frac{\mathbf{y}(0^+)}{\Gamma(1-\alpha)} \times \frac{\mathbf{z}(x) - \mathbf{z}(0^+)}{x^\alpha} + D_\star^\alpha(\mathbf{y})(x) \times \mathbf{z}(x) \\ &\quad + \sum_{k=1}^{\infty} \binom{\alpha}{k} J^{k-\alpha}(\mathbf{y})(x) \times D^k(\mathbf{z})(x), \end{aligned} \tag{10}$$

wherein, unlike the Leibniz product rule for integer-order derivatives, the binomial coefficients

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}, \quad \text{with} \quad \binom{\alpha}{0} = 1, \tag{11}$$

$k \in I\!N$, do not become 0 whenever $k > \alpha$ because $\alpha \notin I\!N$, and as such, the binomial sum is now of infinite extent. An infinite binomial sum is also present in the chain rule for Caputo's derivative. These infinite sums make the chain rule and the Leibniz product rule impractical for use in most applications.

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[2]Actually, Liouville [37, §6, Eq. B] introduced this operator in his historic first paper on the topic. Still, nothing in Liouville's collection of works suggests that he ever saw any difference between $D_*^\alpha = J^{\lceil\alpha\rceil-\alpha}D^{\lceil\alpha\rceil}$ and $D^\alpha = D^{\lceil\alpha\rceil}J^{\lceil\alpha\rceil-\alpha}$, with the latter being his accepted definition (first formula on pg. 10 of [37]) that today is called the Riemann-Liouville differential operator of order α. Liouville freely interchanged the order of integration and differentiation, because the class of problems that he was interested in happened to be a class where such an interchange is legal, and he made only a few terse remarks about the general requirements on the class of functions for which his fractional calculus works (Lützen, private communication, 2001).

Rabotnov [42, p. 129] introduced this same differential operator into the Russian viscoelastic literature the year before Caputo's paper was published.

2. History

In the 1940's, Scott Blair [45] and Gerasimov [31] independently proposed a material model that is bound between a Hookean solid ($\alpha = 0$) and a Newtonian fluid ($\alpha = 1$). Their FDE (fractional differential equation), now called the fractional Newton model, can be written as $\sigma(t) = \mu\tau^\alpha D_\star^\alpha \epsilon(t)$, where σ and ϵ denote stress and strain, respectively, and are considered to be causal functions of time t. The coefficient $\eta = \mu\tau \in I\!R_+$ represents a generalized viscosity (the modulus μ has units of stress, and the characteristic relaxation time τ has units of time) while exponent $\alpha \in (0, 1)$ is a dimensionless material parameter. Experimental results motivated Scott Blair's model development. Mathematics, on the other hand, motivated Gerasimov, who was the first to consider an Abel (i.e., power law) kernel for the relaxation function in Boltzmann's integral equation for viscoelasticity.

Bagley and Torvik [3] demonstrated that the molecular theory of Rouse (derived for dilute solutions of *non-crosslinked* polymer molecules residing in Newtonian solvents) has a polymer contribution to stress that corresponds to a fractional Newton element whose order of evolution is a half, i.e., $\alpha = \frac{1}{2}$. They also state (without proof) that the molecular theory of Zimm (derived for dilute solutions of *crosslinked* polymer molecules residing in Newtonian solvents) has a polymer contribution to stress that corresponds to a fractional Newton element whose order of evolution is two thirds, i.e., $\alpha = \frac{2}{3}$. Most synthetic polymers have an α that lies within this range.

Gemant [30] was the first to propose a viscoelastic model of fractional order. He extended the notion of a Maxwell fluid by replacing its first-order derivative on stress with the semi-derivative, and in doing so, he proposed that $(1 + \sqrt{\eta/\mu} D_\star^{1/2})\sigma(t) = \eta D\epsilon(t)$, where $\eta, \mu \in I\!R_+$ are the material constants. The *FOV* (fractional-order viscoelastic) *fluid*, often referred to as the fractional Maxwell model, is a spring in series with a fractional Newton element, and is described by a slightly different FDE

$$(1 + \tau^\alpha D_\star^\alpha)\sigma(t) = \eta\tau^{\alpha-1} D_\star^\alpha \epsilon(t), \quad \sigma_{0^+} = \frac{\eta}{\tau}\epsilon_{0^+}, \tag{12}$$

where $\eta \in I\!R_+$ is the viscosity, $\tau \in I\!R_+$ is the characteristic relaxation time, and exponent $\alpha \in (0, 1)$ is the fractional order of evolution, which is considered to have the same value for both stress and strain. Parameters σ_{0^+} and ϵ_{0^+} are the initial states of stress and strain at time $t = 0^+$, thereby allowing for a finite inhomogeneity in the stress-strain response at time zero—a characteristic that Gemant's model does not possess. The FOV fluid was first discussed in the manuscript of Caputo and Mainardi [9] as a special case to their viscoelastic solid (Eq. 13 below).

Caputo [8] introduced a fractional Voigt solid $\sigma(t) = E(1+\rho^\alpha D_\star^\alpha)\epsilon(t)$ to model the dynamic response of Earth's crust when excited by earthquakes, which is nearly rate-insensitive over large ranges in frequency, wherein $E, \rho \in I\!R_+$ and $\alpha \in (0,1)$ are the material constants. As a mechanical model, this is a spring in parallel with a fractional Newton element. A more appropriate representation of solid behavior is the *FOV solid*, which is a spring in parallel with a fractional Maxwell element. This material model was introduced by Caputo and Mainardi [9] and has the form

$$(1 + \tau^\alpha D_\star^\alpha)\sigma(t) = E(1 + \rho^\alpha D_\star^\alpha)\epsilon(t), \quad \sigma_{0^+} = E\left(\frac{\rho}{\tau}\right)^\alpha \epsilon_{0^+}, \qquad (13)$$

where $E \in I\!R_+$ is the rubbery modulus, $E(\rho/\tau)^\alpha$ ($> E$) is the glassy modulus, $\tau \in I\!R_+$ is the characteristic relaxation time, ρ ($> \tau$) is the characteristic retardation time, and exponent $\alpha \in (0,1)$ is the fractional order of evolution. This model, unlike Caputo's original model, allows for a finite discontinuity in the stress-strain response at time zero.

Bagley and Torvik [4] have demonstrated that the fractional orders of evolution in stress and strain ought to be the same, as originally proposed in Eqs. (12) and (13) by Caputo and Mainardi, in order that a material model of fractional order comply with the second law of thermodynamics; specifically, to assure non-negative dissipations whenever cyclic loading histories are imposed on a material. Furthermore, Bagley and Calico [2] have also shown that the differential orders need to be the same for both stress and strain in order to ensure that wave fronts, like sound, propagate at finite speed.

Drozdov [20] was the first to construct a FOV constitutive model suitable for finite deformation analysis in 3-space. Specifically, he extended two 1D viscoelastic models: the fluid model $(1+(\eta/\mu)^\alpha D_\star^\alpha)\sigma(t) = \eta D\epsilon(t)$, which is a generalization of Gemant's [30] fractional Maxwell model; and the solid model $\sigma(t) = \mu(1+\rho^\alpha D_\star^\alpha)\epsilon(t)$, which is Caputo's [8] fractional Voigt model. The authors [24, 26] have extended the 1D FOV fluid and solid models of Caputo and Mainardi [9], Eqs. (12) and (13), to 3-space. In these three documents, a fractional-order continuation of the upper-convected derivative of Oldroyd [40] was employed. In another study, Adolfsson and Enelund [1] analytically continued the internal state-variable theory for viscoelasticity of Simo and Hughes [46] to fractional order. In their paper, fractional derivatives were applied to Lagraingian fields. Like Drozdov's solid constitutive model, the constitutive model of Adolfsson and Enelund is an extension of Caputo's fractional Voigt model for 3-space.

3. Introduction

3.1. Basic concepts

The authors of this paper are particularly interested in constructing a viscoelastic model for soft biological tissues. Soft tissues differ from the solid material models listed above in two important ways. First, the strains can be large, and second, the stress-strain curves can be highly non-linear.

Finite strains necessitate that one distinguish between Lagrangian (or engineering) stress $s = f/A_0$ and Eulerian (or true) stress $\sigma = f/A$, where f is the applied force and A is the current cross-sectional area of a loaded specimen whose initial stress-free area was A_0. For incompressible materials, like soft tissues, $A = A_0/\lambda$, and therefore $\sigma = \lambda s$, where $\lambda = \ell/\ell_0$ is the stretch within which ℓ denotes the current gage length whose initial length was ℓ_0. Either set of variables, Lagrangian or Eulerian, can be used in the construction of a constitutive formula; however, one choice may lead to a simpler constitutive relationship over the other, as is the case for biological tissues.

We are guided by Fung's [28] empirical formula for the elastic response of tissue; it being the simple, linear, ODE

$$\frac{\mathrm{d}s(\lambda)}{\mathrm{d}\lambda} = E + \beta s(\lambda), \quad s(1) = 0. \tag{14}$$

This formula is a relationship between Lagrangian stress s and stretch λ, not strain, with two material constants: $E, \beta \in I\!R_+$, where E has units of stress, and β is dimensionless. When integrated, its solution

$$s(\epsilon) = \frac{E}{\beta}(\mathrm{e}^{\beta\epsilon} - 1), \tag{15}$$

which obviously is not linear in strain ϵ, can be expanded as a power series of the form $s(\epsilon) = E\left(\epsilon + \frac{1}{2}\beta\epsilon^2 + \frac{1}{6}\beta^2\epsilon^3 + \cdots\right)$ that associates E with the elastic modulus of infinitesimal strain $\epsilon = \lambda - 1$, while β controls the strength of non-linearity in the stress-strain response, which is an exponential growth in stress with increasing strain.

Equivalently, via the chain rule, Fung's elastic law, Eq. (14), can be rewritten as the hypo-elastic equation

$$Ds(t) = (E + \beta s(t))D\epsilon(t), \tag{16}$$

where $s(0) = \epsilon(0) = 0$ now specifies the initial condition.

Fung's law was derived from experimental observations, and has become the defacto standard for modeling the elastic response of soft tissues. There have been numerous extensions of Fung's one-dimensional law to three-dimensional space by others, e.g., [10, 11, 47, 51, 52], each incorporating a different function when expressing a scalar representation of ϵ in terms of some deformation tensor. Later, in Section 6, we will put forward yet another scenario for extending Fung's law to three-dimensional space.

3.2. The FOV tissue model

Soft tissues are viscoelastic, with a peculiarity of being nearly rate insensitive, yet exhibiting significant stress relaxation. The approach taken herein to construct a suitable viscoelastic model for tissue is very different from the approach taken by Fung [29]. His viscoelastic model, known as QLV (quasi-linear viscoelasticity), employs two exponential integrals in its relaxation function to represent the time-dependent response of tissues. The viscoelastic characteristics of tissue, as shown by Doehring et al. [19], can be handled equally well, if not better, by utilizing a Mittag-Leffler function as the relaxation kernel, which is the relaxation function for the FOV, fluid and solid, material models, Eqs. (12) and (13), [9]. Or, as we have shown [25], by utilizing a modified power law for the relaxation function, which leads to a regularized fractional derivative.

We seek a viscoelastic extension of Fung's elastic law in the form of an ODE, and after that, its analytic continuation into a FDE. We begin by recalling Kelvin's viscoelastic solid

$$(1 + \tau D)\sigma(t) = E(1 + \rho D)\epsilon(t) \tag{17}$$

as our point of departure, otherwise referred to in the literature as the standard viscoelastic solid [54] that Caputo and Mainardi [9] continued to fractional order—see Eq. (13).

We first exchange the quasi-static asymptote in Kelvin's material model $\sigma(t) \asymp E\epsilon(t)$—a Hookean elastic solid—with $s(t) \asymp \frac{E}{\beta}(\mathrm{e}^{\beta\epsilon(t)} - 1)$, which is Fung's elastic solid, Eq. (15). We then exchange the dynamic asymptote in Kelvin's model $D\sigma(t) \asymp E\frac{\rho}{\tau}D\epsilon(t)$ with $Ds(t) \asymp (E + \beta s(t))\frac{\rho}{\tau}D\epsilon(t)$, which is the hypo-elastic representation of Fung's elastic solid, Eq. (16). Through these exchanges, a sense of uniformity is maintained with respect to material non-linearity in the stress response over the frequency range spanning between rubbery and glassy behaviors. Recall that E is the rubbery (or quasi-static) modulus, and that $E\frac{\rho}{\tau}$ is the glassy (or dynamic) modulus in Kelvin's model. Accordingly, we arrive at an ODE that one could use to describe the viscoelastic characteristics of soft tissues

$$(1 + \tau D)s(t) = \frac{E}{\beta}(\mathrm{e}^{\beta\epsilon(t)} - 1) + \rho(E + \beta s(t))D\epsilon(t), \tag{18}$$

which has four material parameters: $E, \beta, \rho, \tau \in \mathbb{R}_+$, where E has units of stress, β is dimensionless, and ρ and τ have units of time with $\rho > \tau$. The problem with this model is that it predicts a greater rate sensitivity than is experimentally observed, because its kinetics are of first order where, in actuality, there is an increasing body of evidence to suggest that the kinetics of tissues are of fractional order—see the references cited in our paper [25].

The quasi-static asymptote of Eq. (18) is in good accord with experimental evidence; it is the dynamic component that needs to be adjusted. Being mindful of the observation of Bagley et al. [2, 4] that the order of differentiation be the same for both variables, while following the example of Caputo et al. [8, 9] for the analytic continuation of an ODE to a FDE, we analytically continue the dynamic asymptote $\tau Ds(t) \asymp \rho(E+\beta s(t))D\epsilon(t)$ of Eq. (18) to one of fractional order $\tau^\alpha D^\alpha_\star s(t) \asymp \rho^\alpha(E+\beta s(t))D^\alpha_\star \epsilon(t)$, thereby resulting in the FDE

$$(1+\tau^\alpha D^\alpha_\star)s(t) = \frac{E}{\beta}(\mathrm{e}^{\beta\epsilon(t)}-1) + \rho^\alpha(E+\beta s(t))D^\alpha_\star \epsilon(t), \tag{19}$$

which is our proposed material model for soft biological tissues. It has five material parameters, namely $E, \alpha, \beta, \rho, \tau$, all of which must be strictly greater than zero. The parameter E has units of stress, α and β are dimensionless with $0 < \alpha < 1$, and ρ and τ have units of time with $\rho > \tau$.

In what follows, some of the characteristics of Eq. (19) are first explored by numerical methods. Then, in Section 5, a new set of fractional-order differential operators are derived that can be used in viscoelastic constitutive developments that the authors believe to be more appropriate, in a physical sense, than those that we and others have used in the past. Finally, in Section 6 these new operators are used to extend Eq. (19) into three-dimensional space, where finite deformations and/or rotations are permitted, with the overall method of development being quite similar to that which we have used in the above one-dimensional construction.

4. Numerical examples

Numerical methods are called upon in order to gain solutions to Eq. (19). Of the numerical methods that exist for solving FDEs, most are restricted to linear FDEs. Our algorithm [16, 17, 18] was the first numerical method to be developed that is capable of solving non-linear FDEs. Since then, other schemes have also been proposed [6, 7, 21]; however, it is not our intent to review these methods here. Actually, it is also not necessary to use a general-purpose routine here because we are interested in computing the dynamic stress response $s(t)$ by solving Eq. (19), where the strain function $\epsilon(t)$ is assumed to be given. A simple rearrangement of terms reveals that the fractional differential equation (19) can be rewritten in the form

$$D_\star^\alpha s(t) = \tau^{-\alpha} E\Big(\frac{1}{\beta}(\mathrm{e}^{\beta\epsilon(t)} - 1) + \rho^\alpha D_\star^\alpha \epsilon(t)\Big) + \tau^{-\alpha}(\rho^\alpha \beta D_\star^\alpha \epsilon(t) - 1)s(t) \quad (20)$$

which, evidently, is a linear differential equation. The second quantity of interest is the quasi-static stress response, also denoted by s, that is obtained from Eq. (15) by direct computation using the same strain function ϵ.

The algorithm of [15] turned out to be a very useful choice to solve the differential equation. The discretization of the Caputo differential operator that is underlying this algorithm has also been used to compute the values of $D_\star^\alpha \epsilon$ on the right-hand side of Eq. (20) numerically.

For our numerical simulations, the five material parameters appearing in (20) have been chosen as:

$$\begin{aligned} \alpha &= 0.33, \\ \beta &= 12.4, \\ E &= 1\,\mathrm{MPa}, \\ \tau &= 10\,\mu\mathrm{s}, \\ \rho &= 10\,\mathrm{ms}. \end{aligned} \quad (21)$$

From many investigations by the authors and their colleagues, e.g., [19], it is known that this value of α is indicative of collagen behavior, which is a class of materials that we were highly interested in when developing our model. Our choice of β has been taken from the experimental data of Fung [28]. The value for the rubbery modulus E was taken so that a quasi-static stress of about a half an MPa is obtained at a strain of 0.15, which agrees well with Fung's original data [28]. The choice of τ was also made in an attempt to be representative of actual experimental data. Finally, the FOV solid predicts that the glassy modulus is $E(\rho/\tau)^\alpha$, which is about 10 times larger than the rubbery modulus E in elastin[3]. With α having the value stated above, this property is achieved by choosing $\rho = 1000\tau$. With this background, we have simulated three classical experiments. All of these experiments were taken to start from a specimen in its virgin state, i.e. the initial condition for the differential equation (20) was

$$s(0) = 0. \quad (22)$$

The first simulation was for a tension test, where

$$\epsilon(t) = kt, \qquad t \in [0, T_{\max}], \quad (23)$$

[3]Unpublished data. Elastin is a biological molecule that often coexists with collagen in tissue structures. We know of no like data for collagen based materials from which to estimate the ratio between rubbery and glassy moduli.

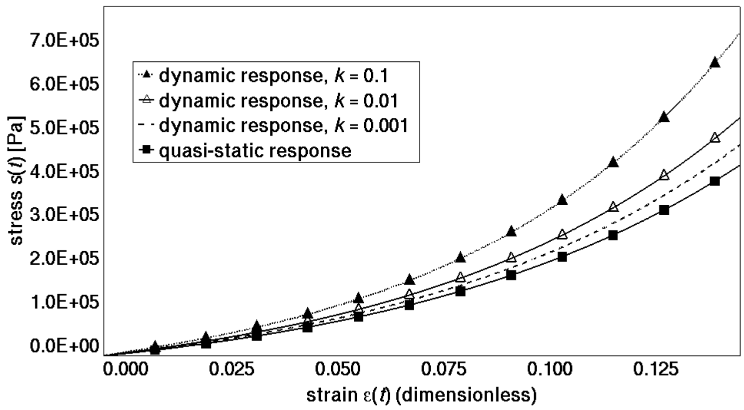

Figure 1: Stress vs. strain for tension test.

with a fixed strain rate k. Actually, this test has been run three times, with $k = 0.1\,\mathrm{s}^{-1}, 0.01\,\mathrm{s}^{-1}$, and $0.001\,\mathrm{s}^{-1}$, respectively. The upper bound $T_{\max}$ of the interval of interest was always chosen such that $\epsilon(T_{\max}) = 0.15$. In Fig. 1 we have displayed the results in the form of a plot of stress vs. strain. We can see very nicely that the quasi-static response exhibits the smallest stresses, and that the stresses in the dynamic responses increase as the strain rate increases. This is well in line with experimental observations.

The second simulation was a stress relaxation test where we have taken the given function in Eq. (20) as

$$\epsilon(t) = \begin{cases} 0.1\,\mathrm{s}^{-1} \cdot t & \text{for } t \in [0\,\mathrm{s}, 1.5\,\mathrm{s}], \\ 0.15 & \text{for } t \in [1.5\,\mathrm{s}, 1000\,\mathrm{s}], \end{cases} \tag{24}$$

i.e., strain was chosen to grow at a constant rate up to a maximum value, once again taken to be 0.15, and then held constant for a very long time. For such an experiment it is common to calculate the normalized dynamic stress response $s(t)/\max\{s(t') : 0 \le t' \le 1000\,\mathrm{s}\}$, with the dynamic stress s computed from Eq. (20), and to plot it against time. When we do so we obtain the result shown in Fig. 2 that, as above, compares very nicely with experimental data, cf., e.g., [19]. The normalized stress rises to its maximum value of 1 during the (relatively short) loading phase, which ends

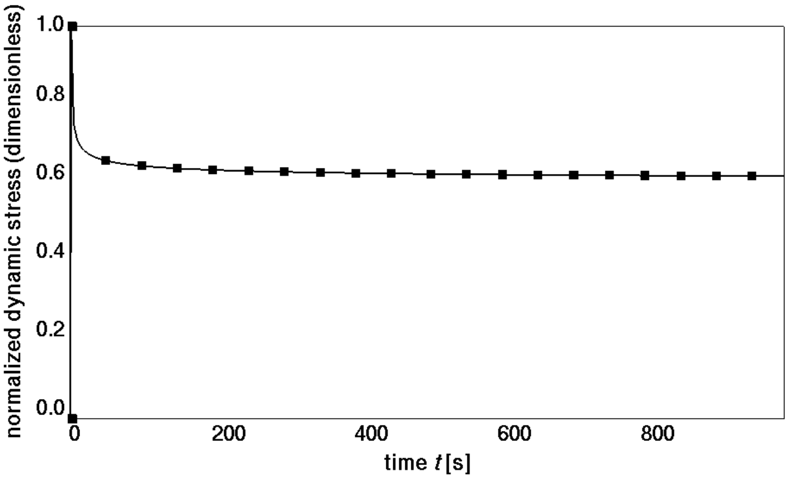

Figure 2: Normalized dynamic stress vs. time for stress relaxation test.

after 1.5 s in a response that mimics the upper curve in Fig. 1, and then a decrease begins which is very fast at first and then rapidly slows down.

Our last simulation refers to a cyclic test, where now the strain is a periodic function oscillating between 0 and the maximum value $\epsilon_{\max} = 0.15$. To be precise, we have chosen

$$\epsilon(t) = \frac{1}{2}\epsilon_{\max}\Big(1 - \cos\frac{2\pi}{T}t\Big). \tag{25}$$

The parameter T, i.e. the length of the period of the oscillations, was taken as $T = 4.7\,\mathrm{s}$ in order to arrive at $\max_t |\mathrm{d}\epsilon(t)/\mathrm{d}t| \approx 0.1$ which coincides with the corresponding value of the stress relaxation test. In Fig. 3 we again compare the dynamic response as computed using Eq. (20) with the quasi-static response of Eq. (15). Both of these functions are plotted against time for 10 loading cycles. As in the two previous simulations, we once again find a good agreement between the numerical solution and the response that was to be expected from the experience gained by experimental observation [19]. Such a testing protocol is often employed in the biomechanics literature before running the 'desired' experiment. It is called preconditioning, with the desired effect being the damping out of the transient behavior observed in the dynamics response from cycle to cycle. The number of preconditioning cycles is usually in the neighborhood of ten, which agrees with our result.

Based on these computations, we strongly believe our model to be a useful tool for investigating the mechanical behavior of viscoelastic tissues.

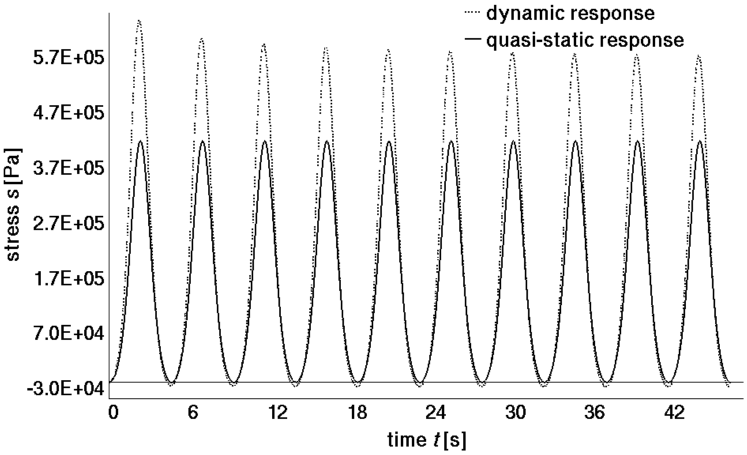

Figure 3: Dynamic and static stress vs. time for cyclic loading test.

We conclude this section by drawing the reader's attention to a special case of Eq. (20). Specifically, by using the strain function

$$\epsilon(t) = \frac{t^\alpha}{\rho^\alpha \beta \Gamma(1+\alpha)}, \tag{26}$$

we find that the coefficient of $s(t)$ on the right-hand side of Eq. (20) vanishes identically. This effectively removes the influence of stress from the response. The fact that such a situation is possible is interesting in its own right. For this special case, our FOV tissue model (20) reduces to

$$D_*^\alpha s(t) = \frac{E}{\beta \tau^\alpha} \exp(\beta \epsilon(t)) \tag{27}$$

which is like a fractional Newton model, but with cause and effect being switched: In the fractional Newton model, stress is a function of strain and its history whereas here we may solve Eq. (27) for $\epsilon(t)$ to find

$$\epsilon(t) = \frac{1}{\beta} \ln \left(\frac{\beta \tau^\alpha}{E} D_*^\alpha s(t) \right) \tag{28}$$

which clearly demonstrates that, in this case, strain can be interpreted as a function of stress and its history.

5. Continuum fields and operators

Scalars are typeset in italics, e.g., z. Vectors are typeset in lower-case bold, e.g., $\mathbf{z} = z_i \mathbf{e}_i$. And tensors are typeset in upper-case bold, e.g., $\mathbf{Z} = Z_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ wherein $\otimes$ signifies a vector product. Components z_i and Z_{ij} are also typeset in italics, and are quantified in a Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Exceptions to this notation are minimal, but they do arise, both from the use of conventional notations, and from the need to introduce multiple variables of similar nature. Components map from one set of coordinates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into another set of coordinates $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ according to the rules $\tilde{z}_i = Q_{ia} z_a$ and $\tilde{Z}_{ij} = Q_{ia} Q_{jb} Z_{ab}$, with $Q_{ik} Q_{jk} = Q_{ki} Q_{kj} = \delta_{ij}$ wherein δ_{ij} denotes the Kronecker delta function.

The identity tensor is defined as $\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$. It maps vectors, e.g., $\mathbf{y}$, into themselves, viz., $\mathbf{y} = \mathbf{I} \cdot \mathbf{y}$, wherein the dot '$\cdot$' denotes a contraction over a pair of indices in that $y_i = \delta_{ij} y_j$. Repeated component indices are summed over from 1 to 3 in accordance with the summation convention of Einstein. We omit the dot in dot notation whenever a tensor contracts with a vector, e.g., $\mathbf{y} = \mathbf{I}\mathbf{y}$, and whenever two tensors are contracted together, e.g., $\mathbf{Y}\mathbf{Z} = Y_{ik} Z_{kj} \mathbf{e}_i \otimes \mathbf{e}_j$. Here contractions are implicit. However, the dot notation is retained whenever two vectors are contracted together, e.g., $\|\mathbf{y}\|_2^2 = \mathbf{y} \cdot \mathbf{y} = y_i y_i$.

A tensor $\mathbf{Y} = Y_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ has a transpose of $\mathbf{Y}^T = Y_{ji} \mathbf{e}_i \otimes \mathbf{e}_j$ and, whenever it exists, an inverse of $\mathbf{Y}^{-1} = Y_{ij}^{-1} \mathbf{e}_i \otimes \mathbf{e}_j$ such that $Y_{ij}^{-1} \neq (Y_{ij})^{-1}$. The symmetric operator, $\mathrm{sym}(\mathbf{Y}) = \frac{1}{2}(Y_{ij} + Y_{ji}) \mathbf{e}_i \otimes \mathbf{e}_j$, and the skew-symmetric operator, $\mathrm{skew}(\mathbf{Y}) = \frac{1}{2}(Y_{ij} - Y_{ji}) \mathbf{e}_i \otimes \mathbf{e}_j$, obey $\mathbf{Y} = \mathrm{sym}(\mathbf{Y}) + \mathrm{skew}(\mathbf{Y})$. The trace operator, $\mathrm{tr}(\mathbf{Y}) = Y_{ii}$, sums the diagonal elements of the tensor, while $\det(\mathbf{Y}) = \frac{1}{3}(\mathrm{tr}\mathbf{Y}^3 - \frac{3}{2}\mathrm{tr}\mathbf{Y}\mathrm{tr}\mathbf{Y}^2 + \frac{1}{2}(\mathrm{tr}\mathbf{Y})^3)$ signifies the determinant of $\mathbf{Y}$.

5.1. Kinematics

Any modern textbook that addresses the mechanics of continua can be used as a reference book. A particularly good one was written by Holzapfel [35].

Consider a mass point in a Cartesian coordinate frame $(1, 2, 3)$ with base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ whose original location in some reference configuration κ_0 of a body affiliated with time $t = 0$, say, is given by the position vector $\mathbf{X} = X_i \mathbf{e}_i$. Its current location affiliated with time t, specified by a different configuration κ of the body, is given by the position vector $\mathbf{x} = x_i \mathbf{e}_i$ that moves with a velocity of $\mathbf{v} = v_i \mathbf{e}_i$ whose components are $v_i = \dot{x}_i = \partial x_i / \partial t$.

It is supposed that the motion of this mass point through space can be described by a continuous one-parameter family of locations $\mathbf{x} = \chi(\mathbf{X}, t)$, which is considered to be sufficiently differentiable so as to allow a deformation gradient to be defined according to

$$\mathbf{F} = F_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \text{with} \quad F_{ij} = \frac{\partial \chi_i(\mathbf{X}, t)}{\partial X_j}, \tag{29}$$

whose inverse $\mathbf{F}^{-1} = F^{-1}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ exists because $J = \det \mathbf{F} = V/V_0 = \rho_0/\rho > 0$ from the conservation of mass, wherein J is the relative volume change at a mass point, with V_0 and V representing volumes, and ρ_0 and ρ denoting mass densities, each referring to the reference or current states, respectively, of the mass point. Therefore, the motion map χ can be inverted to read as $\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$ so that $F^{-1}_{ij} = \partial \chi^{-1}_i(\mathbf{x}, t)/\partial x_j$.

Because $\mathbf{F}$ is represented in any orthogonal spatial triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by a positive-definite matrix F_{ij}, a consequence of $\det \mathbf{F} > 0$, the polar decomposition theorem from linear algebra can be applied to it, allowing one to write

$$\mathbf{F} = \mathbf{VR} = \mathbf{RU}, \tag{30}$$

where

$$\mathbf{V} = \mathbf{V}^T, \mathbf{U} = \mathbf{U}^T, \quad \text{and} \quad \mathbf{R}^{-1} = \mathbf{R}^T \quad \text{with} \quad \det \mathbf{R} = +1, \tag{31}$$

wherein $\mathbf{R}$ is a proper orthogonal tensor, while $\mathbf{V}$ and $\mathbf{U}$ are called the left- and right-stretch tensors, respectively, whose matrix representations are symmetric positive-definite, and are so named because they reside on that particular side of the rotation tensor in the polar decomposition of $\mathbf{F}$.

Associated with the deformation gradient are the respective left- and right-deformation tensors defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \quad \text{and} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \tag{32}$$

which are also symmetric positive-definite, with components $B_{ij} = F_{ik}F_{jk}$ and $C_{ij} = F_{ki}F_{kj}$.

Akin to the deformation gradient tensor $\mathbf{F}$ is the velocity gradient tensor

$$\mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \text{wherein} \quad L_{ij} = \frac{\partial v_i}{\partial x_j} \quad \text{so that} \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{33}$$

whose symmetric and skew-symmetric parts,

$$\mathbf{D} = \mathrm{sym}(\mathbf{L}) \quad \text{and} \quad \mathbf{W} = \mathrm{skew}(\mathbf{L}), \tag{34}$$

are called the stretching and vorticity tensors, respectively, with $\mathbf{W}$ quantifying the rotation rate of the principal axes of $\mathbf{D}$. Tensor $\mathbf{D}$ is also referred to in the literature as the rate-of-deformation tensor, and as the strain-rate tensor.

5.2. Polar fields

From a modeler's perspective, by constructing constitutive equations in what Dienes [13, 14] calls the *polar configuration* $\bar{\kappa}$, a modeler can create material models in terms of kinematic variables that depend only on stretch and stretching, not on the rotation or vorticity, thereby mitigating the need to handle $\mathbf{R}$ and $\mathbf{W}$ within the model itself. This is a huge advantage when developing material models that are suitable for finite deformation analysis. So powerful is this design philosophy that all constitutive models that come with the Pronto finite-element code are solved in the polar configuration [23].

Dienes [13, 14] refers to any spatial field, e.g., vector $\mathbf{y} = y_i \mathbf{e}_i$ or tensor $\mathbf{Y} = Y_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, that is mapped via the rotation tensor $\mathbf{R}$ into an alternate configuration $\bar{\kappa}$ (which, like κ, is affiliated with current time t) as a *polar field*; specifically,

$$\bar{\mathbf{y}} = \mathbf{R}^T \mathbf{y} \quad \text{and} \quad \bar{\mathbf{Y}} = \mathbf{R}^T \mathbf{Y} \mathbf{R}. \tag{35}$$

It immediately follows that

$$\bar{\mathbf{e}}_1 = \mathbf{R}^T \mathbf{e}_1, \quad \bar{\mathbf{e}}_2 = \mathbf{R}^T \mathbf{e}_2, \quad \text{and} \quad \bar{\mathbf{e}}_3 = \mathbf{R}^T \mathbf{e}_3, \tag{36}$$

i.e., $\mathbf{R}$ maps the *polar axes* $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ into the *spatial axes* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; consequently,

$$\bar{\mathbf{y}} = y_i \bar{\mathbf{e}}_i \quad \text{and} \quad \bar{\mathbf{Y}} = Y_{ij} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j. \tag{37}$$

Although Dienes' terminology came later, beginning in [13], the above mathematical constructs appeared earlier in [12]. In all of his documents, Dienes distinguishes polar fields, e.g., $\bar{\mathbf{y}}$ and $\bar{\mathbf{Y}}$, from their spatial counterparts, viz., $\mathbf{y}$ and $\mathbf{Y}$, with a bar. How the three configurations κ_0, $\bar{\kappa}$, and κ relate to one another is depicted in Fig. 4.

5.2.1. Polar rates

In his first document on the topic, Dienes [12] determined that the material time derivatives of polar fields obey the following mappings

$$D\bar{\mathbf{y}} = \mathbf{R}^T \hat{\mathbf{y}} \quad \text{wherein} \quad \hat{\mathbf{y}} = D\mathbf{y} - \boldsymbol{\Omega}\mathbf{y}, \tag{38}$$

$$D\bar{\mathbf{Y}} = \mathbf{R}^T \hat{\mathbf{Y}} \mathbf{R} \quad \text{wherein} \quad \hat{\mathbf{Y}} = D\mathbf{Y} - \boldsymbol{\Omega}\mathbf{Y} + \mathbf{Y}\boldsymbol{\Omega}, \tag{39}$$

with $D\mathbf{y} = \dot{\mathbf{y}} + \nabla\mathbf{y} \cdot \mathbf{v}$ defining the material derivative of $\mathbf{y}$, which contains partial derivatives in time $\dot{\mathbf{y}} = \frac{\partial y_i}{\partial t} \mathbf{e}_i$ and space $\nabla \mathbf{y} = \frac{\partial y_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$, and where

$$\boldsymbol{\Omega} = \dot{\mathbf{R}} \mathbf{R}^T \tag{40}$$

quantifies the rate of polar rotation at a material point, i.e., it is the rate of rigid-body rotation. Like $\mathbf{W}$, $\boldsymbol{\Omega}$ is skew symmetric; therefore, $\boldsymbol{\Omega}^T = -\boldsymbol{\Omega}$.

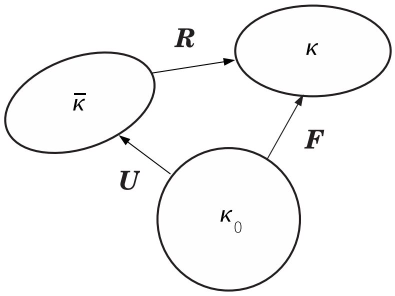

Figure 4: Mappings between the reference κ_0, polar $\bar{\kappa}$, and current κ configurations of a body.

The spatial derivatives $\hat{\mathbf{y}}$ and $\hat{\mathbf{Y}}$ are objective rates, with $\hat{\mathbf{Y}}$ being commonly called the Green-Naghdi rate [32]. Instead, following the naming convention of Dienes [13, 14], we shall call $D\bar{\mathbf{Y}}$ the *material rate of polar tensor* $\bar{\mathbf{Y}}$, and $\hat{\mathbf{Y}}$ the *polar rate of spatial tensor* $\mathbf{Y}$.

Other objective rates that are commonly employed in the standard viscoelastic literature include the corotational rate $(\mathbf{Y})^{\circ} = D\mathbf{Y} - \mathbf{WY} + \mathbf{YW}$ of Zaremba [53] (often credited to Jaumann [36]), and the respective lower- and upper-convected rates $(\mathbf{Y})^{\triangledown} = D\mathbf{Y} + \mathbf{L}^T\mathbf{Y} + \mathbf{YL}$ and $(\mathbf{Y})^{\vartriangle} = D\mathbf{Y} - \mathbf{LY} - \mathbf{YL}^T$ of Oldroyd [40], which are actually Lie derivatives taken in the direction of the velocity vector $\mathbf{v}$.

Remark 2. In accordance with Eqs. (30), (35), and (39), Dienes [12, 13, 14] argues that there is in fact just one *physical*[4] stretch tensor; it being $\mathbf{V}$, because

$$\bar{\mathbf{V}} = \mathbf{R}^T\mathbf{V}\mathbf{R} \equiv \mathbf{U} \quad \text{and} \quad D\bar{\mathbf{V}} = \mathbf{R}^T\hat{\mathbf{V}}\mathbf{R} \equiv D\mathbf{U}, \tag{41}$$

with $\bar{\mathbf{V}}$ and $D\bar{\mathbf{V}}$ denoting the polar stretch and stretching tensors, respectively, i.e., the right-stretch tensor $\mathbf{U}$ is synonymous with the polar stretch tensor $\bar{\mathbf{V}}$, viz., $\mathbf{U} = V_{ij}\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j$. Continuing along this same line of reasoning, Dienes argues that there is just one *physical* deformation tensor; it being the deformation tensor $\mathbf{B}$ of Finger [22] with

[4]In Dienes' papers [12, 13, 14], a 'physical field' is any vector or tensor field defined in the spatial configuration κ that has 'physical' meaning, not just mathematical meaning, e.g., the tensor fields for Cauchy stress and stretching, when contracted together, give the rate at which work is being done by the straining of atomic bonds.

$$\bar{\mathbf{B}} = \mathbf{R}^T\mathbf{B}\mathbf{R} \equiv \mathbf{C} \quad \text{and} \quad D\bar{\mathbf{B}} = \mathbf{R}^T\hat{\mathbf{B}}\mathbf{R} \equiv D\mathbf{C}, \tag{42}$$

wherein

$$\hat{\mathbf{B}} = D\mathbf{B} - \boldsymbol{\Omega}\mathbf{B} + \mathbf{B}\boldsymbol{\Omega} \equiv 2\mathbf{V}\mathbf{D}\mathbf{V}, \tag{43}$$

whereby tensors $\bar{\mathbf{B}}$ and $D\bar{\mathbf{B}}$ denote the polar deformation and its rate, and as such, the right-deformation tensor $\mathbf{C}$ of Green [33] is synonymous with the polar deformation tensor $\bar{\mathbf{B}}$, viz., $\mathbf{C} = B_{ij}\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j$.

In light of the above remark, we will hereafter break with traditional notation and terminology, and adopt the notation and terminology of Dienes, as we find it useful in helping keep straight in one's own mind what configuration a particular field or equation belongs to.

5.2.2. Polar operators of fractional order

In earlier documents [24, 26], the authors considered operators for both the Riemann-Liouville fractional integral and the Caputo fractional derivative of covariant or contravariant body-tensor fields [38] of vector or tensor kind. These operators were then mapped into their associated Eulerian and Lagrangian fields, which are defined in spatial frames of reference. The Caputo derivatives of body fields, when mapped into fractional rates in the Eulerian frame, were found to extend the convected rates of Oldroyd [40] (stated above) into the fractional domain.

In this document, the operators for Riemann-Liouville integration and Caputo differentiation are applied to both vector and tensor polar fields, which are subsequently mapped into their associated spatial fields. The outcome is a new set of objective spatial operators of fractional order that are presented in the theorem below.

Proposition 1. *Because the polar axes $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ of configuration $\bar{\kappa}$ reside in space, but rigidly rotate with the body, and because physical processes 'experienced' by a material are independent of rigid-body rotations, inertial effects withstanding, the polar frame of reference is therefore an ideal choice to select for the purpose of constructing constitutive relationships.*

Definition 1. In accordance with the above proposition, and in accordance with the definitions for polar fields and polar rates given in Eqs. (35)–(39), we consider mappings where the Caputo derivative, Eq. (2), and the Riemann-Liouville integral, Eq. (1), are first applied in the polar frame $\bar{\kappa}$, and then mapped into the spatial frame κ, where such mappings obey

$$D^\alpha_\wedge \mathbf{y}(t) = \mathbf{R}\, D^\alpha_\star \bar{\mathbf{y}}(t) \quad \text{and} \quad D^\alpha_\wedge(\mathbf{Y})(t) = \mathbf{R}\, D^\alpha_\star(\bar{\mathbf{Y}})(t)\mathbf{R}^T, \tag{44}$$

$$J^\alpha_\wedge \mathbf{y}(t) = \mathbf{R}\, J^\alpha \bar{\mathbf{y}}(t) \quad \text{and} \quad J^\alpha_\wedge(\mathbf{Y})(t) = \mathbf{R}\, J^\alpha(\bar{\mathbf{Y}})(t)\mathbf{R}^T. \tag{45}$$

We call $D^\alpha_\wedge$ the **fractional polar derivative of order** α, and we call $J^\alpha_\wedge$ the **fractional polar integral of order** α. These are spatial operators defined in κ that, respectively, associate with Caputo differentiation and Riemann-Liouville integration taking place in the material frame $\bar{\kappa}$.

Definition 2. It turns out that objectivity (in the integrands of $D^\alpha_\wedge \mathbf{y}$, $D^\alpha_\wedge \mathbf{Y}$, $J^\alpha_\wedge \mathbf{y}$, and $J^\alpha_\wedge \mathbf{Y}$ established in Eqs. (47)–(50) below) requires all spatial fields, e.g., $\hat{\mathbf{y}}(t')$, evaluated in $\kappa_{t'} = \kappa(t')$ to be pushed forward through a rotation $\mathbf{R}(t', t)$ into the configuration of integration, viz., κ, so that

$$\mathbf{R}(t) = \mathbf{R}(t', t)\mathbf{R}(t'). \tag{46}$$

In other words, a material rotation $\mathbf{R}(t) = \mathbf{R}(0, t)$ from κ_0 to κ can be decomposed into two rotations: the first rotation $\mathbf{R}(t') = \mathbf{R}(0, t')$ goes from κ_0 to $\kappa_{t'}$, and the second rotation $\mathbf{R}(t', t)$ goes from $\kappa_{t'}$ to κ, wherein $0 \le t' \le t$, while noting that $\mathbf{R}(t', t') = \mathbf{I}\ \forall\ t'$.

Theorem 1. *For present purposes, and only in the case of fractional-order differentiation, it is sufficient to constrain α so that $0 < \alpha < 1$, in which case*[5]

$$D^\alpha_\wedge \mathbf{y}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-t')^\alpha} \mathbf{R}(t', t)\hat{\mathbf{y}}(t')\, \mathrm{d}t', \tag{47}$$

$$D^\alpha_\wedge \mathbf{Y}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-t')^\alpha} \mathbf{R}(t', t)\hat{\mathbf{Y}}(t')\mathbf{R}^T(t', t)\, \mathrm{d}t'. \tag{48}$$

Likewise, for fractional-order integration, with no constraint on α other than $\alpha \in I\!R_+$, one has

$$J^\alpha_\wedge \mathbf{y}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} \mathbf{R}(t', t)\mathbf{y}(t')\, \mathrm{d}t', \tag{49}$$

$$J^\alpha_\wedge \mathbf{Y}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} \mathbf{R}(t', t)\mathbf{Y}(t')\mathbf{R}^T(t', t)\, \mathrm{d}t', \tag{50}$$

with $J^0_\wedge \mathbf{y}(t) = \mathbf{y}(t)$ and $J^0_\wedge \mathbf{Y}(t) = \mathbf{Y}(t)$ because $\mathbf{R}(t, t) = \mathbf{I}$.

[5] It is a straightforward matter to apply Caputo derivatives in the polar frame $\bar{\kappa}$ for those cases where $\alpha > 1$. Their mappings into the spatial frame κ will result in derivatives of the polar rates, i.e., derivatives of the derivatives in Eqs. (38) and (39), appearing in the integrands of integrals that otherwise look like Eqs. (47) and (48) with their parameters relating to α being modified according to Eq. (2).

P r o o f. From the definitions for the Riemann-Liouville fractional-order integral, Eq. (1), and the Caputo fractional-order derivative, Eq. (2), given that $0 < \alpha < 1$, and in accordance with Proposition 1, it follows that the Caputo derivative of an arbitrary polar vector field $\bar{\mathbf{y}}$ satisifes

$$D_\star^\alpha \bar{\mathbf{y}}(t) = J^{1-\alpha} D\bar{\mathbf{y}}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-t')^\alpha} D\bar{\mathbf{y}}(t')\, \mathrm{d}t',$$

which, upon substituting in Eq. (38), becomes

$$D_\star^\alpha \bar{\mathbf{y}}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-t')^\alpha} \mathbf{R}^T(t')\hat{\mathbf{y}}(t')\, \mathrm{d}t'.$$

Using the decomposition for rotations given in Eq. (46) in Definition 2, noting that $\mathbf{R}^T(0,t)$ depends only on the limits of integration and can therefore be brought outside the integral, results in the first mapping found in Eq. (44) of Definition 1, whose associated rate is the first formula in Theorem 1, viz., Eq. (47).

In like manner, also in accordance with Proposition 1, the Riemann-Liouville fractional-order integral of an arbitrary, polar, vector field $\bar{\mathbf{y}}$ satisfies

$$J^\alpha \bar{\mathbf{y}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} \bar{\mathbf{y}}(t')\, \mathrm{d}t',$$

which, upon substituting in Eq. (35), gives

$$J^\alpha \bar{\mathbf{y}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} \mathbf{R}(t')\mathbf{y}(t')\, \mathrm{d}t'.$$

Once again, by using the rotation decompostion of Eq. (46) in Definition 2, one arrives at the first mapping found in Eq. (45) of Definition 1, whose associated integral appears as the third formula in Theorem 1, viz., Eq. (49).

Similar proofs apply to the case of an arbitrary polar tensor field $\bar{\mathbf{Y}}$. ■

The fractional-order operators listed in Theorem 1 are new to the literature.

5.3. Kinetics

When addressing the reference configuration κ_0, the rate at which work is being done per unit volume is well known, and given by [35, 39]

$$\mathrm{d}W = \mathrm{tr}(\mathbf{S} D\mathbf{E}_G) = \frac{1}{2}\mathrm{tr}(\mathbf{S} D\mathbf{C}), \tag{51}$$

where $\mathbf{E}_G = \frac{1}{2}(\mathbf{C}-\mathbf{I})$ is the Green strain, and $\mathbf{S}$ is the second Piola-Kirchhoff stress, which relates to Cauchy stress σ via the identity $\mathbf{S} = J\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}$.

Given that $D\mathbf{C} = D(\bar{\mathbf{V}})\bar{\mathbf{V}} + \bar{\mathbf{V}}D(\bar{\mathbf{V}})$, because $\mathbf{C} = \bar{\mathbf{V}}^2$, Eq. (51) becomes

$$\begin{aligned}
\mathrm{d}W &= \frac{1}{2}\mathrm{tr}(\mathbf{S}D\mathbf{C}) \\
&= \frac{1}{2}\mathrm{tr}\Big(\bar{\mathbf{V}}^{-1}\mathbf{R}^T\mathbf{T}\mathbf{R}\bar{\mathbf{V}}^{-1}(D(\bar{\mathbf{V}})\bar{\mathbf{V}} + \bar{\mathbf{V}}D(\bar{\mathbf{V}}))\Big) \\
&= \frac{1}{2}\mathrm{tr}\Big(\bar{\mathbf{T}}(\bar{\mathbf{V}}^{-1}D(\bar{\mathbf{V}}) + D(\bar{\mathbf{V}})\bar{\mathbf{V}}^{-1})\Big) \\
&= \mathrm{tr}(\bar{\mathbf{T}}D\bar{\mathbf{E}}) \\
&= \mathrm{tr}(\mathbf{R}\bar{\mathbf{T}}\mathbf{R}^T\mathbf{R}D(\bar{\mathbf{E}})\mathbf{R}^T) \\
&= \mathrm{tr}(\mathbf{T}\hat{\mathbf{E}}),
\end{aligned} \tag{52}$$

where the *polar stress* $\bar{\mathbf{T}}$ and *strain-rate* $D\bar{\mathbf{E}}$ *tensors* are defined by

$$\bar{\mathbf{T}} = \mathbf{R}^T\mathbf{T}\mathbf{R} \quad \text{and} \quad D\bar{\mathbf{E}} = \mathrm{sym}(D(\bar{\mathbf{V}})\bar{\mathbf{V}}^{-1}), \tag{53}$$

with $\mathbf{T} = J\sigma$ representing the Kirchhoff stress. Because $\mathrm{d}W = \mathrm{tr}(\mathbf{T}\mathbf{D})$ [35, 39], it necessarily follows that

$$\hat{\mathbf{E}} \equiv \mathbf{D}, \quad \text{and therefore,} \quad D\bar{\mathbf{E}} = \mathbf{R}^T\hat{\mathbf{E}}\mathbf{R} \equiv \mathbf{R}^T\mathbf{D}\mathbf{R}. \tag{54}$$

The *spatial stress* $\mathbf{T}$ and *strain-rate* $\hat{\mathbf{E}}$ *tensors* are therefore synonyms for the Kirchhoff stress and stretching tensors, respectively.

The relationship between $D\bar{\mathbf{E}}$ and $D\bar{\mathbf{V}}$ present in Eq. (53), allowing for differences in notation, can be found in the appendix of Hill [34]. He proved that $D(\ln\bar{\mathbf{V}}) = D\bar{\mathbf{E}} + O(\mathbf{E}_G D(\bar{\mathbf{E}})\mathbf{E}_G)$. In other words, the polar strain-rate tensor $D\bar{\mathbf{E}}$ is distinct from the Hencky strain-rate tensor $D(\ln\bar{\mathbf{V}})$, but $D\bar{\mathbf{E}}$ approximates Hencky's rate whenever $\|\mathbf{E}_G D(\bar{\mathbf{E}})\mathbf{E}_G\|$ is sufficiently small.

An integration of the polar strain-rate tensor $D\bar{\mathbf{E}}$ defines polar strain,

$$\bar{\mathbf{E}} = \mathrm{sym}\int_0^t \bar{\mathbf{V}}^{-1}(t')D\bar{\mathbf{V}}(t')\,\mathrm{d}t' \equiv \int_0^t \mathbf{R}^T(t')\mathbf{D}(t')\mathbf{R}(t')\,\mathrm{d}t'. \tag{55}$$

The stress-strain pair $\{\bar{\mathbf{T}}, \bar{\mathbf{E}}\}$ belonging to the polar configuration $\bar{\kappa}$ constitutes a pair of conjugate thermodynamic variables in the sense that $\mathrm{d}W = \mathrm{tr}(\bar{\mathbf{T}}D\bar{\mathbf{E}})$. The *polar strain tensor* $\bar{\mathbf{E}}$ maps into the spatial configuration κ according to Eqs. (35), (39), and (54), thereby establishing from Eqs. (46) and (55) that

$$\mathbf{E} = \mathbf{R}\bar{\mathbf{E}}\mathbf{R}^T = \int_0^t \mathbf{R}(t', t)\mathbf{D}(t')\mathbf{R}^T(t', t)\,\mathrm{d}t', \tag{56}$$

where, to the best of our knowledge, this spatial strain tensor $\mathbf{E}$ is distinct from other strain measures found in the literature. Strain $\mathbf{E}$ is conjugate to the stress $\mathbf{T}$ of Kirchhoff in the sense that $\mathrm{d}W = \mathrm{tr}(\mathbf{T}\hat{\mathbf{E}})$. An analytic expression for $\bar{\mathbf{E}}(\bar{\mathbf{V}})$, as suggested by Eqs. (53) and (55), does not exist, at least that we are aware of.

6. Constitutive formulæ for isotropic tissues

We follow a similar approach to the one taken to arrive at Eq. (19) to derive a 3D viscoelastic model suitable for soft-tissue analysis. First, a hyper-elastic model is constructed for the quasi-static component. Second, a hypo-elastic model is constructed for the dynamic component, which is then analytically continued into the fractional domain. Finally, a juxtapositioning of these two components leads to the desired formula. It is beyond the scope of this paper to discuss issues regarding stability and thermodynamic admissibility of the proposed constitutive equation, or to present solutions for various boundary-value problems.

6.1. Hyper-elasticity

Experimental evidence indicates that Lagrangian stress is a function of stretch in soft biological tissues [28]. Finite-element codes, on the other hand, typically utilize variational principles that are based on deformation, not stretch. This leads to a dilemma that cannot be completely circumnavigated. Engineering judgment must therefore play a role.

The constitutive relation $\mathbf{S} = 2\partial W/\partial\mathbf{C}$ is the most common representation for a hyper-elastic solid that one will encounter in the literature [39], wherein the scalar field for the work being done W designates a potential function. Recalling that $\mathbf{S} = \mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$, $\mathbf{F} = \mathbf{R}\mathbf{U}$, $\bar{\mathbf{V}} = \mathbf{U}$, $\bar{\mathbf{B}} = \mathbf{C}$, and $\bar{\mathbf{T}} = \mathbf{R}^T\mathbf{T}\mathbf{R}$, it then follows that this classic expression for a Green elastic solid can be rewritten for the polar configuration $\bar{\kappa}$ as

$$\bar{\mathbf{T}} = 2\bar{\mathbf{V}}\frac{\partial W(\bar{\mathbf{B}})}{\partial\bar{\mathbf{B}}}\bar{\mathbf{V}}. \tag{57}$$

For isotropic soft-tissue modeling[6], a commonly used function for the strain-energy potential is

$$W = \frac{\alpha}{\beta^2}(\mathrm{e}^{\beta\mathcal{E}(I_{\mathbf{B}},II_{\mathbf{B}})} - 1) - \wp(III_{\mathbf{B}} - 1), \tag{58}$$

where $\mathcal{E}(I_{\mathbf{B}}, II_{\mathbf{B}})$ is any function of the deformation that mimics strain, wherein $I_{\mathbf{B}} = \mathrm{tr}\mathbf{B} = \mathrm{tr}\bar{\mathbf{B}}$ and $II_{\mathbf{B}} = \mathrm{tr}\mathbf{B}^{-1} = \mathrm{tr}\bar{\mathbf{B}}^{-1}$, with α and β being material parameters, and with $\wp$ denoting a Lagrange multiplier[7] that has

[6]This constitutive formulation is applicable to isotropic materials. The first author is preparing another document for publication that will extend this class of constitutive models to include anisotropic materials, which more accurately reflect biological tissues, but which lies beyond the scope of the present paper.

[7]The Lagrange multiplier $\wp$ is indeterminate. It is a workless constraint of incompressibility, viz., $\det\mathbf{B} = 1$, whose value can only be determined once the boundary conditions become known.

Authors	**Strain Functions** $\mathcal{E}(I_{\mathbf{B}}, I\!I_{\mathbf{B}})$
Demiray [11]	$I\!I_{\mathbf{B}} - 3$
Snyder [47]	$(I_{\mathbf{B}} - I\!I_{\mathbf{B}} + \sqrt{I_{\mathbf{B}}^2 - 3I\!I_{\mathbf{B}}})/I\!I_{\mathbf{B}}$
Veronda & Westmann [51]	$I_{\mathbf{B}} - 3$
Vito [52]	$(I_{\mathbf{B}} - 3) + \gamma(I\!I_{\mathbf{B}} - 3)$

Table 1: Some early strain functions used in the modeling of soft tissues.

been introduced to force a constraint of incompressibility, viz., $\det \bar{\mathbf{B}} = \det \mathbf{B} = I\!I\!I_{\mathbf{B}} = 1$, which is a reasonable assumption to impose on soft tissues. Therefore, the constitutive equation governing the elastic response of soft tissues described by this strain energy function is

$$\bar{\mathbf{T}} + \wp\bar{\mathbf{I}} = \frac{\alpha}{\beta} \mathrm{e}^{\beta\mathcal{E}} (\mathcal{E}_{,I} \bar{\mathbf{B}} - \mathcal{E}_{,II} \bar{\mathbf{B}}^{-1}), \tag{59}$$

wherein $\mathcal{E}_{,I} = \partial\mathcal{E}/\partial I_{\mathbf{B}}$ and $\mathcal{E}_{,II} = \partial\mathcal{E}/\partial I\!I_{\mathbf{B}}$. Tensor $\bar{\mathbf{B}}^{-1}$ arises from the gradient $\bar{\mathbf{V}}(\partial I\!I_{\mathbf{B}}/\partial\bar{\mathbf{B}})\bar{\mathbf{V}}$, because $I\!I_{\mathbf{B}} = \frac{1}{2}((\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\mathbf{B}^2) = \mathrm{tr}\mathbf{B}^{-1} = I_{\mathbf{B}^{-1}}$ whenever $I\!I\!I_{\mathbf{B}} = 1$, which is a direct consequence of the Cayley-Hamilton theorem. When rotated into the spatial configuration κ, Eq. (59) becomes

$$\mathbf{T} + \wp\mathbf{I} = \frac{\alpha}{\beta} \mathrm{e}^{\beta\mathcal{E}} (\mathcal{E}_{,I} \mathbf{B} - \mathcal{E}_{,II} \mathbf{B}^{-1}), \tag{60}$$

which is expressed in terms of the Kirchhoff stress $\mathbf{T}$. Some functions that have been proposed for the strain function $\mathcal{E}(I_{\mathbf{B}}, I\!I_{\mathbf{B}})$ are listed in Table 1.

Remark 3. The contribution $\sqrt{I_{\mathbf{B}}^2 - 3I\!I_{\mathbf{B}}} \in I\!R$ in Snyder's formula is a scalar representation of three-dimensional strain that passes through zero in the reference state $I_{\mathbf{B}} = I\!I_{\mathbf{B}} = 3$ and becomes negative in compression. This introduces a difficulty in its application in that there is no scheme that we are aware of that allows one to determine the correct sign to apply to this square root based on a knowledge of just its two arguments. Consequently, Snyder's function is not a practical function to use for general applications like finite elements.

By requiring that the work potential W be a quadratic function of some arbitrary strain measure ε_{ij}, as is the case in classical elasticity theories, suggests that one consider a strain function of the form

$$\mathcal{E} = \varepsilon_{ij}\varepsilon_{ji} = \mathrm{tr}\varepsilon^2. \tag{61}$$

What remains to do then is to select an admissible measure for defining strain ε, as strain is non-unique in finite deformation analysis. One particular strain measure that we like, in that it is a second-order accurate approximation to Hencky strain, cf. Bazant [5], is

$$\varepsilon = \frac{1}{2}(\mathbf{V} - \mathbf{V}^{-1}) \quad \text{so that} \quad \mathcal{E} = \frac{1}{4}(I_{\mathbf{B}} + II_{\mathbf{B}} - 6), \tag{62}$$

and therefore $\mathcal{E}_{,I} = \mathcal{E}_{,II} = \frac{1}{4}$. Given this definition, Eqs. (59) and (60) become

$$\bar{\mathbf{T}} + \wp\bar{\mathbf{I}} = \frac{2\mu}{\beta} \mathrm{e}^{\beta\mathcal{E}} \frac{1}{4}(\bar{\mathbf{B}} - \bar{\mathbf{B}}^{-1}), \tag{63}$$

$$\mathbf{T} + \wp\mathbf{I} = \frac{2\mu}{\beta} \mathrm{e}^{\beta\mathcal{E}} \frac{1}{4}(\mathbf{B} - \mathbf{B}^{-1}), \tag{64}$$

where $\frac{1}{4}(\mathbf{B} - \mathbf{B}^{-1})$ is yet another strain measure (it is the Eulerian representation of the strain field used in [27]) and where $\alpha = 2\mu$ with μ being the shear modulus. In the above relationship, strain is a tensor with the exponential modifying the modulus; whereas, in most soft-tissue models, strain is a scalar field (an exponential minus one) multiplied by a constant modulus that in turn operates on a deformation tensor.

6.2. Hypo-elasticity

In Dienes' first paper [12], where he introduced the idea of polar fields for use in constitutive development, he proposed the constitutive relationship

$$\begin{aligned} \hat{\mathbf{T}} &= \psi(\mathbf{T}, \hat{\mathbf{E}}) \\ &= \psi(\mathbf{R}\bar{\mathbf{T}}\mathbf{R}^T, \mathbf{R}D(\bar{\mathbf{E}})\mathbf{R}^T) \\ &= \mathbf{R}\psi(\bar{\mathbf{T}}, D\bar{\mathbf{E}})\mathbf{R}^T, \end{aligned} \tag{65}$$

whose equivalent polar representation is the constitutive formula

$$D\bar{\mathbf{T}} = \psi(\bar{\mathbf{T}}, D\bar{\mathbf{E}}). \tag{66}$$

Equations (65) and (66) produce an elastic response (i.e., no relaxation effects) whenever the tensor function ψ is linear in the strain rate, either $\hat{\mathbf{E}}$ or $D\bar{\mathbf{E}}$. Equations (35) and (54) allow line one to become line two in Eq. (65). Invariant theory [48], along with properties of the trace, can be called upon to go from line two to line three in Eq. (65). While Eqs. (39) and (55) enable the third line in Eq. (65) to be rewritten as Eq. (66). Such a material model is said to be *hypo-elastic* in the sense of Truesdell [50], who proposed a similar constitutive class, viz., $(\mathbf{T})^{\triangle} = \varphi(\mathbf{T}, \mathbf{D})$ with φ being linear in $\mathbf{D}$.

A sub-class of constitutive formulæ belonging to Eq. (66) considers tensor function ψ to be described by the gradient $\psi = \partial\Psi(\bar{\mathbf{T}}, D\bar{\mathbf{E}})/\partial(D\bar{\mathbf{E}})$, which is expressed in terms of an otherwise arbitrary, scalar-valued, potential function Ψ; hence,

$$D\bar{\mathbf{T}} = \frac{\partial \Psi(\bar{\mathbf{T}}, D\bar{\mathbf{E}})}{\partial D\bar{\mathbf{E}}}, \tag{67}$$

where, for hypo-elasticitity, Ψ must be quadratic in the strain rate $D\bar{\mathbf{E}}$.

Any potential function that is an isotropic function of two symmetric tensor fields, as is the case in Eq. (67), can be expressed as a function of at most ten scalar invariants (cf., e.g., Spencer [48, p. 81]); specifically: $\mathrm{tr}(\bar{\mathbf{T}})$, $\mathrm{tr}(\bar{\mathbf{T}}^2)$, $\mathrm{tr}(\bar{\mathbf{T}}^3)$, $\mathrm{tr}(D\bar{\mathbf{E}})$, $\mathrm{tr}((D\bar{\mathbf{E}})^2)$, $\mathrm{tr}((D\bar{\mathbf{E}})^3)$, $\mathrm{tr}(\bar{\mathbf{T}}D\bar{\mathbf{E}})$, $\mathrm{tr}(\bar{\mathbf{T}}^2 D\bar{\mathbf{E}})$, $\mathrm{tr}(\bar{\mathbf{T}}(D\bar{\mathbf{E}})^2)$, and $\mathrm{tr}(\bar{\mathbf{T}}^2(D\bar{\mathbf{E}})^2)$. Imposing the elastic constraint that Ψ be quadratic in $D\bar{\mathbf{E}}$, while also assuming that Ψ be at most linear in $\bar{\mathbf{T}}$, in accordance with Fung's law $Ds(t) = (E + \beta s(t))D\epsilon(t)$, allows one to write down a general potential function for this material class of the form

$$\begin{aligned}\Psi \;=\; & \frac{1}{2}\Big(\alpha_1(\mathrm{tr}(D\bar{\mathbf{E}}))^2 + \alpha_2\mathrm{tr}((D\bar{\mathbf{E}})^2) \\ & +\alpha_3\mathrm{tr}(\bar{\mathbf{T}})(\mathrm{tr}(D\bar{\mathbf{E}}))^2 + \alpha_4\mathrm{tr}(\bar{\mathbf{T}})\mathrm{tr}((D\bar{\mathbf{E}})^2) \\ & +2\alpha_5\mathrm{tr}(\bar{\mathbf{T}}D\bar{\mathbf{E}})\mathrm{tr}(D\bar{\mathbf{E}}) + \alpha_6\mathrm{tr}(\bar{\mathbf{T}}(D\bar{\mathbf{E}})^2)\Big),\end{aligned} \tag{68}$$

wherein $\alpha_1, \ldots, \alpha_6$ are material constants.

Soft tissues are, to a good approximation, incompressible; therefore, the above potential function can be further simplified to read as

$$\begin{aligned}\Psi \;=\; & \mu\mathrm{tr}((D\bar{\mathbf{E}})^2) + \frac{\beta_1}{3}\mathrm{tr}(\bar{\mathbf{T}})\mathrm{tr}((D\bar{\mathbf{E}})^2) \\ & +\beta_2\mathrm{tr}(\bar{\mathbf{T}}(D\bar{\mathbf{E}})^2) - \wp(\mathrm{tr}(D\bar{\mathbf{E}}) - 0),\end{aligned} \tag{69}$$

which produces the constitutive equation

$$D\bar{\mathbf{T}} + \wp\bar{\mathbf{I}} = 2(\mu - \beta_1 p)D\bar{\mathbf{E}} + 2\beta_2\mathrm{sym}(\bar{\mathbf{T}}D\bar{\mathbf{E}}), \tag{70}$$

where μ, β_1, β_2 are now the material constants, and where

$$p = -\frac{1}{3}\mathrm{tr}\bar{\mathbf{T}} \equiv -\frac{1}{3}\mathrm{tr}\mathbf{T} \tag{71}$$

defines hydrostatic pressure. As before, the Lagrange multiplier $\wp$ is an indeterminate constraint of incompressibility, viz., $\mathrm{tr}(D\bar{\mathbf{E}}) = 0$. A clear distinction between the hydrostatic pressure p and the Lagrange multiplier $\wp$ exists in this construction, which does not occur in other classic constitutive constructions, cf., e.g., Lodge [38].

Remark 4. By assigning $\beta = \beta_1 = \beta_2$, the material model in Eq. (70) predicts the same stiffening ratios for β and μ as the neo-Hookean solid does for μ, as they pertain to differences between the experiments of pure shear, uniaxial extension, and equi-biaxial extension (viz., 2, 3, and 6 [49]). At present, no consistent data set for tissue exists that suggests otherwise. So, in what follows, we shall assume that $\beta_1 = \beta_2 = \beta$, keeping in mind that hypo-elasticity permits a flexibility to adjust for hydrostatic effects that goes beyond that of the classical neo-Hookean solid.

6.3. Viscoelasticity

A straightforward analytic continuation of Eq. (70) to fractional order, taking into account Remark 4, and being mindful of the observations of Bagley et al. [2, 4] regarding the fractional order of evolution, leads one to consider $\tau^\alpha D^\alpha_\star \bar{\mathbf{T}} \asymp 2\rho^\alpha((\mu - \beta p)D^\alpha_\star \bar{\mathbf{E}} + \beta \mathrm{sym}(\bar{\mathbf{T}} D^\alpha_\star \bar{\mathbf{E}}))$ for the dynamic component of our model. A juxtaposition of this hypo-elastic asymptote for the dynamic component with the hyper-elastic formula in Eq. (60) for the quasi-static component, in a manner that is consistent with the formulation of our 1D model, Eq. (19), leads to the constitutive equation

$$\begin{aligned}(1 + \tau^\alpha D^\alpha_\star)\bar{\mathbf{T}} + \wp \bar{\mathbf{I}} \;&=\; \frac{2\mu}{\beta} \mathrm{e}^{\beta(I_{\mathbf{B}} + I_{\mathbf{B}^{-1}} - 6)/4} \, \frac{1}{4}(\bar{\mathbf{B}} - \bar{\mathbf{B}}^{-1}) \\ &\quad + 2\rho^\alpha((\mu - \beta p)D^\alpha_\star \bar{\mathbf{E}} + \beta \mathrm{sym}(\bar{\mathbf{T}} D^\alpha_\star \bar{\mathbf{E}})), \qquad (72)\end{aligned}$$

which is but one possible extension of Eq. (19) to 3-space. Equation (72), like Eq. (19), has five material constants: $\alpha, \beta, \mu, \tau, \rho \in I\!R_+$, where α and β are dimensionless with $0 < \alpha < 1$, μ has units of stress, and τ and ρ have units of time with $\rho > \tau$. As in Eqs. (59 and 70), $\wp$ is a Lagrange multiplier used to enforce a constraint of incompressiblity, viz., $\det \bar{\mathbf{B}} = 1$, or equivalently, $\mathrm{tr}(D\bar{\mathbf{E}}) = 0$. Recall that $\mathcal{E} = \frac{1}{4}(I_{\mathbf{B}} + I\!I_{\mathbf{B}} - 6)$ and that $I\!I_{\mathbf{B}} \equiv I_{\mathbf{B}^{-1}}$.

From Eqs. (1) and (2), the fractional polar strain-rate is defined by

$$D^\alpha_\star \bar{\mathbf{E}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-t')^\alpha} D\bar{\mathbf{E}}(t')\,\mathrm{d}t', \qquad (73)$$

hence, there is no need to integrate $D\bar{\mathbf{E}}$ to get $\bar{\mathbf{E}}$ whenever Caputo's definition for fractional differentiation is adopted. Said differently, $\bar{\mathbf{B}}$ and $D\bar{\mathbf{E}}$ are the primative kinematic variables of our theory.

The FDE stated in Eq. (72) is written for the polar configuration $\bar{\kappa}$. Although it is not necessary to do so, we suggest that it be solved in this configuration, with the resulting value for polar stress $\bar{\mathbf{T}}$ then being mapped into the spatial configuration κ via Eq. (35) to get the Kirchhoff (or Cauchy) stress $\mathbf{T}$ ($= J\sigma$, where $J = 1$ is a direct consequence of incompressibility).

Nevertheless, in accordance with Theorem 1, Eq. (72) maps into the spatial configuration κ as

$$(1+\tau^\alpha D^\alpha_\wedge)\mathbf{T}+\wp\mathbf{I} = \frac{2\mu}{\beta}\mathrm{e}^{\beta(I_\mathbf{B}+I_{\mathbf{B}^{-1}}-6)/4}\,\frac{1}{4}(\mathbf{B}-\mathbf{B}^{-1}) + 2\rho^\alpha((\mu-\beta p)D^\alpha_\wedge\mathbf{E}+\beta\mathrm{sym}(\mathbf{T}D^\alpha_\wedge\mathbf{E})). \quad (74)$$

Equations (72) and (74) are equivalent material models. They are just defined in different configurations, while employing alternate, but equivalent, expressions for fractional rates.

7. Closing remarks

In this paper, Fung's elastic law has been extended to one that is appropriate for the viscoelastic representation of soft biological tissues, and whose kinetics are of fractional order. To be able to use this one-dimensional material model in applications, like a finite element-based simulation of a mechanical process, this material model had to be generalized for three-dimensional analysis, allowing for both finite deformations and finite rotations in an objective manner. To accomplish this, a set of fractional-order operators were derived that align themselves better with the physics of this problem than the fractional operators that have been used previously in the literature.

References

[1] K. Adolfsson and M. Enelund, Fractional derivative viscoelasticity at large deformations. *Nonlinear Dynamics* **33** (2003), 301-321.

[2] R. L. Bagley and R. A. Calico, Fractional order state equations for the control of viscoelastically damped structures. *AIAA Journal of Guidance, Control, and Dynamics* **14** (1991), 304-311.

[3] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology* **27** (1983), 201-210.

[4] R. L. Bagley and P. J. Torvik, On the fractional calculus model of viscoelastic behavior. *Journal of Rheology* **30** (1986), 133-155.

[5] Z. P. Bazant, Easy-to-compute tensors with symmetric inverse approximating Hencky finite strain and its rate. *Journal of Engineering Materials and Technology* **120** (1998), 131-136.

[6] H. B r u n n e r, *Collocation Methods for Volterra Integral and Related Functional Equations*. Cambridge University Press, Cambridge (2004).

[7] X. C a o, K. B u r r a g e and F. A b d u l l a h, A variable stepsize implementation for fractional differential equations. Technical Report, University of Queensland, Brisbane (2006).

[8] M. C a p u t o, Linear models of dissipation whose Q is almost frequency independent–II. *Geophysical Journal of the Royal Astronomical Society* **13** (1967), 529-539.

[9] M. C a p u t o and F. M a i n a r d i. Linear models of dissipation in anelastic solids. *Rivista del Nuovo Cimento* **1** (1971), 161-198.

[10] H. D e m i r a y, A note on the elasticity of soft biological tissues. *Journal of Biomechanics* **5** (1972), 309-311.

[11] H. D e m i r a y, Stresses in ventricular wall. *Journal of Applied Mechanics* **43** (1976) 194-197.

[12] J. K. D i e n e s, On the analysis of rotation and stress rate in deforming bodies. *Acta Mechanica* **32** (1979), 217-232.

[13] J. K. D i e n e s, Theory of Deformation: Part 1, Kinematics. Technical Report LA–11063–MS Vol. 1, Los Alamos National Laboratory, December 1987.

[14] J. K. D i e n e s, Finite Deformation of Materials with an Ensemble of Defects. Technical Report LA–13994–MS, Los Alamos National Laboratory, January 2003.

[15] K. D i e t h e l m, An algorithm for the numerical solution of differential equations of fractional order. *Electronic Transactions on Numerical Analysis* **5** (1997), 1-6.

[16] K. D i e t h e l m, N. J. F o r d and A. D. F r e e d, A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics* **29** (2002), 3-22.

[17] K. D i e t h e l m, N. J. F o r d and A. D. F r e e d, Detailed error analysis for a fractional Adams method. *Numerical Algorithms* **36** (2004), 31-52.

[18] K. D i e t h e l m, N. J. F o r d, A. D. F r e e d and Yu. L u c h k o, Algorithms for the fractional calculus: A selection of numerical methods. *Computer Methods in Applied Mechanics and Engineering* **194** (2005), 743-773.

[19] T. C. D o e h r i n g, A. D. F r e e d, E. O. C a r e w and I. V e s e l y, Fractional order viscoelasticity of the aortic valve cusp: an alternative to quasilinear viscoelasticity. *Journal of Biomechanical Engineering* **127** (2005), 700-708.

[20] A. D. D r o z d o v, Fractional differential models in finite viscoelasticity. *Acta Mechanica* **124** (1997), 155-180.

[21] F. D u b o i s and S. M e n g u é, Mixed collocation for fractional differential equations. *Numerical Algorithms* **34** (2003), 303-312.

[22] J. F i n g e r, Über die allgemeinsten Beziehungen zwischen endlichen Deformationen und den zugehörigen Spannungen in aeolotropen und isotropen Substanzen. *Sitzungsberichte der Akademie der Wissenschaften, Wien* **103** (1894), 1073-1100.

[23] D. P. F l a n a g a n and L. M. T a y l o r, An accurate numerical algorithm for stress integration with finite rotations. *Computer Methods in Applied Mechanics and Engineering* **62** (1987), 305-320.

[24] A. D. F r e e d and K. D i e t h e l m, Tensor fields for use in fractional-order viscoelasticity. In: A. Le Mehauté, J. A. Tenreiro Machado, J. C. Trigeassou, and J. Sabatier (Eds), *Fractional Differentiation and Its Applications*, UBooks Verlag, Neusäß (2005), 169-182.

[25] A. D. F r e e d and K. D i e t h e l m, Fractional calculus in biomechanics: a 3D viscoelastic model using regularized fractional derivative kernels with application to the human calcaneal fat pad. *Biomechanics and Modeling in Mechanobiology* **5** (2006), 203-215.

[26] A. D. F r e e d, K. D i e t h e l m and Yu. L u c h k o, Fractional-order viscoelasticity (FOV): constitutive development using the fractional calculus: First annual report. Technical Memorandum 2002–211914, NASA Glenn Res. Center at Lewis Field, Cleveland, Dec. 2002.

[27] A. D. F r e e d, D. R. E i n s t e i n and I. V e s e l y, Invariant formulation for dispersed transverse isotropy in aortic heart valves: an efficient means for modeling fiber splay. *Biomechanics and Modeling in Mechanobiology* **4** (2005), 100-117.

[28] Y.-C. F u n g, Elasticity of soft tissues in simple elongation. *American Journal of Physiology*, **28** (1967), 1532-1544.

[29] Y.-C. F u n g, Stress–strain-history relations of soft tissues in simple elongation. In: Y.-C. Fung, N. Perrone, and M. Anliker (Eds), *Biomechanics: Its Foundations and Objectives*, Prentice-Hall, Englewood Cliffs (1971), Chapter 7, 181-208.

[30] A. G e m a n t, A method of analyzing experimental results obtained from elasto-viscous bodies. *Physics* **7** (1936), 311-317.

[31] A. N. G e r a s i m o v, A generalization of linear laws of deformation and its application to inner friction problems. *Prikladnaya Matematika i Mekhanika* **12** (1948), 251-260. In Russian.

[32] A. E. G r e e n and P. M. N a g h d i, A general theory of an elastic-plastic continuum. *Archives for Rational Mechanics and Analysis* **18** (1965), 251-281.

[33] G. G r e e n, On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society* **7** (1841), 121-140.

[34] R. H i l l, On constitutive inequalites for simple materials–I. *Journal of the Mechanics and Physics of Solids* **16** (1968), 229-242.

[35] G. A. H o l z a p f e l, *Nonlinear Solid Mechanics, a Continuum Approach for Engineering*. Wiley, Chichester (2000).

[36] G. J a u m a n n, Geschlossenes System physikalischer und chemischer Differentialgesetze. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften: Mathematisch-naturwissenschaftliche Klasse* **120** (1911), 385-530.

[37] J. L i o u v i l l e, Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions. *Journal de l'École Polytechnique* **13** (cahier 21) (1832), 1-66.

[38] A. S. L o d g e, *Elastic Liquids: An Introductory Vector Treatment of Finite-Strain Polymer Rheology*. Academic Press, London (1964).

[39] R. W. O g d e n, *Non-Linear Elastic Deformations*. John Wiley, New York (1984). Republished by Dover Publications, Mineola, NY (1997).

[40] J. G. O l d r o y d, On the formulation of rheological equations of state. *Proceedings of the Royal Society, London, Series A* **200** (1950), 523-541.

[41] I. P o d l u b n y, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[42] Yu. N. R a b o t n o v, *Creep Problems in Structural Members.* North-Holland Publishing, Amsterdam, (1969). Originally published in Russian as: *Polzuchest Elementov Konstruktsii*, Nauka, Moscow (1966).

[43] B. R o s s, The development of fractional calculus 1695–1900. *Historia Mathematica* **4** (1977), 75-89.

[44] S. G. S a m k o, A. K i l b a s and O. I. M a r i c h e v, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon (1993).

[45] G. W. S c o t t B l a i r, Analytical and integrative aspects of the stress-strain-time problem. *J. Scientific Instruments* **21** (1944), 80-84.

[46] J. C. S i m o and T. J. R. H u g h e s, *Computational Inelasticity*. Springer, New York (1998).

[47] R. W. S n y d e r, Large deformation of isotropic biological tissue. *Journal of Biomechanics* **5** (1972), 601-606.

[48] A. J. M. S p e n c e r, *Deformations in Fibre-reinforced Materials*. Clarendon Press, Oxford (1972).

[49] L. R. G. T r e l o a r, *The Physics of Rubber Elasticity*, 3rd edition. Clarendon Press, Oxford (1975).

[50] C. T r u e s d e l l, Hypoelasticity. *Archives for Rational Mechanics and Analysis*, **4** (1955), 83-133.

[51] D. R. V e r o n d a and R. A. W e s t m a n n, Mechanical characterization of skin–finite deformations. *Journal of Biomechanics* **3** (1970), 111-124.

[52] R. V i t o, A note on arterial elasticity. *Journal of Biomechanics* **6** (1973), 561-564.

[53] S. Z a r e m b a, Sur une forme perfectionnée de la théorie de la relaxation. *Bulletin de l'Académie de Cracovie* (June 1903), 594-614.

[54] C. Z e n e r, *Elasticity and Anelasticity of Metals*. University of Chicago Press, Chicago (1948).

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