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GENERALIZED CONVOLUTION TRANSFORMS AND TOEPLITZ PLUS HANKEL INTEGRAL EQUATIONS

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Dedicated to Professor S.L. Kalla
on the occasion of his 70th birthday

Abstract

We study integral transforms of the form

$$g(x) = \int_0^\infty K_1(u)[f(|x+u-1|) + f(|x-u-1|) - f(x+u+1) - f(|x-u+1|)]du + \int_0^\infty K_2(u)[f(|x-u|) - f(x+u)]du$$

from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$, $(1 \leqslant p \leqslant 2,\ p^{-1}+q^{-1}=1)$ with the help of a generalized convolution and prove Watson's and Plancherel's theorems. Using generalized convolutions a class of Toeplitz plus Hankel integral equations, and also a system of integro-differential equations are solved in closed form.

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Key Words and Phrases: convolution, Fourier sine transform, Fourier cosine transform, Toeplitz plus Hankel integral equation, Plancherel theorem

1. Introduction

The integral equation with the Toeplitz plus Hankel kernel is of the form ([8], [9], [19])

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)]f(y)dy = g(x). \tag{1.1}$$

This equation has many useful applications, see [8], [19]. However, this integral equation can be solved in closed form only in some particular cases of the Hankel kernel k_1 and the Toeplitz kernel k_2. The solution in closed form in general case is still open.

Let F_c and F_s denote the Fourier cosine and Fourier sine transforms [2]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y) dy, \quad (F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y) dy,$$

in case $f \in L_1(\mathbb{R}_+)$, and

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty f(y) \frac{\sin xy}{y} dy, \quad (F_s f)(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty f(y) \frac{1-\cos xy}{y} dy,$$

in case $f \in L_2(\mathbb{R}_+)$. These two definitions are identical, if $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$.

In 1941, R.V. Churchill introduced the convolution of two functions f and g for the Fourier cosine transform [12]

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[g(|x-u|) + g(x+u)] du, \quad x > 0, \tag{1.2}$$

which satisfies the following factorization equality

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y), \qquad \forall y > 0. \tag{1.3}$$

Using this factorization property one can easily solve the integral equation with the Toeplitz plus Hankel kernel (1.1) in case the Toeplitz kernel $k_2(x)$ and the Hankel kernel $k_1(x)$ are equal $k_2(x) = k_1(|x|)$ (see [8], [9] for other methods).

Churchill was also the first author who introduced the convolution for two different integral transforms. Namely, in 1941 he defined the convolution of two functions f and g for the Fourier sine and Fourier cosine transforms ([12])

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[g(|x-u|) - g(x+u)] du, \quad x > 0, \tag{1.4}$$

and showed that this convolution satisfy the following factorization identity (see [12])

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y).(F_c g)(y), \quad \forall y > 0. \tag{1.5}$$

In this factorization equality, there are more than one integral transform (F_c and F_s). Convolutions of this kind are called generalized convolutions (see [6]).

So the Toeplitz plus Hankel integral equation (1.1) with $k_2(x) = -k_1(|x|)$ can be solved easily with the help of factorization equality (1.5). In fact, equation (1.1) can be rewritten in the form

$$f(x) + \sqrt{2\pi}(f \underset{F_c}{*} h_1)(x) + \sqrt{2\pi}(f \underset{1}{*} h_2)(x) = g(x),$$

where $h_1(x) = \frac{1}{2}(k_1 + k_2)(x)$ and $h_2(x) = \frac{1}{2}(k_2 - k_1)(x)$. From $k_2(x) = -k_1(|x|)$ we see that $h_1 = 0$, and therefore (1.5) can be applied.

In 1998, Kakichev and Nguyen Xuan Thao proposed a constructive method for defining a generalized convolution for three arbitrary integral transforms (see [6]). For instance, the generalized convolution of two functions f and g for the Fourier cosine and sine transforms has been obtained based on this method [7]

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[\mathrm{sign}(u-x)g(|u-x|) + g(u+x)]du, \ x > 0. \tag{1.6}$$

For this generalized convolution the following factorization equality holds ([7]):

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \ \forall y > 0. \tag{1.7}$$

With $k_2(x) = \mathrm{sign}\, x.k_1(|x|)$, the integrand of integral equation (1.1) will have the form as in (1.6).

The following generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier cosine and Fourier sine transforms has been studied in [16]:

$$\begin{aligned}(f \overset{\gamma}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty & f(u)[g(|x+u-1|) + g(|x-u+1|) \\ & - g(x+u+1) - g(|x-u-1|)]du, \quad x > 0.\end{aligned} \tag{1.8}$$

It satisfies the factorization property ([16])

$$F_c(f \overset{\gamma}{\underset{2}{*}} g)(y) = \sin y(F_s f)(y)(F_c g)(y), \ \forall y > 0. \tag{1.9}$$

Interchanging the variables one can easily see the generalized convolution (1.8) becomes

$$(f \overset{\gamma}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty g(u)[f(x+u+1)+\text{sign}(x-u+1)f(|x-u+1|) \quad (1.10)$$

$$- \text{sign}(x+u-1)f(|x+u-1|) - \text{sign}(x-u-1)f(|x-u-1|)]du, \quad x>0.$$

Thus one can easily solve the Toeplitz plus Hankel integral equation (1.1) when the kernel is defined by

$$k_1(x) = k(x+1) - \text{sign}(x-1)k(|x-1|) \qquad \text{and}$$

$$k_2(x) = \text{sign}(x+1)k(|x+1|) - \text{sign}(x-1)k(|x-1|)$$

for some k. So studying generalized convolutions may extend the class of integral equations with Toeplitz plus Hankel kernel (1.1) that can be solved in a closed form.

Another generalized convolution of two functions f and g with the weight function $\gamma(y) = \sin y$ for the Fourier sine and cosine transforms has the form

$$(f \overset{\gamma}{\underset{1}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) \tag{1.11}$$

$$- g(x+u+1) - g(|x-u+1|)]du, \quad x>0$$

for which the following factorization identity holds ([15]):

$$F_s(f \overset{\gamma}{\underset{1}{*}} g)(y) = \sin y (F_c f)(y)(F_c g)(y), \ \ \forall y > 0. \tag{1.12}$$

In any convolution of two functions f and g, if we fix one function, say g, as the kernel, and allow the other function f to vary in a certain function space, we get an integral transform of convolution type. The most famous integral transforms constructed in that way are the Watson transforms that are related to the Mellin convolution and the Mellin transform ([17]):

$$f(x) \longmapsto g(x) = \int_0^\infty k(xy)f(y)dy.$$

Recently, several classes of integral transforms that are related to generalized convolutions (1.4), (1.6) have been investigated in [11], [10]. In this paper we consider a class of integral transform which has a connection with the generalized convolution (1.11), namely, the transforms of the form

$$g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int\limits_0^\infty k_1(u)[f(|x+u-1|) + f(|x-u-1|) - f(x+u+1)$$
$$- f(|x-u+1|)]du + \int\limits_0^\infty k_2(u)[f(|x-u|) - f(x+u)]du\Big\}, \quad x > 0. \quad (1.13)$$

We obtain necessary and sufficient conditions on the functions $k_1, k_2 \in L_2(\mathbb{R}_+)$ to ensure that the transformation (1.13) is unitary on $L_2(\mathbb{R}_+)$, and define the inverse transformation. The Plancherel type theorem is also obtained. Furthermore, the boundedness of the transformation (1.13) on $L_p(\mathbb{R}_+)$ for $1 \leqslant p \leqslant 2$ is studied. Next, we give several examples of these transformations and apply on solving a system of integro-differential equations. In the last section we consider a class of integral equation with the Toeplitz plus Hankel kernel that can be solved in closed form with the help of generalized convolutions.

2. A Watson type theorem

LEMMA 1. *Let $f, g \in L_2(\mathbb{R}_+)$. Then for any $x > 0$ the following Parseval formulas hold:*

$$\int\limits_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)]du$$
$$= 2\sqrt{2\pi} F_s\big(\sin y (F_c f)(y)(F_c g)(y)\big)(x), \quad (2.1)$$

and

$$\int\limits_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du$$
$$= 2\sqrt{2\pi} F_c\big(\sin y (F_s f)(y)(F_c g)(y)\big)(x). \quad (2.2)$$

P r o o f. We will prove only formula (2.1), as the proof of formula (2.2) is quite similar. Let f_1 and g_1 be even extensions of f and g from $\mathbb{R}_+$ to $\mathbb{R}$. Then $Ff_1 = F_c f$ and $Fg_1 = F_c g$, where F stands for the Fourier transform

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y)dy.$$

For the Fourier transform we have the well-known Parseval identity

$$\int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} (Ff)(y)(Fg)(y)e^{ixy}dy.$$

We have

$$\int_0^\infty f(u)[g(|x+u-1|)+g(|x-u-1|)-g(x+u+1)-g(|x-u+1|)]du$$

$$= \int_0^\infty f_1(u)g_1(x+u-1)du + \int_0^\infty f_1(u)g_1(x-u-1)du$$

$$- \int_0^\infty f_1(u)g_1(x+u+1)du - \int_0^\infty f_1(u)g_1(x-u+1)du$$

$$= \int_{-\infty}^\infty f_1(u)g_1(x-u-1)du - \int_{-\infty}^\infty f_1(u)g_1(x-u+1)du$$

$$= \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)e^{i(x-1)y}dy - \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)e^{i(x+1)y}dy$$

$$= \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)[\cos(x-1)y + i\sin(x-1)y]dy$$

$$- \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)[\cos(x+1)y + i\sin(x+1)y]dy.$$

On the other hand, note that

$$(Ff_1)(y)(Fg_1)(y)\sin(x-1)y \quad \text{and} \quad (Ff_1)(y)(Fg_1)(y)\sin(x+1)y$$

are odd functions in y. Hence, their integrals over $\mathbb{R}$ vanish, and therefore

$$\int_0^\infty f(u)[g(|x+u-1|)+g(|x-u-1|)-g(x+u+1)-g(|x-u+1|)]du$$

$$= \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)\cos(x-1)ydy - \int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)\cos(x+1)ydy$$

$$= 2\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)\sin y\sin(xy)dy = 4\int_0^\infty (Ff)(y)(Fg)(y)\sin y\sin(xy)dy$$

$$= 2\sqrt{2\pi}F_s\big(\sin y(F_cf)(y)(F_cg)(y)\big)(x).$$

This completes the proof. We have assumed that all the integrals over $\mathbb{R}$ are interpreted as Cauchy principal value integrals, if necessary. ∎

THEOREM 1. *Let $k_1, k_2 \in L_2(\mathbb{R}_+)$. Then*

$$|2\sin y(F_ck_1)(y) + (F_sk_2)(y)| = \frac{1}{\sqrt{2\pi}(1+y^2)} \tag{2.3}$$

is a necessary and sufficient condition to ensure that the integral transform $f \longmapsto g$:

$$g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{\int_0^\infty k_1(u)[f(|x+u-1|) + f(|x-u-1|) - f(x+u+1)$$
$$- f(|x-u+1|)]du + \int_0^\infty k_2(u)[f(|x-u|) - f(x+u)]du\Big\} \tag{2.4}$$

is unitary on $L_2(\mathbb{R}_+)$ and the reciprocal transform has the form

$$f(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{\int_0^\infty g(u)[\overline{k}_1(|x+u-1|) + \overline{k}_1(|x-u+1|) - \overline{k}_1(x+u+1)$$
$$- \overline{k}_1(|x-u-1|)]du + \int_0^\infty g(u)[\operatorname{sign}(u-x)\overline{k}_2(|u-x|) + \overline{k}_2(u+x)]du\Big\}. \tag{2.5}$$

P r o o f. *Necessity.* Suppose that k_1 and k_2 satisfy condition (2.3). It is well-known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R}_+)$ if and only if $(Fh)(x)$, $\frac{d}{dx}(Fh)(x)$, $\frac{d^2}{dx^2}(Fh)(x) \in L_2(\mathbb{R}_+)$ (Theorem 68, [17]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = F\big((-iy)^2h(y)\big)(x).$$

In particular, in case h is an even or odd function such that $(1+y^2)h(y) \in L_2(\mathbb{R}_+)$, the following equalities hold

$$\Big(1 - \frac{d^2}{dx^2}\Big)(F_ch)(x) = F_c\big((1+y^2)h(y)\big)(x),$$
$$\Big(1 - \frac{d^2}{dx^2}\Big)(F_sh)(x) = F_s\big((1+y^2)h(y)\big)(x). \tag{2.6}$$

Using Lemma 1 and the factorization equalities for the generalized convolutions (1.4), (1.8) we have

$$g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)F_s\big(2\sqrt{2\pi}\sin y(F_ck_1)(y)(F_cf)(y) + \sqrt{2\pi}(F_sk_2)(y)(F_cf)(y)\big)(x)$$
$$= F_s\Big(\sqrt{2\pi}(1+y^2)\big(2\sin y(F_ck_1)(y) + (F_sk_2)(y)\big)(F_cf)(y)\Big)(x).$$

By virtue of the Parseval identity for the Fourier cosine and sine transforms $\|f\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)}$ and from condition (2.3) we get

$$\|g\|_{L_2(\mathbb{R}_+)} = \Big\| \sqrt{2\pi}(1+y^2)\big[2\sin y(F_c k_1)(y) + (F_s k_2)(y)\big](F_c f)(y)\Big\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

It shows that the transformation (2.4) is unitary.

On the other hand, formula (2.3) implies that $(1+y^2)\big(2\sin y(F_c k_1)(y) + (F_s k_2)(y)\big)$ is bounded on $\mathbb{R}_+$, hence $(1+y^2)\big[2\sin y(F_c k_1)(y) + (F_s k_2)(y)\big]$ $(F_c f)(y) \in L_2(\mathbb{R}_+)$. We have

$$(F_s g)(y) = \sqrt{2\pi}(1+y^2)\big[2\sin y(F_c k_1)(y) + (F_s k_2)(y)\big](F_c f)(y).$$

Using condition (2.3) we obtain

$$(F_s f)(y) = \sqrt{2\pi}(1+y^2)[2\sin y(F_c \overline{k_1})(y) + (F_s \overline{k_2})(y)](F_c g)(y).$$

Again, condition (2.3) shows that $(1+y^2)[2\sin y(F_c \overline{k_1})(y) + (F_s \overline{k_2})(y)](F_c g)(y) \in L_2(\mathbb{R}_+)$, then formula (2.6) yields

$$f(x) = F_c\big((1+y^2)[2\sin y(F_c \overline{k_1})(y) + (F_s \overline{k_2})(y)](F_c g)(y)\big)(x)$$
$$= \Big(1 - \frac{d^2}{dx^2}\Big) F_c\big(2\sqrt{2\pi}\sin y(F_c \overline{k}_1)(y)(F_s g)(y) + \sqrt{2\pi}(F_s \overline{k}_2)(y)(F_s g)(y)\big)(x).$$

Using formula (2.2) and the factorization equality for the generalized convolution (1.6) we have

$$f(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{\int_0^\infty g(u)[\overline{k}_1(|x+u-1|) + \overline{k}_1(|x-u+1|) - \overline{k}_1(x+u+1) - \overline{k}_1(|x-u-1|)]du + \int_0^\infty g(u)[\mathrm{sign}(u-x)\overline{k}_2(|u-x|) + \overline{k}_2(u+x)]du\Big\}.$$

Therefore the transformation (2.4) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation is defined by (2.5).

Sufficiency. If transform (2.4) is unitary on $\mathbb{R}_+$, then the Parseval identities for the Fourier sine and cosine transforms yield

$$\|g\|_{L_2(\mathbb{R}_+)} = \|\sqrt{2\pi}(1+y^2)[2\sin y(F_c k_1)(y) + (F_s k_2)(y)](F_c f)(y)\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

By virtue of the Hahn-Banach Theorem, the middle equality hold for all $f \in L_2(\mathbb{R}_+)$ if and only if

$$|\sqrt{2\pi}(1+y^2)[2\sin y(F_c k_1)(y) + (F_s k_2)(y)](F_c f)(y)| = |(F_c f)(y)|.$$

It shows that k_1 and k_2 satisfy condition (2.3). This completes the proof of Theorem 1. ∎

We show now the existence of functions k_1 and k_2 satisfying (2.3). Let $h_1, h_2 \in L_2(\mathbb{R}_+)$ satisfy

$$|(F_s h_1)(y)(F_c h_2)(y)| = \frac{1}{(1+y^2)(1+\sin^2 y)}. \tag{2.7}$$

The existence of such functions h_1, h_2 that satisfy (2.7) is clear. For instance,

$$h_1(x)=F_s\Big(\frac{e^{iv_1(y)}}{\sqrt{(1+y^2)(1+\sin^2 y)}}\Big)(x),\ \ h_1(x)=F_c\Big(\frac{e^{iv_2(y)}}{\sqrt{(1+y^2)(1+\sin^2 y)}}\Big)(x),$$

where v_1, v_2 are arbitrary real-valued functions defined on $\mathbb{R}_+$. Let $k_1, k_2 \in L_2(\mathbb{R}_+)$ be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \overset{\gamma}{\underset{2}{*}} h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 \underset{1}{*} h_2)(x).$$

We have

$$\begin{aligned} &|2\sin y(F_c k_1)(y) + (F_s k_2)(y)| \\ &= |\frac{2}{2\sqrt{2\pi}}\sin^2 y(F_s h_1)(y)(F_c h_2)(y) + \frac{1}{\sqrt{2\pi}}(F_s h_1)(y)(F_c h_2)(y)| \\ &= \frac{1}{\sqrt{2\pi}}|(F_s h_1)(y)(F_c h_2)(y)|(1+\sin^2 y) = \frac{1}{\sqrt{2\pi}(1+y^2)}. \end{aligned}$$

Thus k_1 and k_2 satisfy condition (2.3).

3. A Plancherel type theorem

THEOREM 3. *Let k_1 and k_2 be functions satisfying condition* (2.3) *and suppose that $K_1(x) = \left(1 - \frac{d^2}{dx^2}\right)k_1(x)$ and $K_2(x) = \left(1 - \frac{d^2}{dx^2}\right)k_2(x)$ are locally bounded. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer N, put*

$$\begin{aligned} g_N(x) = &\int_0^\infty K_1(u)[f^N(|x+u-1|) + f^N(|x-u-1|) - f^N(x+u+1) \\ &- f^N(|x-u+1|)]du + \int_0^\infty K_2(u)[f^N(|x-u|) - f^N(x+u)]du, \end{aligned} \tag{3.1}$$

where $f^N = f.\chi_{(0,N)}$, the restriction of f over $(0,N)$. Then:

1) $g_N \in L_2(\mathbb{R}_+)$ and as $N \to \infty$, g_N converges in $L_2(\mathbb{R}_+)$ norm to a function $g \in L_2(\mathbb{R}_+)$, moreover, $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$.

2) Reciprocally,

$$f_N(x) = \int_0^N g(u)[\overline{K}_1(|x+u-1|) + \overline{K}_1(|x-u+1|) - \overline{K}_1(x+u+1)$$

$$-\overline{K}_1(|x-u-1|)]du + \int_0^N g(u)[\text{sign}(u-x)\overline{K}_2(|u-x|) + \overline{K}_2(u+x)]du \quad (3.2)$$

belongs to $L_2(\mathbb{R}_+)$ *and converges in* $L_2(\mathbb{R}_+)$ *norm to* f *as* $N \to \infty$.

Remark 1. Because of the definitions of g_N and f_N these integrals are over finite intervals and therefore converge.

Remark 2. The convolution (1.4) can be rewritten as follows

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(u)[\text{sign}(x-u)f(|x-u|) + f(x+u)]du.$$

Moreover, by the commutativity of the generalized convolution (1.11) (see [15]), we can interchange f and g without changing the value of the generalized convolution,

$$\int_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) - g(|x-u+1|)]du$$

$$= \int_0^\infty g(u)[f(|x+u-1|) + f(|x-u-1|) - f(x+u+1) - f(|x-u+1|)]du.$$

Proof of Theorem 2. Applying Remark 2, we have

$$g_N(x) = \int_0^\infty f^N(u)[K_1(|x+u-1|) + K_1(|x-u-1|) - K_1(x+u+1)$$

$$- K_1(|x-u+1|)]du + \int_0^\infty f^N(u)[\text{sign}(x-u)K_2(|x-u|) + K_2(x+u)]du$$

$$= \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int_0^\infty f^N(u)[k_1(|x+u-1|) + k_1(|x-u-1|) - k_1(x+u+1)$$

$$-k_1(|x-u+1|)]du + \int_0^\infty f^N(u)[\text{sign}(x-u)k_2(|x-u|) + k_2(x+u)]du\Big\}.$$

It is legitimate to interchange the order of integration and differentiation since the integrals are actually over finite intervals. By applying Remark 2 one more time, we obtain

$$g_N(x)=\Big(1-\frac{d^2}{dx^2}\Big)\Big\{\int\limits_0^\infty k_1(u)[f^N(|x+u-1|)+f^N(|x-u-1|)-f^N(x+u+1)$$

$$-f^N(|x-u+1|)]du+\int\limits_0^\infty k_2(u)[\text{sign}(x-u)f^N(|x-u|)+f^N(x+u)]du\Big\}.$$

From this and in view of Theorem 1, we conclude that $g_N \in L_2(\mathbb{R}_+)$. Let g be the transform of f under the transformation (2.4). Then Theorem 1 implies that $g \in L_2(\mathbb{R}_+)$, and $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$. Furthermore, the reciprocal formula (2.5) holds. We have

$$(g-g_N)(x) = \Big(1-\frac{d^2}{dx^2}\Big)\Big\{\int\limits_0^\infty k_1(u)\big[(f-f^N)(|x+u-1|)+(f-f^N)(|x-u-1|)$$

$$-(f-f^N)(x+u+1)-(f-f^N)(|x-u+1|)\big]du$$

$$+\int\limits_0^\infty k_2(u)[\text{sign}(x-u)(f-f^N)(|x-u|)+(f-f^N)(x+u)]du\Big\}.$$

Again by Theorem 1, $g-g_N \in L_2(\mathbb{R}_+)$, and

$$\|g-g_N\|_{L_2(\mathbb{R}_+)} = \|f-f^N\|_{L_2(\mathbb{R}_+)}.$$

Since $\|g-g_N\|_{L_2(\mathbb{R}_+)} \to \infty$ as $N \to \infty$, then g_N converges in $L_2(\mathbb{R}_+)$ norm to g as $N \to \infty$.

The second part of the theorem can be obtained by the similar way. ■

Remark 3. Theorem 1 and Theorem 2 show that the integral transform (2.4) is unitary in $L_2(\mathbb{R}_+)$ and the inverse transform is defined by formula (2.5). Moreover, integral operators (2.4) and (2.5) can be approximated in $L_2(\mathbb{R}_+)$ norm by operators (3.1) and (3.2), respectively.

If we assume in addition that $K_1(x) = \Big(1-\frac{d^2}{dx^2}\Big)k_1(x)$ and $K_2(x) = \Big(1-\frac{d^2}{dx^2}\Big)k_2(x)$ are bounded on $\mathbb{R}_+$, then the transformations (2.4) and (2.5) are bounded operators from $L_1(\mathbb{R}_+)$ into $L_\infty(\mathbb{R}_+)$.

On the other hand, Theorem 2 implies that the transformations (2.4) and (2.5) are bounded on $L_2(\mathbb{R}_+)$. Then the Riesz's interpolation theorem yields

Theorem 3. *Let k_1, k_2 be functions satisfying condition* (2.3) *and suppose that $K_1(x)$ and $K_2(x)$, defined as in Theorem 2, are bounded on $\mathbb{R}_+$. Let $1 \leqslant p \leqslant 2$ and q be its conjugate exponent ($\frac{1}{p} + \frac{1}{q} = 1$). Then the transformations*

$$f(x) \longmapsto g(x) = \lim_{N\to\infty} \Big\{ \int_0^\infty K_1(u)[f^N(|x+u-1|) + f^N(|x-u-1|) - f^N(x+u+1)$$

$$- f^N(|x-u+1|)]du + \int_0^\infty K_2(u)[f^N(|x-u|) - f^N(x+u)]du \Big\} \tag{3.3}$$

and

$$f(x) \longmapsto g(x) = \lim_{N\to\infty} \Big\{ \int_0^N f(u)[\overline{K}_1(|x+u-1|) + \overline{K}_1(|x-u+1|) - \overline{K}_1(x+u+1)$$

$$-\overline{K}_1(|x-u-1|)]du + \int_0^N f(u)[\mathrm{sign}(u-x)\overline{K}_2(|u-x|) + \overline{K}_2(u+x)]du \Big\} \tag{3.4}$$

are bounded operators from $L_p(\mathbb{R}_+)$ into $L_q(\mathbb{R}_+)$. Here the limits are understood in $L_q(\mathbb{R}_+)$ norm.

4. Examples

Now we consider some examples of k_1 and k_2 for which the condition (2.3) holds.

Example 1. It is obvious that the kernels k_1, k_2 defined as follows satisfy condition (2.3):

$$F_c k_1(y) = \frac{\sin y}{2\sqrt{2\pi}(1+y^2)}, \qquad F_s k_2(y) = \frac{\cos^2 y}{\sqrt{2\pi}(1+y^2)}.$$

Then

$$k_1(x) = F_c\Big(\frac{\sin y}{2\sqrt{2\pi}(1+y^2)}\Big)(x)$$
$$= \frac{1}{2\pi}\int_0^\infty \frac{\sin y \cos xy}{(1+y^2)}dy = \frac{1}{4\pi}\int_0^\infty \frac{\sin(x+1)y - \sin(x-1)y}{(1+y^2)}dy.$$

Using formula (2.2.14) from [4], we get

$$k_1(x) = \frac{1}{8}\big[e^{-(x+1)}\overline{Ei}(x+1) - e^{(x+1)}Ei(-x-1) \tag{4.1}$$

$$-e^{-(x-1)}\overline{Ei}(x-1) - e^{(x-1)}Ei(-x+1)\big],$$

where $Ei(x)$ is the exponential integral (formula 5.1.2, [1])

$$Ei(x) = \int_{-x}^{\infty} \frac{e^{-t}}{t}dt = \int_{-\infty}^{x} \frac{e^{t}}{t}dt, \tag{4.2}$$

and the integral here is understood as the Cauchy principle value integral.

On the other hand,

$$k_2(x) = F_s\Big(\frac{\cos^2 y}{\sqrt{2\pi}(1+y^2)}\Big)(x) = \frac{1}{\pi}\int\limits_0^\infty \frac{(1+\cos 2y)\sin xy}{2(1+y^2)}dy$$

$$= \frac{1}{4\pi}\int\limits_0^\infty \frac{2\sin xy + \sin(x+2)y + \sin(x-2)y}{(1+y^2)}dy.$$

Using formula (1.2.11) from [4], we have

$$k_2(x) = \frac{1}{8}\big[2e^{-x}\overline{Ei}(x) - 2e^{(x)}Ei(-x) + e^{-(x+2)}\overline{Ei}(x+2) \tag{4.3}$$

$$-e^{(x+2)}Ei(-x-2) + e^{-(x-2)}\overline{Ei}(x+2) - e^{(x+1)}Ei(-x-2)\big].$$

Following formula (5.1.10) from [1], K_1 and K_2 defined as in Theorem 2 are obviously locally bounded, so Theorem 1 shows that the reciprocal transformations (2.4) and (2.5) with functions k_1 and k_2 defined by (4.1) and (4.3) are unitary on $L_2(\mathbb{R}_+)$, and by Theorem 2, they can be approximated by sequences of operators defined by (3.1), (3.2).

Example 2. Let

$$(F_c k_1)(y) = \frac{i\cos y}{\sqrt{2\pi}(1+y^2)} \qquad \text{and} \qquad (F_s k_2)(y) = \frac{\cos 2y}{\sqrt{2\pi}(1+y^2)}.$$

One can check easily that k_1 and k_2 defined as above satisfy condition (2.3), since

$$|2\sin y(F_c k_1)(y) + (F_s k_2)(y)| = \Big|\frac{\cos 2y + i\sin 2y}{\sqrt{2\pi}(1+y^2)}\Big|$$

$$= \Big|\frac{e^{i2y}}{\sqrt{2\pi}(1+y^2)}\Big| = \frac{1}{\sqrt{2\pi}(1+y^2)}.$$

Since $(F_c k_1)(y)$ and $(F_s k_2)(y)$ are functions in $L_2(\mathbb{R}_+)$, we have

$$k_1(x) = F_c\Big(\frac{i\cos y}{\sqrt{2\pi}(1+y^2)}\Big) = \frac{i}{\pi}\int\limits_0^\infty \frac{\cos y\cos xy}{1+y^2}dy$$

$$= \frac{i}{2\pi}\int_0^\infty \frac{\cos y(x+1)+\cos y(x-1)}{1+y^2}dy.$$

Using formula 1.2.11 in [4], we get

$$k_1(x) = \frac{i}{4}(e^{-(x+1)} + e^{-(x-1)}). \tag{4.4}$$

On the other hand,

$$k_2(x) = F_s\Big(\frac{\cos 2y}{\sqrt{2\pi}(1+y^2)}\Big) = \frac{1}{\pi}\int_0^\infty \frac{\cos 2y \sin xy}{1+y^2}dy$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\sin y(x+2)+\sin y(x-2)}{1+y^2}dy.$$

From formula 2.2.14 in [4], we have

$$k_2(x) = \frac{1}{4\pi}\big[e^{-(x+2)}\overline{Ei}(x+2) - e^{(x+2)}Ei(-x-2) + e^{-(x-2)}\overline{Ei}(x-2) - e^{(x-2)}Ei(-x+2)\big]. \tag{4.5}$$

So, Theorem 1 shows that the transformations (2.4) and (2.5) with functions (4.4) and (4.5) define unitary operators on $L_2(\mathbb{R}_+)$. Theorem 2 shows that one can approximate them in $L_2(\mathbb{R}_+)$ norm by sequences of operators (3.1), (3.2).

Example 3. Finally, let us choose

$$(F_ck_1)(y) = \frac{1}{2\sqrt{2\pi}(1+y^2)} \qquad \text{and} \qquad (F_sk_2)(y) = \frac{\cos y}{\sqrt{2\pi}(1+y^2)}.$$

Obviously, the condition (2.3) holds with these functions. Moreover, formula 1.2.11 in [4] give us

$$k_1(x) = F_c\Big(\frac{1}{2\sqrt{2\pi}(1+y^2)}\Big) = \frac{i}{2\pi}\int_0^\infty \frac{\cos xy}{1+y^2}dy = \frac{i}{4}e^{-x}. \tag{4.6}$$

And following formula 2.2.14 in [4], we have

$$k_2(x) = F_s\Big(\frac{\cos y}{\sqrt{2\pi}(1+y^2)}\Big) = \frac{1}{2\pi}\int_0^\infty \frac{\sin y(x+1)+\sin y(x-1)}{1+y^2}dy \tag{4.7}$$

$$= \frac{1}{4\pi}\big[e^{-(x+1)}\overline{Ei}(x+1) - e^{(x+1)}Ei(-x-1) + e^{-(x-1)}\overline{Ei}(x-1) - e^{(x-1)}Ei(-x+1)\big].$$

Transformations (2.4) and (2.5) with functions (4.6) and (4.7) are unitary on $L_2(\mathbb{R}_+)$ and can be approximated by operators defined as in Theorem 2.

5. Application to system of integro-differential equations

Consider the system of integro-differential equations

$$
\begin{aligned}
f(x) - \frac{d^2}{dx^2} f(x) + \lambda_1 (K_{\varphi,\psi} g)(x) &= h(x), \\
\lambda_2 \int_0^\infty f(u)\theta_1(x,u)du + \lambda_3 \int_0^\infty f(u)\theta_2(x,u)du + g(x) &= k(x),
\end{aligned} \tag{5.1}
$$

where $K_{\varphi,\psi}(.)$ is the transformation (2.4) with $k_1 \equiv \varphi \equiv \varphi_1 \underset{2}{*} \varphi_2$; $k_2 \equiv \psi$;

$$\theta_1(x,u) = \operatorname{sign}(u-x)\xi(|u-x|) + \xi(u+x),$$

$$\theta_2(x,u) = \eta(|x+u-1|) + \eta(|x-u+1|) - \eta(x+u+1) - \eta(|x-u-1|);$$

$\varphi_1, \varphi_2, \psi, \xi, \mu \in L_1(\mathbb{R}_+)$ are given function; f and g are unknown functions such that $f', f'', g', g'' \in L_1(\mathbb{R}_+)$ and $f(0) = 0$.

Since $f, f' \in L_1(\mathbb{R}_+)$, then there exist the Fourier sine and Fourier cosine transforms of f, f'. Furthermore,

$$
\begin{aligned}
(F_s f')(y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty f'(x) \sin xy dx \\
&= \frac{1}{\sqrt{2\pi}} \Big\{ f(x) \sin xy \Big|_0^\infty - y \int_0^\infty f(x) \cos xy dx \Big\} = -y(F_c f)(y),
\end{aligned} \tag{5.2}
$$

and

$$
\begin{aligned}
(F_c f')(y) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty f'(x) \cos xy dx \\
&= \frac{1}{\sqrt{2\pi}} \Big\{ f(x) \cos xy \Big|_0^\infty + y \int_0^\infty f(x) \sin xy dx \Big\} = f(0) + y(F_s f)(y).
\end{aligned} \tag{5.3}
$$

Theorem 4. *Suppose the following condition holds:*

$$
\begin{aligned}
&1 - 4\pi\lambda_1\lambda_2 F_c(\xi \overset{\gamma}{\underset{2}{*}} \varphi)(y) - 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y) \\
&- 8\pi\lambda_1\lambda_3 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} (\varphi_2 \overset{\gamma}{\underset{2}{*}} \mu))(y) - 4\pi\lambda_1\lambda_3 F_c(\psi \overset{\gamma}{\underset{2}{*}} \mu)(y) \neq 0, \qquad \forall y > 0,
\end{aligned} \tag{5.4}
$$

and

$$f(x) = \sqrt{\frac{\pi}{2}} (h(u) \underset{1}{*} e^{-u})(x) - 2\sqrt{2\pi}\lambda_1 (\varphi \overset{\gamma}{\underset{1}{*}} k)(x) - \sqrt{2\pi}(\psi \underset{1}{*} k)(x)$$

$$+\sqrt{\frac{\pi}{2}}((h(u)\underset{1}{*}e^{-u})\underset{1}{*}l)(x)-2\sqrt{2\pi}\lambda_1((\varphi\overset{\gamma}{\underset{1}{*}}k)\underset{1}{*}l)(x)-\sqrt{2\pi}((\psi\underset{1}{*}k)\underset{1}{*}l)(x),$$

$$g(x)=k(x)-\pi\lambda_2((h(u)\underset{1}{*}e^{-u})\underset{2}{*}\xi)(x)-2\pi((h(u)\underset{1}{*}e^{-u})\overset{\gamma}{\underset{2}{*}}\mu)(x)+(k\underset{F_c}{*}l)(x)$$

$$-\pi\lambda_2(((h(u)\underset{1}{*}e^{-u})\underset{2}{*}\xi)\underset{F_c}{*}l)(x)-2\pi(((h(u)\underset{1}{*}e^{-u})\overset{\gamma}{\underset{2}{*}}\mu)\underset{F_c}{*}l)(x)$$

satisfy that $f', f'', g', g'' \in L_1(\mathbb{R}_+)$. *Then* (f,g) *defined as above is the unique solution of system* (5.1) *in* $L_1(\mathbb{R}_+)$.

P r o o f. One can rewrite system (5.1) in the form

$$\Big(1-\frac{d^2}{dx^2}\Big)\Big\{f(x)+2\sqrt{2\pi}\lambda_1(\varphi\overset{\gamma}{\underset{1}{*}}g)(x)+\sqrt{2\pi}\lambda_1(\psi\underset{1}{*}g)(x)\Big\}=h(x),$$
$$\lambda_2\sqrt{2\pi}(f\underset{2}{*}\xi)(x)+2\sqrt{2\pi}\lambda_3(f\overset{\gamma}{\underset{2}{*}}\mu)(x)+g(x)=k(x).$$

With condition that g' and g'' belong to $L_1(\mathbb{R}_+)$, we conclude that $\frac{d}{dx}(\varphi\overset{\gamma}{\underset{1}{*}}g)(x)$, $\frac{d}{dx}(\psi\underset{1}{*}g)(x)$, $\frac{d^2}{dx^2}(\varphi\overset{\gamma}{\underset{1}{*}}g)(x)$ and $\frac{d^2}{dx^2}(\psi\underset{1}{*}g)(x)$ also belong to $L_1(\mathbb{R}_+)$. Moreover,

$$(\varphi\overset{\gamma}{\underset{1}{*}}g)(0)=(\psi\underset{1}{*}g)(0)=0.$$

Applying F_s and F_c respectively on the first and the second equations of system (5.1), and in view of the factorization equalities (1.5), (1.7), (1.9), (1.12) and formulas (5.2), (5.3) we obtain

$$\begin{aligned}(1+y^2)F_sf(y)+\sqrt{2\pi}\lambda_1(1+y^2)\big[2\sin yF_c\varphi(y)+F_s\psi(y)\big]F_cg(y)=&F_sh(y),\\ \sqrt{2\pi}\lambda_2F_sf(y)F_s\xi(y)+2\sqrt{2\pi}\lambda_3\sin yF_sf(y)F_c\mu(y)+F_cg(y)=&F_ck(y).\end{aligned}\tag{5.5}$$

Note that (see [4], formula 1.4.1)

$$\frac{1}{1+y^2}=\sqrt{\frac{\pi}{2}}F_c(e^{-u})(y).$$

So the system is equivalent to

$$F_sf(y)+\sqrt{2\pi}\lambda_1\big[2\sin yF_c\varphi(y)+F_s\psi(y)\big]F_cg(y)=\sqrt{\frac{\pi}{2}}F_s(h(u)\underset{1}{*}e^{-u})(y),$$

$$\sqrt{2\pi}\lambda_2F_sf(y)F_s\xi(y)+2\sqrt{2\pi}\lambda_3\sin yF_sf(y)F_c\mu(y)+F_cg(y)=F_ck(y).$$

We have

$$\Delta = \begin{vmatrix} 1 & \sqrt{2\pi}\lambda_1(2\sin y F_c\varphi(y) + F_s\psi(y)) \\ \sqrt{2\pi}\lambda_2 F_s\xi(y) + 2\sqrt{2\pi}\lambda_3 \sin y F_c\mu(y) & 1 \end{vmatrix}$$

$$= 1 - 4\pi\lambda_1\lambda_2 F_c(\xi \overset{\gamma}{\underset{2}{*}} \varphi)(y) - 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y) - 8\pi\lambda_1\lambda_3 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} (\varphi_2 \overset{\gamma}{\underset{2}{*}} \mu))(y)$$
$$-4\pi\lambda_1\lambda_3 F_c(\psi \overset{\gamma}{\underset{2}{*}} \mu)(y).$$

With condition (5.4) and applying the Wiener-Levy theorem, we obtain

$$\frac{1}{\Delta} = 1\times\Big[1 - 4\pi\lambda_1\lambda_2 F_c(\xi \overset{\gamma}{\underset{2}{*}} \varphi)(y) - 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y) - 8\pi\lambda_1\lambda_3 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} (\varphi_2 \overset{\gamma}{\underset{2}{*}} \mu))(y)$$
$$- 4\pi\lambda_1\lambda_3 F_c(\psi \overset{\gamma}{\underset{2}{*}} \mu)(y)\Big]^{-1} = 1 + \Big[4\pi\lambda_1\lambda_2 F_c(\xi \overset{\gamma}{\underset{2}{*}} \varphi)(y) + 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y)$$
$$+8\pi\lambda_1\lambda_3 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} (\varphi_2 \overset{\gamma}{\underset{2}{*}} \mu))(y) + 4\pi\lambda_1\lambda_3 F_c(\psi \overset{\gamma}{\underset{2}{*}} \mu)(y)\Big] \times \Big[1 - 4\pi\lambda_1\lambda_2 F_c(\xi \overset{\gamma}{\underset{2}{*}} \varphi)(y)$$
$$-2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y) - 8\pi\lambda_1\lambda_3 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} (\varphi_2 \overset{\gamma}{\underset{2}{*}} \mu))(y) - 4\pi\lambda_1\lambda_3 F_c(\psi \overset{\gamma}{\underset{2}{*}} \mu)(y)\Big]^{-1}$$
$$= 1 + (F_c l)(y),$$

for some $l \in L_1(\mathbb{R}_+)$.

We have

$$\Delta_1 = \begin{vmatrix} \sqrt{\frac{\pi}{2}} F_s(h(u) \underset{1}{*} e^{-u})(y) & \sqrt{2\pi}\lambda_1(2\sin y F_c\varphi(y) + F_s\psi(y)) \\ F_c k(y) & 1 \end{vmatrix}$$

$$= \sqrt{\frac{\pi}{2}} F_s(h(u) \underset{1}{*} e^{-u})(y) - 2\sqrt{2\pi}\lambda_1 F_s(\varphi \overset{\gamma}{\underset{1}{*}} k)(y) - \sqrt{2\pi} F_s(\psi \underset{1}{*} k)(y);$$

$$\Delta_2 = \begin{vmatrix} 1 & \sqrt{\frac{\pi}{2}} F_s(h(u) \underset{1}{*} e^{-u})(y) \\ \sqrt{2\pi}\lambda_2 F_s\xi(y) + 2\sqrt{2\pi}\lambda_3 \sin y F_c\mu(y) & F_c k(y) \end{vmatrix}$$

$$= F_c k(y) - \pi\lambda_2 F_c\big((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi\big)(y) - 2\pi F_c\big((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu\big)(y).$$

Hence,

$$(F_s f)(y) = \frac{\Delta_1}{\Delta} = \big[\sqrt{\frac{\pi}{2}} F_s(h(u) \underset{1}{*} e^{-u})(y) - 2\sqrt{2\pi}\lambda_1 F_s(\varphi \overset{\gamma}{\underset{1}{*}} k)(y)$$
$$-\sqrt{2\pi} F_s(\psi \underset{1}{*} k)(y)\big](1 + (F_c l)(y)) = F_s\Big(\sqrt{\frac{\pi}{2}}(h(u) \underset{1}{*} e^{-u}) - 2\sqrt{2\pi}\lambda_1(\varphi \overset{\gamma}{\underset{1}{*}} k)$$
$$-\sqrt{2\pi}(\psi \underset{1}{*} k) + \sqrt{\frac{\pi}{2}}\big((h(u) \underset{1}{*} e^{-u}) \underset{1}{*} l\big)$$

$$-2\sqrt{2\pi}\lambda_1\big((\varphi \overset{\gamma}{\underset{1}{*}} k) \underset{1}{*} l\big) - \sqrt{2\pi}\big((\psi \underset{1}{*} k) \underset{1}{*} l)\Big)(y).$$

This shows that

$$f(x) = \sqrt{\frac{\pi}{2}}(h(u) \underset{1}{*} e^{-u})(x) - 2\sqrt{2\pi}\lambda_1(\varphi \overset{\gamma}{\underset{1}{*}} k)(x) - \sqrt{2\pi}(\psi \underset{1}{*} k)(x)$$

$$+\sqrt{\frac{\pi}{2}}\big((h(u) \underset{1}{*} e^{-u}) \underset{1}{*} l\big)(x) - 2\sqrt{2\pi}\lambda_1\big((\varphi \overset{\gamma}{\underset{1}{*}} k) \underset{1}{*} l\big)(x) - \sqrt{2\pi}\big((\psi \underset{1}{*} k) \underset{1}{*} l\big)(x).$$

One can check easily that $f(0) = 0$.

Similarly,

$$(F_c g)(y) = \frac{\Delta_2}{\Delta}$$

$$= \big[F_c k(y) - \pi\lambda_2 F_c\big((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi\big)(y) - 2\pi F_c\big((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu\big)(y)\big](1 + F_c l(y))$$

$$= F_c k(y) - \pi\lambda_2 F_c\big((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi\big)(y) - 2\pi F_c\big((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu\big)(y) + F_c(k \underset{F_c}{*} l)(y)$$

$$-\pi\lambda_2 F_c\big(((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi) \underset{F_c}{*} l\big)(y) - 2\pi F_c\big(((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu) \underset{F_c}{*} l\big)(y).$$

Therefore,

$$g(x) = k(x) - \pi\lambda_2\big((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi\big)(x) - 2\pi\big((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu\big)(x) + (k \underset{F_c}{*} l)(x)$$

$$-\pi\lambda_2\big(((h(u) \underset{1}{*} e^{-u}) \underset{2}{*} \xi) \underset{F_c}{*} l\big)(x) - 2\pi\big(((h(u) \underset{1}{*} e^{-u}) \overset{\gamma}{\underset{2}{*}} \mu) \underset{F_c}{*} l\big)(x).$$

Hence, if $f', f'', g', g'' \in L_1(\mathbb{R}_+)$ then (f, g) is the solution of system (5.1). ■

6. A class of Toeplitz plus Hankel integral equations

Finally, we consider a class of integral equations with the Toeplitz plus Hankel kernel (1.1), that can be solved in closed form with the use of generalized convolutions.

First, let us recall the generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier sine transform of two functions f and g [5], [14]

$$(f \overset{\gamma}{\underset{F_s}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u)[\operatorname{sign}(x+u-1)g(|x+u-1|) + \operatorname{sign}(x-u+1) \tag{6.1}$$

$$\times g(|x-u+1|) - g(x+u+1) - \operatorname{sign}(x-u-1)g(|x-u-1|)]du, \quad x > 0.$$

This convolution satisfies the following factorization equality [5], [14]:

$$F_s(f \overset{\gamma}{\underset{F_s}{*}} g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{6.2}$$

Consider the integral equation with the Toeplitz plus Hankel kernel

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)]f(y)dy = g(x), \tag{6.3}$$

where the Toeplitz kernel k_1 and the Hankel kernel k_2 are defined as follows:

$$k_1(t) = \frac{1}{2\sqrt{2}}\,\mathrm{sign}(t-1)h_1(|t-1|) - \frac{1}{2\sqrt{2}}\,\mathrm{sign}(t+1)h_1(|t+1|) - \frac{1}{\sqrt{2}}h_2(t), \tag{6.4}$$

$$k_2(t) = \frac{1}{2\sqrt{2}}\,\mathrm{sign}(t-1)h_1(|t-1|) - \frac{1}{2\sqrt{2}}\,\mathrm{sign}(t+1)h_1(|t+1|) + \frac{1}{\sqrt{2}}h_2(|t|). \tag{6.5}$$

Moreover, we assume that $h_1(x) = (\varphi_1 \underset{1}{*} \varphi_2)(x)$, the generalized convolution of φ_1 and φ_2 for the Fourier sine and cosine transforms (1.4), and that φ_1, φ_2 and h_2 are functions in $L_1(\mathbb{R}_+)$.

THEOREM 5. *Suppose that the condition*

$$1 + \lambda\big(\sin y(F_s h_1)(y) + (F_c h_2)(y)\big) \neq 0, \quad \forall y > 0, \tag{6.6}$$

holds. Then the integral equation with Toeplitz plus Hankel kernel (6.3) *has a unique solution in* $L_1(\mathbb{R}_+)$ *of the form*

$$f(x) = g(x) + (g \underset{1}{*} l)(x). \tag{6.7}$$

Here $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{\lambda\big(F_c(\varphi_1 \underset{1}{*} \varphi_2)(y) + (F_c h_2)(y)\big)}{1 + \lambda\big(F_c(\varphi_1 \underset{1}{*} \varphi_2)(y) + (F_c h_2)(y)\big)}.$$

P r o o f. We can rewrite the Toeplitz plus Hankel equation (6.3) as follows:

$$f(x) + \lambda\int_0^\infty f(u)\Big\{\Big[\frac{1}{2\sqrt{2}}(\mathrm{sign}(x+u-1)h_1(|x+u-1|) - h_1(x+u+1)) - \frac{1}{\sqrt{2}}h_2(x+u)\Big]$$
$$+ \Big[\frac{1}{2\sqrt{2}}(\mathrm{sign}(x-u-1)h_1(|x-u-1|) - \mathrm{sign}(x-u+1)h_1(|x-u+1|))$$
$$+ \frac{1}{\sqrt{2}}h_2(|x-u|)\Big]\Big\}du = g(x). \tag{6.8}$$

Rearranging the terms in the integrand, we obtain

$$f(x)+\lambda\frac{1}{2\sqrt{2}}\int\limits_0^\infty f(u)[\operatorname{sign}(x+u-1)h_1(|x+u-1|)-h_1(x+u+1)$$
$$+\operatorname{sign}(x-u-1)h_1(|x-u-1|)-\operatorname{sign}(x-u+1)h_1(|x-u+1|)]du$$
$$+\lambda\frac{1}{\sqrt{2}}\int\limits_0^\infty f(u)[h_2(|x-u|)-h_2(x+u)]du=g(x). \tag{6.9}$$

Applying the Fourier sine transform on two side of equation (6.9) and using the factorization equalities (6.2), (1.5) we get

$$(F_sf)(y)+\lambda\sin y(F_sf)(y)(F_sh_1)(y)+\lambda(F_sf)(y)(F_ch_2)(y)=(F_sg)(y).$$

Using the assumption that $h_1(x)=(\varphi_1\underset{1}{*}\varphi_2)(x)$ and condition (6.6) we obtain

$$\begin{aligned}(F_sf)(y)=&(F_sg)(y)\Big[1-\frac{\lambda\big(\sin y(F_sh_1)(y)+(F_ch_2)(y)\big)}{1+\lambda\big(\sin y(F_sh_1)(y)+(F_ch_2)(y)\big)}\Big]\\ =&(F_sg)(y)\Big[1-\frac{\lambda\big(\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y)+(F_ch_2)(y)\big)}{1+\lambda\big(\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y)+(F_ch_2)(y)\big)}\Big]\\ =&(F_sg)(y)\Big[1-\frac{\lambda\big(F_c(\varphi_1\underset{1}{*}\varphi_2)(y)+(F_ch_2)(y)\big)}{1+\lambda\big(F_c(\varphi_1\underset{1}{*}\varphi_2)(y)+(F_ch_2)(y)\big)}\Big].\end{aligned}$$

In view of the Wiener-Levy theorem, there exits a function $l\in L_1(\mathbb{R}_+)$ such that

$$(F_cl)(y)=\frac{\lambda\big(F_c(\varphi_1\underset{1}{*}\varphi_2)(y)+(F_ch_2)(y)\big)}{1+\lambda\big(F_c(\varphi_1\underset{1}{*}\varphi_2)(y)+(F_ch_2)(y)\big)}.$$

Then,

$$(F_sf)(y)=(F_sg)(y)[1-(F_cl)(y)]$$
$$=(F_sg)(y)-F_s(g\underset{1}{*}l)(y)=F_s(g-g\underset{1}{*}l)(y).$$

Therefore,

$$f(x)=g(x)-(g\underset{1}{*}l)(x).$$

So we obtain the solution of the integral equation with the Toeplitz plus Hankel kernel (6.3), where the Toeplitz kernel k_1 and the Hankel kernel k_2 are defined by (6.4) and (6.5), in closed form. It completes the proof of Theorem 5. ∎

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