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ON q–ANALOGUES OF CAPUTO DERIVATIVE AND MITTAG–LEFFLER FUNCTION

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Abstract

Based on the fractional q–integral with the parametric lower limit of integration, we consider the fractional q–derivative of Caputo type. Especially, its applications to q-exponential functions allow us to introduce q–analogues of the Mittag–Leffler function. Vice versa, those functions can be used for defining generalized operators in fractional q–calculus.

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1. Introduction

The calculus of the real order derivatives and integrals has become very suitable apparatus in describing and solving a lot of problems in numerous sciences, such as physics, electrochemistry and material science (see, for example [12]). Their treatment from the point of view of the q–calculus can additionally open new perspectives as it did, for example, in optimal control problems [5].

Starting from the q–integral

$$\big(I_{q,0}f\big)(x) = \int_0^x f(t)\, d_qt = x(1-q)\sum_{k=0}^{\infty} f(xq^k)\, q^k \quad (0 \le |q| < 1)\ , \qquad (1)$$

and

$$\big(I_{q,a}f\big)(x)=\int_a^x f(t)\,d_qt=\int_0^x f(t)\,d_qt-\int_0^a f(t)\,d_qt\ , \tag{2}$$

the iterated q–integral operator $I^n_{q,a}$ is defined by

$$I^0_{q,a}f=f,\qquad I^n_{q,a}f=I_{q,a}\big(I^{n-1}_{q,a}f\big)\quad (n=1,2,3,\ldots)\ .$$

The reduction of this iterated q–integral to a single integral of one variable was considered by Al-Salam [3], by a q–analogue of the Cauchy formula

$$\big(I^n_{q,a}f\big)(x)=\frac{x^{n-1}}{[n-1]_q!}\int_a^x (qt/x;q)_{n-1}\ f(t)\ d_qt\qquad (n\in\mathbb{N})\ , \tag{3}$$

where

$$[a]_q:=\frac{1-q^a}{1-q}\quad (a\in\mathbb{R})\ ,\qquad [n]_q!=\prod_{k=1}^n [k]_q!\quad (n\in\mathbb{N}),$$

and

$$(a;q)_\infty=\prod_{i=0}^\infty (1-aq^i)\ ,\qquad (a;q)_\alpha=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}\quad (a,\alpha\in\mathbb{R})\ . \tag{4}$$

Al-Salam [2] and Agarwal [1] introduced several types of fractional q–integral operators and fractional q–derivatives, always with the lower limit of integration being zero. However, in some considerations, such as the construction of a q–Taylor formula or solving of q–differential equation of fractional order, it is interesting to allow nonzero lower limit of integration.

Therefore, we define the fractional q–integral in the following way.

Definition 1.1. The *fractional q–integral* is

$$\big(I^\alpha_{q,c}f\big)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_c^x (qt/x;q)_{\alpha-1}\ f(t)\ d_qt\qquad (\alpha\in\mathbb{R}^+)\ , \tag{5}$$

where the q–Gamma function is defined by

$$\Gamma_q(x)=(q;q)_{x-1}(1-q)^{1-x}\ . \tag{6}$$

Lemma 1.1. *The fractional q–integral (5) can be written in the equivalent form*

$$\big(I^\alpha_{q,c}f\big)(x)=\int_c^x\ f(t)\ d_qw_\alpha(x,t)\qquad (\alpha\in\mathbb{R}^+)\ ,$$

where the function $w_\alpha(x,t)$ *is defined by*

$$w_\alpha(x,t)=\frac{1}{\Gamma_q(\alpha+1)}\big(x^\alpha-x^\alpha(t/x;q)_\alpha\big)\qquad(\alpha\in\mathbb{R}^+)\ .$$

The permission for the lower limit of integration to take some nonzero value, brings a lot of troubles while working with fractional q–calculus (see [13]).

Here are some of the properties of the previously defined integral.

THEOREM 1.2. *Let* $\alpha,\beta\in\mathbb{R}^+$*. The* q*–fractional integration has the following semigroup property*

$$\big(I_{q,c}^{\beta}I_{q,c}^{\alpha}f\big)(x)=\big(I_{q,c}^{\alpha+\beta}f\big)(x)\qquad(0<c<x)\ .$$

In [13] the next useful statement is proven.

LEMMA 1.3. *For* $\alpha\in\mathbb{R}^+,\ \lambda,\lambda+\alpha\in\mathbb{R}\setminus\{-1,-2,\dots\}$*, the following fractional* q*–integral is valid:*

$$I_{q,c}^{\alpha}\big(x^\lambda(c/x;q)_\lambda\big)=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}\ x^{\alpha+\lambda}(c/x;q)_{\alpha+\lambda}\qquad(0<c<x)\ .\tag{7}$$

2. The fractional q–derivative of Riemann-Liouville type

The q–derivative of a function $f(x)$ is defined by

$$\big(D_qf\big)(x)=\frac{f(x)-f(qx)}{x-qx}\quad(x\neq0)\ ,\quad\big(D_qf\big)(0)=\lim_{x\to0}\big(D_qf\big)(x)\ ,$$

and the q–derivatives of higher order, resp.:

$$D_q^0f=f\ ,\qquad D_q^nf=D_q\big(D_q^{n-1}f\big)\quad(n=1,2,3,\dots)\ .\tag{8}$$

On the basis of fractional q–integral, we can define q–derivative of real order.

DEFINITION 2.1. The *fractional* q*–derivative of Riemann–Liouville type* of order $\alpha\in\mathbb{R}^+$ is

$$\big(D_{q,c}^{\alpha}f\big)(x)=\big(D_q^{\lceil\alpha\rceil}I_{q,c}^{\lceil\alpha\rceil-\alpha}f\big)(x),\tag{9}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α.

Some useful relations are true:

Lemma 2.1. *For $0 < c < x$, the operators $D^{\alpha}_{q,c}$ and $I^{\alpha}_{q,c}$ satisfy:*

$$\big(D^{\alpha}_{q,c} I^{\alpha}_{q,c} f\big)(x) = f(x)\ .$$

Lemma 2.2. *For $\lambda \in \mathbb{R} \setminus \{-1,-2,\dots\}$ and $0 < c < x$, the following relation is valid:*

$$D^{\alpha}_{q,c}\big(x^{\lambda}(c/x;q)_{\lambda}\big) = \begin{cases} \dfrac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}\, x^{\lambda-\alpha}(c/x;q)_{\lambda-\alpha}\ , & \alpha-\lambda \in \mathbb{R}\setminus\mathbb{N}\ , \\ 0\ , & \alpha-\lambda\in\mathbb{N}\ . \end{cases}$$

3. The fractional q–derivative of Caputo type

If we change the order of operators in (9), we can introduce another type of fractional q–derivative.

Definition 3.1. The *fractional q-derivative of Caputo type* of order $\alpha \in \mathbb{R}^{+}$ is defined as

$$\big({}_{\star}D^{\alpha}_{q,c} f\big)(x) = \Big(I^{\lceil\alpha\rceil-\alpha}_{q,c} D^{\lceil\alpha\rceil}_{q} f\Big)(x)\ . \tag{10}$$

It is important to establish the connection between the two types of the fractional q–derivatives (9) and (10).

Theorem 3.1. *Let $\alpha \in \mathbb{R}^{+} \setminus \mathbb{N}_0$ and $0 < c < x$. The following connection between the Caputo type and the Riemann-Liouville type fractional integral holds true:*

$$\big(D^{\alpha}_{q,c} f\big)(x) = \big({}_{\star}D^{\alpha}_{q,c} f\big)(x) + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(D^k_q f)(c)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha}(c/x;q)_{k-\alpha}\ .$$

The proof is pretty long, so we will omit it here.

Lemma 3.2. *For $\alpha \in \mathbb{R}^{+}\setminus\mathbb{N}$ and $\lambda \in (-1,\infty)$, the following is valid:*

$${}_{\star}D^{\alpha}_{q,c}\big(x^{\lambda}(c/x;q)_{\lambda}\big) = \begin{cases} D^{\alpha}_{q,c}\big(x^{\lambda}(c/x;q)_{\lambda}\big)\ , & \lceil\alpha\rceil-\lambda\in\mathbb{R}\setminus\mathbb{N}\ , \\ 0\ , & \lceil\alpha\rceil-\lambda\in\mathbb{N}\ . \end{cases}$$

THEOREM 3.3. *For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and $a < x$, we have the following relation:*

$$\big({}_\star D_{q,a}^{\alpha+1} f\big)(x) = \big({}_\star D_{q,a}^{\alpha} D_q f\big)(x).$$

P r o o f. Since $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, we can write $\alpha = n + \varepsilon$, $n \in \mathbb{N}_0$, $0 < \varepsilon < 1$. Then $\alpha + 1 \in (n+1, n+2)$, so we get

$$\begin{aligned}({}_\star D_{q,a}^{\alpha+1} f)(x) &= \big(I_{q,a}^{1-\varepsilon} D_q^{n+2} f\big)(x)\\ &= \big(I_{q,a}^{1-\varepsilon} D_q^{n+1} D_q f\big)(x) = ({}_\star D_{q,a}^{\alpha} D_q f)(x)\ .\end{aligned}$$

■

THEOREM 3.4. *For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$ and $a < x$,*

$$\big(D_q\, {}_\star D_{q,a}^{\alpha} f\big)(x) - \big({}_\star D_{q,a}^{\alpha+1} f\big)(x) = \frac{\big(D_q^{\lceil\alpha\rceil} f\big)(a)}{\Gamma_q(\lceil\alpha\rceil - \alpha)}\ x^{\lceil\alpha\rceil-\alpha-1}(a/x;q)_{\lceil\alpha\rceil-\alpha-1}\ .$$

THEOREM 3.5. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, for $a < x$, the following is valid:*

$$\big(I_{q,a}^{\alpha}\, {}_\star D_{q,a}^{\alpha} f\big)(x) = f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\big(D_q^k f\big)(a)}{[k]_q!}\ x^k (a/x;q)_k\ .$$

P r o o f. Really, we have

$$\begin{aligned}\big(I_{q,a}^{\alpha}\, {}_\star D_{q,a}^{\alpha} f\big)(x) &= \big(I_{q,a}^{\alpha} I_{q,a}^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f\big)(x) = \big(I_{q,a}^{\lceil\alpha\rceil} D_q^{\lceil\alpha\rceil} f\big)(x)\\ &= f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\big(D_q^k f\big)(a)}{[k]_q!}\ x^k (a/x;q)_k\ .\end{aligned}$$

■

THEOREM 3.6. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, for $a < x$, the following is valid:*

$$\big({}_\star D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) = f(x)\ .$$

P r o o f. According to the previous theorems, we can write

$$\begin{aligned}\big({}_\star D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) &= \big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(D_q^k I_{q,a}^{\alpha} f)(a)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha}(a/x;q)_{k-\alpha}\\ &= f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(I_{q,a}^{\alpha-k} f)(a)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha}(a/x;q)_{k-\alpha} = f(x)\ .\end{aligned}$$

■

THEOREM 3.7. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $\beta \in \mathbb{R}^+$ and $\alpha > \beta$. Then, for $a < x$, the following relations hold:*

$$\begin{aligned}
&\big({}_\star D^{\alpha}_{q,a} I^{\beta}_{q,a} f\big)(x) \\
&\qquad = \big({}_\star D^{\alpha-\beta}_{q,a} f\big)(x) + \sum_{k=0}^{\lceil \alpha-\beta\rceil -1} \frac{\big(D_q^k f\big)(a)}{\Gamma_q(k-\alpha+\beta+1)} x^{k-\alpha+\beta} (a/x;q)_{k-\alpha+\beta} \ , \\
&\big(I^{\beta}_{q,a}\ {}_\star D^{\alpha}_{q,a} f\big)(x) \\
&\qquad = \big({}_\star D^{\alpha-\beta}_{q,a} f\big)(x) - \sum_{k=\lceil \alpha-\beta\rceil}^{\lceil \alpha\rceil -1} \frac{\big(D_q^k f\big)(a)}{\Gamma_q(k-\alpha+\beta+1)} x^{k-\alpha+\beta} (a/x;q)_{k-\alpha+\beta} \ .
\end{aligned}$$

For ideas of the proofs of the propositions in this section, one can see in [13] and [14].

4. On q–analogs of the Mittag–Leffler function

In the mathematical literature (see for example, see [12], [8], [9]) the Mittag-Leffler (M-L) function (of two indices) is already well known and is defined by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha\, n+\beta)} \qquad \big(\alpha,\beta \in \mathbb{C}:\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0\big) \ . \tag{11}$$

The function $E_\alpha(x) = E_{\alpha,1}(x)$, as a direct generalization of the exponential and trigonometric functions, was introduced first in the paper [11] by G. Mittag–Leffler in 1903. It always appears when solving fractional order differential or integral equations. Nowadays, it is involved in the treatment of many concrete problems in various applied sciences.

The q–exponential functions (see [6]) can be written by the power series

$$\begin{aligned}
e_q(x) &= \frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} \qquad (|x|<1) \ , \\
E_q(x) &= (-x;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n}\ x^n \ ,
\end{aligned}$$

or, applying the q–form of the Taylor theorem, by

$$e_q(x) = e_q(c)\sum_{n=0}^{\infty}\frac{x^n(c/x;q)_n}{(q;q)_n} \qquad (c\in\mathbb{R};\ |c|<|x|)\ , \tag{12}$$

$$E_q(x) = E_q(c)\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(-c;q)_n}\ \frac{x^n(c/x;q)_n}{(q;q)_n} \qquad (c,x\in\mathbb{R})\ . \tag{13}$$

COROLLARY 4.1. *For $\alpha\in\mathbb{R}^+$ and $0<c<x$, the following q–fractional integrals are valid:*

$$I^{\alpha}_{q,c}\big(e_q(x)\big) = (1-q)^{\alpha}\ e_q(c)\sum_{n=0}^{\infty}\frac{x^{\alpha+n}(c/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}}\ ,$$

$$I^{\alpha}_{q,c}\big(E_q(x)\big) = (1-q)^{\alpha}\ E_q(c)\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(-c;q)_n}\ \frac{x^{\alpha+n}(c/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}}\ .$$

P r o o f. According to Lemma 1.3, we have

$$\frac{I^{\alpha}_{q,c}\big(x^n(c/x;q)_n\big)}{(q;q)_n} = \frac{1}{(q;q)_n}\ \frac{\Gamma_q(n+1)}{\Gamma_q(n+\alpha+1)}\ x^{n+\alpha}(c/x;q)_{n+\alpha}$$
$$= (1-q)^{\alpha}\ \frac{x^{n+\alpha}(c/x;q)_{n+\alpha}}{(q;q)_{\alpha+n}}\ .$$

Applying this to formulas (12) and (13) we get the required identities. ∎

The previous corollary indicates that it is necessary to define functions which are q–analogues of the Mittag–Leffler function (11).

DEFINITION 4.1. The function

$$e_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty}\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha\,n+\beta-1}} \qquad (|c|<|x|)\ , \tag{14}$$

$$\big(q,x,c,\alpha,\beta\in\mathbb{C};\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0,\ \ |q|<1\big) \tag{15}$$

we shall call *the small q–Mittag–Leffler function*. Similarly, *the big q–Mittag–Leffler function* is introduced as

$$E_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty}\frac{q^{\binom{\alpha n+\beta-1}{2}}}{(-c;q)_{\alpha\,n+\beta-1}}\ \frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha\,n+\beta-1}}\ , \tag{16}$$

with the same conditions (15).

Evidently, in the limit case, we get

$$\lim_{q\to1}\lim_{c\to0} e_{q;\alpha,\beta}((1-q)x;c) = \lim_{q\to1}\lim_{c\to0} E_{q;\alpha,\beta}((1-q)x;c) = x^{\beta-1}E_{\alpha,\beta}(x^{\alpha}) \ ,$$

and especially,

$$e_{q;1,1}(x;0) = e_q(x), \qquad E_{q;1,1}(x;0) = E_q(x) \ .$$

THEOREM 4.2. *The following q–fractional integrals are valid*

$$\begin{aligned} I^{\alpha}_{q,c}\big(e_q(x)\big) &= (1-q)^{\alpha}\ e_q(c)\ e_{q;1,\alpha+1}(x;c) \\ I^{\alpha}_{q,c}\big(E_q(x)\big) &= (1-q)^{\alpha} q^{\binom{\alpha+1}{2}}\ E_q\big(cq^{-\alpha}\big)\ E_{q;1,\,\alpha+1}\big(xq^{-\alpha};cq^{-\alpha}\big) \ . \end{aligned}$$

P r o o f. The first statement is obvious. The second one requires some additional simplification.

Applying the relation (4) and (4), we can write

$$\begin{aligned} (-c;q)_n &= \frac{(-cq^{-\alpha};q)_{n+\alpha}}{(-cq^{-\alpha};q)_{\alpha}} = \frac{(-cq^{-\alpha};q)_{n+\alpha}\ (-c;q)_{\infty}}{(-cq^{-\alpha};q)_{\infty}} \\ &= (-cq^{-\alpha};q)_{n+\alpha}\ \frac{E_q(c)}{E_q(cq^{-\alpha})} \ . \end{aligned}$$

The second statement of Corollary 4.1 can be written in the form

$$\begin{aligned} I^{\alpha}_{q,c}\big(E_q(x)\big) &= (1-q)^{\alpha}\ E_q(c)\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(-c;q)_n}\ \frac{x^{\alpha+n}\ (c/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}} \\ &= (1-q)^{\alpha}\ q^{\binom{\alpha+1}{2}}\ E_q(cq^{-\alpha}) \\ &\quad\times\sum_{n=0}^{\infty}\frac{q^{\binom{n+\alpha}{2}}}{(-cq^{-\alpha};q)_{n+\alpha}}\ \frac{(xq^{-\alpha})^{\alpha+n}\ (cq^{-\alpha}/xq^{-\alpha};q)_{\alpha+n}}{(q;q)_{\alpha+n}} \ , \end{aligned}$$

wherefrom the second relation follows. ■

COROLLARY 4.3. *For $\alpha\in\mathbb{R}\backslash\mathbb{N}_0$, the Riemann-Liouville (R-L) fractional q–derivatives of the q–exponential functions are:*

$$D^{\alpha}_{q,c}(e_q(x)) = \frac{e_q(c)}{(1-q)^{\alpha}}\sum_{n=0}^{\infty}\frac{x^{n-\alpha}(c/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}}$$

$$= \frac{e_q(c)}{(1-q)^\alpha} \left\{ \sum_{n=0}^{\lceil\alpha\rceil-1} \frac{x^{n-\alpha}(c/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}} + e_{q;1,\lceil\alpha\rceil-\alpha+1}(x;c) \right\} ,$$

and

$$D^\alpha_{q,c}\big(E_q(x)\big) = \frac{E_q(c)}{(1-q)^\alpha} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(-c;q)_n} \frac{x^{n-\alpha}(c/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}}$$

$$= \frac{q^{\binom{\alpha}{2}} E_q(cq^\alpha)}{(1-q)^\alpha} \left\{ \sum_{n=0}^{\lceil\alpha\rceil-1} \frac{q^{\binom{n-\alpha}{2}}(xq^\alpha)^{n-\alpha}(c/x;q)_{n-\alpha}}{(-cq^\alpha;q)_{n-\alpha}\ (q;q)_{n-\alpha}} + E_{q;1,\lceil\alpha\rceil-\alpha+1}(xq^\alpha;cq^\alpha) \right\} .$$

Corollary 4.4. *For $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, the Caputo (C) fractional q–derivatives of the q–exponential functions are:*

$$\begin{aligned} {}_\star D^\alpha_{q,c}(e_q(x)) &= \frac{e_q(c)}{(1-q)^\alpha} \sum_{n=\lceil\alpha\rceil}^{\infty} \frac{x^{n-\alpha}(c/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}} \\ &= \frac{e_q(c)}{(1-q)^\alpha}\ e_{q;1,\lceil\alpha\rceil-\alpha+1}(x;c) , \end{aligned}$$

and

$$\begin{aligned} {}_\star D^\alpha_{q,c}\big(E_q(x)\big) &= \frac{E_q(c)}{(1-q)^\alpha} \sum_{n=\lceil\alpha\rceil}^{\infty} \frac{q^{\binom{n}{2}}}{(-c;q)_n} \frac{x^{n-\alpha}(c/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}} \\ &= \frac{q^{\binom{\alpha}{2}} E_q(cq^\alpha)}{(1-q)^\alpha}\ E_{q;1,\lceil\alpha\rceil-\alpha+1}(xq^\alpha;cq^\alpha) . \end{aligned}$$

From now on, we suppose that $0 < c < x$ and $\alpha, \beta \in \mathbb{R}$.

Theorem 4.5. *For $\alpha, \beta > 0$, the following is true:*

$$\begin{aligned} e_{q;\alpha,\beta}(x;c) &= \frac{x^{\beta-1}(c/x;q)_{\beta-1}}{(q;q)_{\beta-1}} + e_{q;\alpha,\beta+\alpha}(x;c) , \\ E_{q;\alpha,\beta}(x;c) &= \frac{q^{\binom{\beta-1}{2}}}{(-c;q)_{\beta-1}} \frac{x^{\beta-1}(c/x;q)_{\beta-1}}{(q;q)_{\beta-1}} + E_{q;\alpha,\beta+\alpha}(x;c) . \end{aligned}$$

P r o o f. Starting from (14), by shifting of the counter index, we have

$$\begin{aligned} e_{q;\alpha,\beta+\alpha}(x;c) &= \sum_{n=0}^{\infty} \frac{x^{\alpha(n+1)+\beta-1}(c/x;q)_{\alpha(n+1)+\beta-1}}{(q;q)_{\alpha(n+1)+\beta-1}} \\ &= \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}} - \frac{x^{\beta-1}(c/x;q)_{\beta-1}}{(q;q)_{\beta-1}} . \end{aligned}$$

We get the second identity in the same way, starting from (16). ∎

THEOREM 4.6. *For $m \in \mathbb{N}$ and $\beta > m$, the following differentiation formulas hold:*

$$\begin{aligned} D_q^m\big(e_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{-m}\ e_{q;\alpha,\beta-m}(x;c)\ , \\ D_q^m\big(E_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{-m}\ \frac{q^{\binom{m}{2}}}{(-c;q)_m} E_{q;\alpha,\beta-m}(xq^m;cq^m)\ . \end{aligned}$$

The fractional q–integrals of the q–Mittag–Leffler functions are given in the next statement.

THEOREM 4.7. *For $\alpha,\beta,\mu > 0$, the following q–integrals are valid:*

$$\begin{aligned} I_{q,c}^{\mu}\big(e_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{\mu}\ e_{q;\alpha,\beta+\mu}(x;c)\ , \\ I_{q,c}^{\mu}\big(E_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{\mu}\ q^{\binom{\mu+1}{2}}(-cq^{-\mu};q)_{\mu}\ E_{q;\alpha,\beta+\mu}(xq^{-\mu};cq^{-\mu})\ . \end{aligned}$$

The following two special cases are interesting. The second one follows from Theorem 4.5.

COROLLARY 4.8. *For $\alpha,\beta > 0$, the following formulas hold:*

$$\begin{aligned} I_{q,c}^{\beta}\big(e_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{\beta}\ e_{q;\alpha,2\beta}(x;c)\ , \\ I_{q,c}^{\beta}\big(E_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{\beta}\ q^{\binom{\beta+1}{2}}(-cq^{-\beta};q)_{\beta}\ E_{q;\alpha,2\beta}(xq^{-\beta};cq^{-\beta})\ . \end{aligned}$$

COROLLARY 4.9. *For $\alpha,\beta > 0$, the following is true:*

$$I_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(x;c)\big) = (1-q)^{\alpha}\Big(e_{q;\alpha,\beta}(x;c) - \frac{x^{\beta-1}(c/x;q)_{\beta-1}}{(q;q)_{\beta-1}}\Big)\ ,$$

$$\begin{aligned} I_{q,c}^{\alpha}\big(E_{q;\alpha,\beta}(x;c)\big) = {}& (1-q)^{\alpha}\ q^{\binom{\alpha+1}{2}}(-cq^{-\alpha};q)_{\alpha}\ E_{q;\alpha,\beta}(xq^{-\alpha};cq^{-\alpha}) \\ & - (1-q)^{\alpha}\ \frac{q^{\binom{-\alpha+\beta-1}{2}}}{(-c;q)_{-\alpha+\beta-1}}\ \frac{x^{\beta-1}(c/x;q)_{\beta-1}}{(q;q)_{\beta-1}}\ . \end{aligned}$$

COROLLARY 4.10. *For $\alpha,\beta > 0$, the following differentiation formulas are true:*

$$\begin{aligned} D_{q,c}^{\beta}\big(e_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{-\beta}\ e_{q;\alpha,\alpha}(x;c)\ , \\ D_{q,c}^{\beta}\big(E_{q;\alpha,\beta}(x;c)\big) &= (1-q)^{-\beta}\ \frac{q^{\binom{\beta}{2}}}{(-c;q)_{\beta}}\ E_{q;\alpha,\alpha}(xq^{\beta};cq^{\beta})\ . \end{aligned}$$

COROLLARY 4.11. *For* $\alpha,\beta>0$, $\alpha\neq\beta$, *the following is true:*

$$D_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(x;c)\big)=(1-q)^{-\alpha}\left(e_{q;\alpha,\beta}(x;c)+\frac{x^{\beta-\alpha-1}(c/x;q)_{\beta-\alpha-1}}{(q;q)_{\beta-\alpha-1}}\right),$$

$$D_{q,c}^{\alpha}\big(E_{q;\alpha,\beta}(x;c)\big)=(1-q)^{-\alpha}\frac{q^{\binom{\alpha}{2}}}{(-c;q)_{\alpha}}$$
$$\times\left(E_{q;\alpha,\beta}(xq^{\alpha};cq^{\alpha})-\frac{q^{\binom{\beta-\alpha-1}{2}}}{(-cq^{\alpha};q)_{\beta-\alpha-1}}\ \frac{(xq^{\alpha})^{\beta-\alpha-1}(c/x;q)_{\beta-\alpha-1}}{(q;q)_{\beta-\alpha-1}}\right).$$

THEOREM 4.12. *For* $\alpha,\beta>0$, *such* $\alpha-\beta\in\mathbb{N}_0$, *the function* $e_{q;\alpha,\beta}(\tau x;\tau c)$ $(\tau\in\mathbb{R})$ *is an eigenfunction of the operator* $D_{q,c}^{\alpha}$, *i.e.*

$$D_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(\tau x;\tau c)\big)=\frac{\tau^{\alpha}}{(1-q)^{\alpha}}\ e_{q;\alpha,\beta}(\tau x;\tau c)\ .$$

P r o o f. Using (14) and Lemma 2.2, and by shifting of the counter index, we get

$$\begin{aligned}
D_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(\tau x;\tau c)\big)&=\sum_{n=0}^{\infty}\tau^{\alpha n+\beta-1}\ D_{q,c}^{\alpha}\Big(\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}\Big)\\
&=\frac{\tau^{\alpha}}{(1-q)^{\alpha}}\ \sum_{n=1}^{\infty}\tau^{\alpha n+\beta-\alpha-1}\frac{x^{\alpha n+\beta-\alpha-1}(c/x;q)_{\alpha n+\beta-\alpha-1}}{(q;q)_{\alpha n+\beta-\alpha-1}}\\
&=\frac{\tau^{\alpha}}{(1-q)^{\alpha}}\ \sum_{n=0}^{\infty}\frac{(\tau x)^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}\\
&=\frac{\tau^{\alpha}}{(1-q)^{\alpha}}\ e_{q;\alpha,\beta}(\tau x;\tau c)\ .
\end{aligned}$$

∎

THEOREM 4.13. *For* $\alpha,\beta>0$, *such that* $\lceil\alpha\rceil-\beta\in\mathbb{N}_0$, *the function* $e_{q;\alpha,\beta}(\tau x;\tau c)$ $(\tau\in\mathbb{R})$ *is an eigenfunction of the operator* ${}_{\star}D_{q,c}^{\alpha}$, *i.e.*

$${}_{\star}D_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(\tau x;\tau c)\big)=\frac{\tau^{\alpha}}{(1-q)^{\alpha}}\ e_{q;\alpha,\beta}(\tau x;\tau c)\ .$$

P r o o f. By using (14) we get

$${}_{\star}D_{q,c}^{\alpha}\big(e_{q;\alpha,\beta}(\tau x;\tau c)\big)={}_{\star}D_{q,c}^{\alpha}\Big(\sum_{n=0}^{\infty}\frac{(\tau x)^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}\Big)$$
$$=\sum_{n=0}^{\infty}\tau^{\alpha n+\beta-1}\ {}_{\star}D_{q,c}^{\alpha}\Big(\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}\Big)\ .$$

If $\lceil\alpha\rceil - \beta = m \in \mathbb{N}_0$, according to Lemma 3.2, we have that

$$ {}_\star D^{\alpha}_{q,c}\big(x^{\beta-1}(c/x;q)_{\beta-1}\big) = 0 \; . $$

Therefore,

$$ {}_\star D^{\alpha}_{q,c}\big(e_{q;\alpha,\beta}(\tau x;\tau c)\big) = \sum_{n=1}^{\infty} \tau^{\alpha n+\beta-\alpha-1} \; {}_\star D^{\alpha}_{q,c}\Big(\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}\Big) $$

$$ = \frac{\tau^{\alpha}}{(1-q)^{\alpha}} \sum_{n=1}^{\infty} \tau^{\alpha n+\beta-\alpha-1} \; \frac{x^{\alpha n+\beta-\alpha-1}(c/x;q)_{\alpha n+\beta-\alpha-1}}{(q;q)_{\alpha n+\beta-\alpha-1}} $$

$$ = \frac{\tau^{\alpha}}{(1-q)^{\alpha}} \sum_{n=0}^{\infty} \frac{(\tau x)^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}} = \frac{\tau^{\alpha}}{(1-q)^{\alpha}} \; e_{q;\alpha,\beta}(\tau x;\tau c) \; . $$

■

5. From the q–analogues of the Mittag–Leffler function to q–integrals and q–derivatives

Following the notions and considerations in Kiryakova [8, Ch.2, Ch.5] and [9], we can define the fractional q–integral and fractional q–derivative of any function, using the q-analogues of Mittag-Leffler functions as generating functions of some Gelfond-Leontiev (G-L) operators for generalized integration and differentiation.

Let us consider a function $\varphi(x;q)$ of the form

$$ \varphi(x;q) = \sum_{n=0}^{\infty} \varphi_n(q) x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1} \; . \tag{17} $$

By means of its coefficients $\varphi_n(q)$, we can define two G-L operators: of integration $LI^{\alpha,\beta}_{\varphi,q}$ and resp., of differentiation $LD^{\alpha,\beta}_{\varphi,q}$, in the following way: if a function $f(x)$ is given by the expansion

$$ f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1} \, , $$

then

$$ LI^{\alpha,\beta}_{\varphi,q} f(x) := \sum_{n=0}^{\infty} a_n \frac{\varphi_{n+1}(q)}{\varphi_n(q)} x^{\alpha(n+1)+\beta-1}(c/x;q)_{\alpha(n+1)+\beta-1} \; , \tag{18} $$

and

$$LD^{\alpha,\beta}_{\varphi,q} f(x) := \sum_{n=1}^{\infty} a_n \frac{\varphi_{n-1}(q)}{\varphi_n(q)} x^{\alpha(n-1)+\beta-1} (c/x;q)_{\alpha(n-1)+\beta-1} \ . \qquad (19)$$

The following statement is then true.

THEOREM 5.1. *If* $\varphi(x;q) = e_{q;\alpha,1}\big((1-q)x;(1-q)c\big)$, *then* $LI^{\alpha,1}_{\varphi,q}$ *is a fractional* q*–integral operator* $I^{\alpha}_{q,c}$, *and* $LD^{\alpha,1}_{\varphi,q}$ *is a Caputo fractional* q*–derivative operator* ${}_\star D^{\alpha}_{q,c}$.

P r o o f. The expansion of the function $\varphi(x;q) = e_{q;\alpha,1}\big((1-q)x;(1-q)c\big)$ in the form (17) has coefficients

$$\varphi_n(q) = \frac{(1-q)^{\alpha n}}{(q;q)_{\alpha n}} \qquad (n \in \mathbb{N}_0) \ .$$

Hence, for

$$f(x) = \sum_{n=0}^{\infty} a_n \ x^{\alpha n} (c/x;q)_{\alpha n} \ ,$$

we have

$$\begin{aligned} LI^{\alpha,1}_{\varphi,q} f(x) &= \sum_{n=0}^{\infty} a_n \frac{(1-q)^{\alpha(n+1)}}{(q;q)_{\alpha(n+1)}} \ \frac{(q;q)_{\alpha n}}{(1-q)^{\alpha n}} \ x^{\alpha(n+1)} (c/x;q)_{\alpha(n+1)} \\ &= (1-q)^{\alpha} \sum_{n=0}^{\infty} a_n \ \frac{(q;q)_{\alpha n}}{(q;q)_{\alpha(n+1)}} \ x^{\alpha(n+1)} (c/x;q)_{\alpha(n+1)} \\ &= \sum_{n=0}^{\infty} a_n \ I^{\alpha}_{q,c}\big(x^{\alpha n}(c/x;q)_{\alpha n}\big) = I^{\alpha}_{q,c} f(x) \ . \end{aligned}$$

Similarly,

$$\begin{aligned} LD^{\alpha,1}_{\varphi,q} f(x) &= \sum_{n=1}^{\infty} a_n \frac{(1-q)^{\alpha(n-1)}}{(q;q)_{\alpha(n-1)}} \ \frac{(q;q)_{\alpha n}}{(1-q)^{\alpha n}} \ x^{\alpha(n-1)} (c/x;q)_{\alpha(n-1)} \\ &= (1-q)^{-\alpha} \sum_{n=1}^{\infty} a_n \frac{(q;q)_{\alpha n}}{(q;q)_{\alpha(n-1)}} \ x^{\alpha(n-1)} (c/x;q)_{\alpha(n-1)} \\ &= \sum_{n=1}^{\infty} a_n \ {}_\star D^{\alpha}_{q,c}\big(x^{\alpha n}(c/x;q)_{\alpha n}\big) \\ &= {}_\star D^{\alpha}_{q,c} \left(\sum_{n=0}^{\infty} a_n x^{\alpha n} (c/x;q)_{\alpha n} \right) = {}_\star D^{\alpha}_{q,c} f(x) \ . \end{aligned}$$

■

Example 5.1. Especially, if $\alpha = 1$, the operators $LI^{\alpha,1}_{\varphi,q}$ and $LD^{\alpha,1}_{\varphi,q}$ become fractional q–integral operator $I_{q,c}$ and fractional q–derivative operator D_q, respectively.

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References

[1] R.P. Agarwal, Certain fractional q–integrals and q–derivatives. *Proc. Camb. Phil. Soc.* **66** (1969), 365-370.

[2] W.A. Al-Salam, Some fractional q–integrals and q–derivatives. *Proc. Edinburgh Math. Soc.* **15** (1966), 135-140.

[3] W.A. Al-Salam, q–Analogues of Cauchy's Formulas. *Proc. Amer. Math. Soc.* **17**, No. 3 (1966), 616-621.

[4] W.A. Al-Salam, A. Verma, A fractional Leibniz q–formula. *Pacific J. of Mathematics* **60**, No 2 (1975), 1-9.

[5] G. Bangerezako, Variational calculus on q–nonuniform lattices. *J. Math. Anal. and Appl.* **306**, No 1 (2005), 161-179.

[6] G. Gasper, M. Rahman, *Basic Hypergeometric Series, 2nd Ed*, Encyclopedia of Math. and its Appl. **96**. Cambridge University Press, Cambridge (2004).

[7] H. Gauchman, Integral inequalities in q–calculus. *Computers and Math. with Appl.* **47** (2004), 281-300.

[8] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman (Pitman Res. Notes in Math. Ser. **301**), Harlow & J. Wiley Ltd., N. York (1994).

[9] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comp. Appl. Math.* (Special Issue: "Higher Transcendental Functions and Their Applications") **118** (2000), 214-259.

[10] R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order. In: A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Wien (1997), 223276 (Available at http://www.fracalmo.org).

[11] G.M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris* **137** (1903), 554558.

[12] I. Podlubny, *Fractional Differential Equations (An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications).* Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto (1999).

[13] P.M. Rajković, S.D. Marinković, M.S. Stanković, Fractional integrals and derivatives in q–calculus. *Applicable Anal. and Discrete Mathematics* **1** (2007), 311-323.

[14] M.S. Stanković, P.M. Rajković, S.D. Marinković, On q–fractional derivatives of Riemann–Liouville and Caputo type. *To appear.*

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