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A BRIEF STORY ABOUT THE OPERATORS OF THE GENERALIZED FRACTIONAL CALCULUS

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Dedicated to Professor Shyam L. Kalla,
on the occasion of his 70th anniversary

Abstract

In this survey we present a brief history and the basic ideas of the generalized fractional calculus (GFC). The notion "generalized operator of fractional integration" appeared in the papers of the jubilarian Prof. S.L. Kalla in the years 1969-1979 when he suggested the general form of these operators and studied examples of them whose kernels were special functions as the Gauss and generalized hypergeometric functions, including arbitrary G- and H-functions. His ideas provoked the author to choose a more peculiar case of such kernels and to develop a theory of the corresponding GFC that featured many applications. All known fractional integrals and derivatives and other generalized integration and differential operators in various areas of analysis happened to fall in the scheme of this GFC.

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1. Introduction to the classical fractional calculus

One of the trends of the contemporary fractional calculus is the so-called generalized fractional calculus (GFC). Along with the expansion of numerous and even unexpected recent applications of the operators of the classical fractional calculus (FC), the GFC is another powerful tool stimulating the development of this field.

It is also generating new classes of special functions (special functions of fractional calculus) and integral transforms, as well as providing new transmutation operators applicable to solve more complicated problems in analysis via their reduction to simpler ones. The GFC poses also: new challenges for interpretations of its operators, similar to the recently found for the classical fractional integrals and derivatives (see e.g. Podlubny [27]); and new open problems for their applications in solving not only theoretical models of fractional (multi-)order differential and integral eqations, but mathematical models of real phenomena and events (as it is now well illustrated for the classical FC).

The classical FC is based on several (almost equivalent) definitions for the operators of integration and differentiation of arbitrary (including real fractional or complex) order, as continuation of the classical integration and differentiation operators and their integer order powers ($n \in \mathbb{N}$), namely - the n-fold integration

$$R^n f(z) = \int_0^z dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-2}} dt_{n-1} \int_0^{t_{n-1}} f(t_n) dt_n \\ = \frac{1}{(n-1)!} \int_0^z (z-t)^{n-1} f(t) dt, \qquad (1)$$

and n-th order derivatives $D^n f(z) = f^{(n)}(z)$. For definiteness, further we have in mind the so-called *Riemann-Liouville (R-L) definition.* The essence of the mathematical problem for defining integrals and derivatives of fractional order consists in the following: for each function $f(z), z = x + iy$, of sufficiently large class and for each number δ (rational, irrational, complex), to set up a correspondence to a function $g(z) = D^\delta f(z)$ satisfying the conditions:

- If $f(z)$ is an analytic function of z, the derivative $D^\delta f(z)$ is an analytic function of z and δ.
- The operation D^δ gives the same result as the usual differentiation of order n, when $\delta = n$ is a positive integer, and the same effect as the n-fold integration, if $\delta = -n$ is a negative integer (i.e. $D^{-n} = R^n$). Moreover, $D^\delta f(z)$ should vanish at the initial point $z = 0$ (or $z = c$) together with its first $(n-1)$ derivatives.
- The operator of order $\delta = 0$ is the identity operator.
- The fractional operators are linear:

$$D^\delta a f(z) + b g(z) = a D^\delta f(z) + b D^\delta g(z).$$

- For fractional integrations of arbitrary orders $\alpha > 0, \beta > 0$ ($\Re\alpha > 0$, $\Re\beta > 0$) the additive index law (semigroup property) holds:

$$D^{-\alpha} D^{-\beta} f(z) = D^{-(\alpha+\beta)} f(z), \text{ i.e. } R^{\alpha} R^{\beta} = R^{\beta} R^{\alpha} = R^{\alpha+\beta},$$

as the denotation $R^{\delta} f(z) := D^{-\delta} f(z), \Re\delta > 0$ will be used further in the case of derivative of negative (or with negative real part) order.

The definition for the *Riemann-Liouville (R-L) fractional integral* (in the whole survey we discuss only the so-called left-hand sided variants of operators, denoted in the literature also as I^{δ}_{0+}):

$$R^{\delta} f(z) = D^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1} f(t) dt = z^{\delta} \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} f(z\sigma) d\sigma, \tag{2}$$

is easily seen to satisfy all the above conditions, and in particular, coincides with the repeated (n-fold) integration represented by the Dirichlet formula in the form (1). If $z = x + iy$ is a complex variable, the above representation can be modified to the Cauchy integral formula. The R-L definition (2) concerns integrations of (real part) positive orders and could not be used directly for a differentiation ($\Re\delta < 0$). However, a little trick is helpful for a suitable expression. For noninteger $\delta > 0$ we set $n := [\delta] + 1$ (the smallest integer greater than δ), then we can define properly the *R-L fractional derivative* by means of the differ-integral expression

$$\begin{aligned} D^{\delta} f(z) &= D^n D^{\delta-n} f(z) = \left(\frac{d}{dz}\right)^n R^{n-\delta} f(z) \\ &= \left(\frac{d}{dz}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int_0^z (z-t)^{n-\delta-1} f(t) dt \right\}, \end{aligned} \tag{3}$$

since $n - \delta > 0$. In suitable functional spaces,

$$D^{\delta} R^{\delta} f(z) = f(z), \quad \text{i.e. the inversion formula holds:} \quad \left\{R^{\delta}\right\}^{-1} = D^{\delta}.$$

One more interesting fact, to compare this with the classical calculus, follows from the formula

$$D^{\delta} \{z^{\alpha}\} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\delta)} z^{\alpha-\delta}, \ \delta > 0, \ \alpha > -1,$$

whence, for $\alpha = 0$ we obtain:

$$D^{\delta}\{c\} = c \frac{z^{-\delta}}{\Gamma(1-\delta)},$$

i.e. a fractional derivative of a constant is zero *only* for positive integer values $\delta = n = 1, 2, 3, \ldots$.

The classical fractional calculus can be thought as originated as early as probably in 1695, when l'Hospital asked Leibnitz in a letter: *"What if n be 1/2 in $d^n y/dx^n$?"* and he replied, *"It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn"*. Since then, many known analysts and applied scientists contributed to the development of this "strange" calculus, but the first book and the first conference dedicated specially to that topic took place 279 years after the mentioned correspondence! The detailed history, theory and its various applications, by the years of 1987-1993 can be seen in the "FC Encyclopedia" [33].

2. Introduction to the generalized fractional calculi

Besides the R-L definitions of the fractional calculus operators, several modifications and their generalizations are widely used. The most useful classical fractional integrals however seem to be the *Erdélyi-Kober (E-K) operators* (see e.g. Sneddon [34] for the case $\beta = 2$), whose generalizations in the form

$$I_\beta^{\gamma,\delta} f(z) = z^{-\beta(\gamma+\delta)} \int_0^z \frac{(z^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta\gamma} f(\tau)\, d(\tau^\beta)$$
$$= \int_0^1 \frac{(1-\sigma)^{\delta-1}\sigma^\gamma}{\Gamma(\delta)} f(z\sigma^{\frac{1}{\beta}})\, d\sigma, \quad \gamma \in \mathbb{R},\ \beta > 0, \tag{4}$$

are *used essentially in our works and in the present survey.*

Several authors, like Love [22], Saxena [32], Kalla and Saxena [11], Saigo [29, 30], McBride [26], also Tricomi, Sprinkhuizen-Kuiper, Koornwinder, etc., have studied and used different modifications of the so-called *hypergeometric operators of fractional integration*

$$\mathcal{H} f(z) = \frac{\mu z^{-\gamma-1}}{\Gamma(1-\delta)} \int_0^z {}_2F_1\left(\delta, \beta+m;\ \eta; a(\frac{t}{z})^\mu\right) t^\gamma f(t) dt, \tag{5}$$

involving the Gauss hypergeometric function.

An example of *fractional integration operators involving other special functions*, is given by the operators of Lowndes [23, 24]:

$$I_\lambda(\eta, \nu+1) f(z) = \frac{2^{\nu+1}}{\lambda^\nu} z^{-(\nu+\eta+1)} \int_0^z t^{2\eta+1}(z^2-t^2)^{\frac{\nu}{2}} J_\nu(\lambda\sqrt{z^2-t^2}) f(t) dt, \tag{6}$$

related to the second order Bessel type differential operator $B_\eta = z^{-2\eta-1}(d/dz) z^{2\eta+1}(d/dz)$.

One of the most general fractional integration operators of type (2) can be obtained when the *kernel-function is an arbitrary Meijer G-function*, as in Kalla [7], also in Parashar, Rooney, etc.:

$$\mathcal{I}_G f(z) = z^{-\gamma-1} \int_0^z G_{p,q}^{m,n} \left[a(\frac{t}{z})^r \,\middle|\, \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right] t^\gamma f(t) dt, \tag{7}$$

or its further generalization,the *Fox H-function*, as in Kalla [6], also in Srivastava and Buschman [35], and others:

$$\mathcal{I}_H f(z) = z^{-\gamma-1} \int_0^z H_{p,q}^{m,n} \left[a(\frac{t}{z})^r \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] t^\gamma f(t) dt. \tag{8}$$

In his papers [7, 8] of years 1970-1979, Kalla suggested that all the above operators of R-L type (2) can be considered as "generalized operators of fractional integration" of the general form (here we mention only the left-hand sided type integrations):

$$\mathcal{I} f(z) = z^{-\gamma-1} \int_0^z \Phi(\frac{t}{z})\, t^\gamma f(t) dt, \tag{9}$$

where the kernel $\Phi(z)$ is an arbitrary continuous function so that the above integral makes sense in sufficiently large functional spaces. Kalla established a series of their general properties, analogous to those of the classical fractional integrals, and also studied their special cases. By suitable choices of the kernel-function Φ, operators (9) can be shown to include all the other known fractional integrals as particular cases.

Nowadays, there exist a great number of articles, surveys and books, and proceedings of conferences, entirely devoted to fractional calculus, its generalizations and applications. For those before 1987 one can see detailed references in the encyclopaedic book of Samko, Kilbas and Marichev [33]. Afterwards, several new books and volumes of collected papers appeared, and many newer research papers, surveys and references can be seen in the "FCAA" journal [5].

3. Some definitions

Definition 1. (see [36],[28],[17, App.]) By a *Fox's H-function* we mean the generalized hypergeometric function defined by means of the contour integral

$$\begin{aligned} H_{p,q}^{m,n}(\sigma) &= H_{p,q}^{m,n} \left[\sigma \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right] \\ &= H_{p,q}^{m,n} \left[\sigma \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) \sigma^s ds, \end{aligned} \tag{10}$$

where the integrand in (10) has the form

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - B_k s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + B_k s) \prod\limits_{j=n+1}^{p} \Gamma(a_j - A_j s)}$$

and $\mathcal{L}$ is a suitable contour in $\mathbb{C}$; the orders $(m, n; p, q)$ are nonnegative integers such that $0 \le m \le q,\ 0 \le n \le q$; the parameters $A_j, j = 1, \dots, p$ and $B_k, k = 1, \dots, q$ are *positive* and $a_j, j = 1, \dots, p,\ b_k, k = 1, \dots, q$ are arbitrary complex numbers such that
$A_j(b_k + l) \neq B_k(a_j - l' - 1); \quad l,\ l' = 0, 1, 2, \dots\ ; \quad j = 1, \dots, p,\ k = 1, \dots, q.$

In particular, when all $A_j = B_k = 1$, we obtain the so-called *Meijer's G-function* ([4, vol.1]):

$$H_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} (a_j, 1)_1^p \\ (b_k, 1)_1^q \end{matrix}\right] = G_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix}\right], \tag{11}$$

that is,

$$G_{p,q}^{m,n}\left[\sigma \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + s) \prod\limits_{j=n+1}^{p} \Gamma(a_j - s)} \sigma^s ds. \tag{12}$$

The operators (7) and (8) involve arbitrary Meijer's G-functions and Fox's H-functions, and thus appear as the most general operators of GFC, being in the classes of the convolutional type G- and H-integral transforms. However, in order to develop a meaningful detailed theory with practical applications, we have chosen the kernel-functions Φ in (9) (resp. in (7),(8)) as suitable peculiar cases of the G- and H-functions:

$$G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix}\right],\ H_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix}\right] \tag{13}$$

Thus, in Kiryakova [17] (also [13]-[16]) we defined a class of *generalized fractional derivatives and integrals* by means of single (differ-)integrals involving the generalized hypergeometric functions (13). This allowed to develop a detailed *theory of GFC*, with operational properties analogous to these of the classical R-L and E-K fractional integrals and derivatives, *with numerous applications* in solving problems for differential and integral equations (including integer ordered), for classes of analytic functions in geometric functions theory, in operational calculus and integral transforms, and with a great impact in the theory of special functions, see [17], [18], [20], [19], [1].

Definition 2. ([13, 14, 3, 17] Let $m \geq 1$ be integer, $\beta > 0$, $\gamma_1, ..., \gamma_m$ and $\delta_1 \geq 0, ..., \delta_m \geq 0$ be arbitrary real numbers. By a *generalized (multiple, m-tuple) Erdélyi-Kober (E.-K.) operator of integration of multi-order* $\delta = (\delta_1, ..., \delta_m)$ we mean an integral operator

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) = \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{array} \right. \right] f(z\sigma^{\frac{1}{\beta}})\, d\sigma. \tag{14}$$

Then, each operator of the form

$$\mathcal{R}f(z) = z^{\beta\delta_0} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) \quad \text{with arbitrary} \quad \delta_0 \geq 0, \tag{15}$$

is said to be a *generalized (m-tuple) operator of fractional integration of Riemann-Liouville type*, or briefly: a *generalized (R.-L.) fractional integral.*

Generalizing further the operators of fractional calculus, in Kiryakova [15, 16, 9, 10, 17] we introduced also operators involving classes of Fox's H-functions instead of the G-functions in (14),(15). They are called in the same way, namely *generalized (multiple) E.-K. operators (fractional integrals)*:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = \begin{cases} \int\limits_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right. \right] f(z\sigma) d\sigma, & \text{if } \sum\limits_{k=1}^m \delta_k > 0, \\ f(z), \text{ if } \delta_1 = \delta_2 = \cdots = \delta_m = 0. \end{cases} \tag{16}$$

Thus, along with the *multi-order of integration* $(\delta_1, ..., \delta_m)$ and the *multi-weight* $(\gamma_1, ..., \gamma_m)$, we introduced also a *multi-parameter* $(\beta_1 > 0, ..., \beta_m > 0)$ (different β_k's) instead of the same $\beta > 0$ in the case with a G-function. Note that due to relation (a generalization of (11)),

$$H_{p,q}^{m,n} \left[\sigma \left| \begin{array}{c} (a_1, \frac{1}{\beta}), \ldots, (a_p, \frac{1}{\beta}) \\ (b_1, \frac{1}{\beta}), \ldots, (b_q, \frac{1}{\beta}) \end{array} \right. \right] = \beta\, G_{p,q}^{m,n} \left[\sigma^\beta \left| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right. \right], \quad \beta > 0, \tag{17}$$

operator (16) involving a H-function reduces to its simpler form (14)

$$\text{for} \quad \beta_1 = \beta_2 = ... = \beta_m = \beta > 0: \quad I_{(\beta,\beta,...,\beta),m}^{(\gamma_k),(\delta_k)} = I_{\beta,m}^{(\gamma_k),(\delta_k)}. \tag{18}$$

Now let us introduce the corresponding generalizations of the classical Riemann-Liouville fractional derivatives (3).

Definition 3. With the same parameters as in Def. 2 and the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer,} \\ [\delta_k] + 1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \qquad k = 1, \ldots, m, \tag{19}$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta_r} z \frac{d}{dz} + \gamma_r + j \right) \right]. \tag{20}$$

Then, we define the *multiple (m-tuple) Erdélyi-Kober fractional derivative* of multi-order $\delta = (\delta_1 \geq 0, \ldots, \delta_m \geq 0)$ by means of the differ-integral operator:

$$\begin{aligned} D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) &= D_\eta \, I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(z) \\ &= D_\eta \int_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] f(z\sigma) \, d\sigma. \end{aligned} \tag{21}$$

In the case (18) of equal β_k's, we obtain a simpler representation involving the Meijer's G-function, corresponding to generalized fractional integral (14):

$$\begin{aligned} D_{\beta,m}^{(\gamma_k),(\delta_k)} f(z) &= D_\eta \, I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} \\ &= \left[\prod_{r=1}^{m} \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} z \frac{d}{dz} + \gamma_r + j \right) \right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(z). \end{aligned} \tag{22}$$

More generally, all differ-integral operators of the form

$$\mathcal{D} f(z) = D_{\beta,m}^{(\gamma_k),(\delta_k)} z^{-\delta_0} f(z) = z^{-\delta_0} D_{\beta,m}^{(\gamma_k - \frac{\delta_0}{\beta}),(\delta_k)} f(z) \ \text{ with } \ \delta_0 \geq 0, \tag{23}$$

are called *generalized (multiple, multi-order) fractional derivatives.*

Generalized derivatives (22),(23) are the counterparts of the generalized fractional integrals (14),(15).

Let us note that the hint for introducing the operators below (yet in Kiryakova [13, 14]) was the extended study on the *Bessel-type differential operators* of arbitrary (integer!) order $m > 1$ (called afterwards as *hyper-Bessel operators*, [17, Ch.3]) and on the related integral transform of Laplace type. In a series of papers, started by [2], Dimovski introduced these operators and developed operational calculi for them, based on the Mikusinski's algebraic scheme as well as on the Obrechkoff integral transform. The continuation of these studies in papers by Dimovski-Kiryakova, and Kiryakova, lead to involving the Meijer G-functions into the theory of hyper-Bessel operators and equations, as their solutions and as kernel-functions of related integral transforms and transmutation operators. Especially, the integral representations for fractional powers L^λ of the hyper-Bessel operators (see Dimovski and Kiryakova [3]) resulted in an integral operator of the form (14) with all equal $\delta_k = \lambda, k = 1, ..., m$, the operator L himself being thus an integration of multi-integer order (1,...,1)! Similar results were predicted also by an earlier paper by McBride [26]. More details and references related to the theory and applications of the hyper-Bessel operators and Obrechkoff integral transform, can be seen in [17, Ch.3].

4. Basic results of the GFC

The particular choice (13) of the kernel-functions ensures a decomposition of our operators (called also *multiple Erdélyi-Kober operators*) into *products of commuting classical Erdélyi-Kober (E.-K.) operators* (4). Thus, complicated multiple integrals or differ-integral expressions can be represented alternatively, by means of single integrals involving special functions. The beauty and succinctness of notations and properties of these functions allow the development of a *full chain of operational rules, mapping properties and convolutional structure of the generalized fractional integrals* as well as *an appropriate explicit definition of the corresponding generalized derivatives.* On the other hand, the frequent appearance of compositions of classical Riemann-Liouville and Erdélyi-Kober fractional operators in various problems of applied analysis gives *the key to the great number of applications and known special cases* of our *generalized fractional differ-integrals.*

The main *functional spaces* discussed in our papers on GFC are the weighted spaces of continuous, Lebesgue integrable or analytic functions: Let α, μ be arbitrary real, $k \geq 0$ and $1 \leq p < \infty$ be integers, the variables x, z be real or complex, running resp. over the interval $[0, \infty)$ or in the domain $\Omega \subset \mathbb{C}$, starlike with respect to the origin $z = 0$ and let $\mathcal{H}(\Omega)$ stand for the space of analytic functions in Ω. We use the denotations:

$$C_\alpha^{(k)} := \left\{ f(x) = x^p \tilde{f}(x); p > \alpha, \tilde{f} \in C^{(k)}[0, \infty) \right\}, \quad C_\alpha^{(0)} := C_\alpha;$$

$$L_{\mu,p}(0, \infty) := \left\{ f(x) : \ ||f||_{\mu,p} = \Big[\int_0^\infty x^{\mu-1} |f(x)| dx \Big]^{\frac{1}{p}} < \infty \right\};$$

$$\mathcal{H}_\mu(\Omega) = \left\{ f(z) = z^\mu \tilde{f}(z); \tilde{f}(z) \in \mathcal{H}(\Omega) \right\}, \quad \mathcal{H}_0(\Omega) := \mathcal{H}(\Omega).$$

To study the generalized fractional integrals, we have used essentially the *theory of the G- and H-functions*, appearing as kernel-functions of (14),(16). To this end, we refer the reader to the recently appeared books on special functions and fractional calculus, for example [12] as well as to "classics" [4, vol. 1, Ch.5], [28], [36]; or [17, App.]. Note also, that the $G^{m,0}_{m,m}$- and $H^{m,0}_{m,m}$-functions have three regular singular points $\sigma = 0, 1$ and ∞, they vanish for $|\sigma| > 1$ and are analytic functions in the unit disk $|\sigma| < 1$. Their asymptotic behaviour near $\sigma = 0, 1$ is already well-known (see e.g. [28], [12]) and ensures the correctness of definitions (14),(16) in the above spaces, under suitable conditions on parameters.

Most of the basic results for the operators of the generalized fractional calculus have been stated in Kiryakova [17] separately for the cases of G- and H-functions and for all the above mentioned spaces. Here we expose them in one version only and in the most characteristic space and only mention the analogues for the others.

Theorem 1. *Each multiple E.-K. fractional integral (16) preserves the power functions in $C_\alpha, \alpha \geq \max_k [-\beta(\gamma_k+1)]$ up to a constant multiplier:*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{x^p\} = c_p x^p,\ p > \alpha, \quad \text{where } c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + \frac{p}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{p}{\beta_k} + 1)}, \tag{24}$$

and it is an invertible mapping $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} : C_\alpha \mapsto C_\alpha^{(\eta_1+\cdots+\eta_m)} \subset C_\alpha$. If the index α of C_α is fixed, then the conditions on the parameters are as follows:

$$\gamma_k \geq -\frac{\alpha}{\beta_k} - 1, \delta_k > 0,\ \eta_k := \begin{cases} [\delta_k] + 1, & \text{for noninteger } \delta_k, \\ \delta_k, & \text{for integer } \delta_k, \end{cases} \quad k = 1, \ldots, m. \tag{25}$$

Similar proposition holds also in the space $\mathcal{H}_\mu(\Omega)$, stated as follows.

Theorem 2. *Let the conditions*

$$\gamma_k > -\frac{\mu}{\beta_k} - 1,\ \delta_k > 0,\ k = 1, \ldots, m \tag{26}$$

be satisfied. Then, the multiple E.-K. operator (16) maps the class $\mathcal{H}_\mu(\Omega)$ into itself, preserving the power functions up to constant multipliers like in (24) and the image of a power series

$$f(z) = z^\mu \sum_{n=0}^{\infty} a_n z^n = z^\mu (a_0 + a_1 z + \ldots) \in \mathcal{H}_\mu(\Delta_R),\ \Delta_R = \{|z| < R\},$$

where $R = \left\{\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right\}^{-1}$, is given by the series

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = z^\mu \sum_{n=0}^{\infty} \left\{ a_n \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + \frac{n+\mu}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k + \frac{n+\mu}{\beta_k} + 1)} \right\} z^n, \tag{27}$$

having the same radius of convergence $R > 0$ and the same signs of the coefficients.

Operator (16) can be rewritten in the form

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \frac{1}{z} \int\limits_0^z H^{m,0}_{m,m} \left[\frac{t}{z} \left| \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] f(t) dt,$$

and thus, it can be put in the form of a *convolutional type integral transform*, namely:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \int\limits_0^\infty k \frac{z}{t} f(t) \frac{dt}{t} = (k \circ f)(z),$$

where $\circ$ denotes the Mellin convolution. Thus we obtain the following

LEMMA 3. *Multiple E.-K. fractional integral (16) has the following convolutional type representation in* $L_{\mu,p}$*:*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = H^{m,0}_{m,m} \left[\frac{1}{z} \left| \begin{matrix} (\gamma_k + \delta_k + 1, \ \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1, \ \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] \circ f(z) \,, \tag{28}$$

and for $1 \le p \le 2$ *its Mellin transformation is given by the equality*

$$\mathcal{M} \left\{ I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z); s \right\} = \left[\prod_{k=1}^m \frac{\Gamma(\gamma_k - \frac{s}{\beta_k} + 1)}{\Gamma(\gamma_k + \delta_k - \frac{s}{\beta_k} + 1)} \right] \mathcal{M}\{f(z); s\}. \tag{29}$$

Using representation (28) and following the pattern of [33] (a lemma descending from Hardy, Littlewood and Polya), it is easy to prove the following proposition.

THEOREM 4. *Let the parameters of the multiple E.-K. fractional integral (16) satisfy the conditions*

$$\beta_k(\gamma_k + 1) > \frac{\mu}{p}, \ \delta_k > 0, \ k = 1, \dots, m. \tag{30}$$

Then, $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z)$ *exists almost everywhere on* $(0, \infty)$ *and it is a bounded linear operator from the Banach space* $L_{\mu,p}$ *into itself. More exactly,*

$$\left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\|_{\mu,p} \le h_{\mu,p} \, \|f\|_{\mu,p}, \qquad \text{i.e.} \quad \left\| I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f \right\| \le h_{\mu,p} \tag{31}$$

with $h_{\mu,p} = \prod\limits_{k=1}^m \Gamma(\gamma_k - \frac{\mu}{p\beta_k} + 1) \ / \ \Gamma(\gamma_k + \delta_k - \frac{\mu}{p\beta_k} + 1) < \infty.$

From the properties of the H- and G-functions some immediate corollaries of definitions (14),(16) follow.

THEOREM 5. *Suppose conditions (25) for C_α, (26) for $\mathcal{H}_\mu$ or (30) for $L_{\mu,p}$ hold. Then, in the above spaces the following* basic operational rules *of the multiple E.-K. fractional integrals hold:*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{\lambda f(cz)+\eta g(cz)\}=\lambda\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}f\right\}(cz)+\eta\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}g\right\}(cz) \quad (32)$$

(bilinearity of (16));

$$I^{(\gamma_1,\ldots,\gamma_s,\gamma_{s+1},\ldots,\gamma_m),(0,\ldots,0,\delta_{s+1},\ldots,\delta_m)}_{(\beta_1,\ldots,\beta_m),m}f(z)=I^{(\gamma_{s+1},\ldots,\gamma_m)(\delta_{s+1},\ldots,\delta_m)}_{(\beta_{s+1},\ldots,\beta_m),m-s}f(z) \quad (33)$$

(if $\delta_1=\delta_2=\cdots=\delta_s=0$, then the multiplicity reduces to $(m-s)$);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}z^\lambda f(z)=z^\lambda I^{(\gamma_k+\frac{\lambda}{\beta_k}),(\delta_k)}_{(\beta_k),m}f(z),\quad \lambda\in\mathbb{R} \quad (34)$$

(generalized commutability with power functions);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n}f(z)=I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n}I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}f(z) \quad (35)$$

(commutability of operators of form (16));

$$\text{the left-hand side of (35)}=I^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n}f(z) \quad (36)$$

(compositions of m-tuple and n-tuple integrals (16) give $(m+n)$-tuple integrals of same form);

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m}I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}f(z)=I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m}f(z),\ \text{if}\delta_k>0,\ \sigma_k>0,\ k=1,\ldots,m \quad (37)$$

(law of indices, product rule or semigroup property);

$$\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\right\}^{-1}f(z)=I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m}f(z) \quad (38)$$

(formal inversion formula).

The above inversion formula follows from the index law (37) for $\sigma_k=-\delta_k<0$, $k=1,\ldots,m$ and definition (16) for zero multi-order of integration, since:

$$I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m}I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}f(z)=I^{(\gamma_k),(0,\ldots,0)}_{\beta_k,m}f(z)=f(z).$$

But symbols (16) have not yet been defined for negative multi-orders of integration $-\delta_k<0$, $k=1,\ldots,m$. The problem has been to propose an appropriate meaning for them and hence to avoid the divergent integrals appearing in (38). The situation is exactly the same as in the classical case when the R.-L. and E.-K. operators of fractional order $\delta>0$ can be inverted by appealing to an additional differentiation of suitable integer order $\eta = [\delta]+1$. In this case, we have used the following differential formula for the kernel H-function (see Kiryakova [17, Lemma 5.1.7 or Lemma B.3, App.], resp. for the H- or the G-function): Let $\eta_k \geq 0, k = 1, ..., m$ be arbitrary integers, then

$$H_{m,m}^{m,0}\left[\frac{t}{z}\left|\begin{matrix}(a_k, \frac{1}{\beta_k})_1^m \\ (b_k, \frac{1}{\beta_k})_1^m\end{matrix}\right.\right] = D_\eta H_{m,m}^{m,0}\left[\frac{t}{z}\left|\begin{matrix}(a_k+\eta_k, \frac{1}{\beta_k})_1^m \\ (b_k, \frac{1}{\beta_k})_1^m\end{matrix}\right.\right], \qquad (39)$$

with differential operator D_η being polynomial of $z(d/dz)$ of degree $\eta = \eta_1 + \cdots + \eta_m$:

$$D_\eta = \prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} z \frac{z}{dz} + a_r - 1 + j\right).$$

This formula is helping to increase the parameters a_k, $k = 1, \ldots, m$ of the H-function in the upper row by arbitrary integers $\eta_k \geq 0$, $k = 1, \ldots, m$, by using a suitable operator D_η. Choosing appropriately the necessary parameters, as in Def. 3, we have proved that $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, the operator of form (21), is indeed a *generalized fractional derivative* with a linear right inverse operator $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, namely:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = f(z), \qquad f \in L_{\mu,p},\ C_\alpha \ \text{ or } \ \mathcal{H}_\mu. \qquad (40)$$

In other words, we have for example in $L_{\mu,p}$ the following

THEOREM 6. *Let $f \in L_{\mu,p}$, let conditions (30) be satisfied and $g(z) = I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z)$. Then, the following inversion formula holds with the generalized fractional derivative defined in (21): $f(z) = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} g(z)$, i.e.*

$$f(z) = \left\{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\right\}^{-1} g(z) = D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} g(z) \quad \text{for} \quad g \in I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left(L_{\mu,p}\right). \quad (41)$$

Next, we shall state the *basic result* for the generalized fractional integrals (14),(16) *suggesting their alternative name "multiple (m-tuple)" fractional integrals.*

THEOREM 7. (Composition/Decomposition theorem) *Under the conditions (30) (resp. (25),(26)), the classical E.-K. fractional integrals of form (4): $I_{\beta_k}^{\gamma_k,\delta_k}, k = 1, \ldots, m$, commute in the space $L_{\mu,p}$ and their product*

$$I_{\beta_m}^{\gamma_m,\delta_m}\left\{I_{\beta_{m-1}}^{\gamma_{m-1},\delta_{m-1}} \ldots \left(I_{\beta_1}^{\gamma_1,\delta_1} f(z)\right)\right\} = \left[\prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}\right] f(z)$$

$$= \int\limits_0^1 \ldots_{(m)} \int\limits_0^1 \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f\left(z \sigma_1^{\frac{1}{\beta_1}} \ldots \sigma_m^{\frac{1}{\beta_m}} \right) d\sigma_1 \ldots d\sigma_m \quad (42)$$

can be represented as an m-tuple E.-K. operator (16), i.e. by means of a single integral involving the H-function:

$$\left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k, \delta_k} \right] f(z) = I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) \quad (43)$$

$$= \int\limits_0^1 H_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] f(z\sigma) d\sigma, \; f \in L_{\mu,p} \, (resp. \, C_\alpha, \mathcal{H}_\mu).$$

Conversely, under the same conditions, each multiple E.-K. operator of form (16) can be represented as a product (42).

Let us note that the same proposition, under additional restrictions, holds for the generalized fractional derivatives (21),(22) as well: they can be seen as products of *E.-K. fractional derivatives* (analogues of R-L derivatives (3) corresponding to E-K integrals (4)) of the form

$$D_\beta^{\gamma,\delta} f(z) := D_{\beta,1}^{\gamma,\delta} f(z) = D_\eta I_\beta^{\gamma+\delta, \eta-\delta} f(z)$$
$$= \left[\prod_{j=1}^\eta \left(\frac{1}{\beta} z \frac{d}{dz} + \gamma + j \right) \right] \int\limits_0^1 \frac{(1-\sigma)^{\eta-\delta-1} \sigma^{\gamma+\delta}}{\Gamma(\eta-\delta)} f(z\sigma^{\frac{1}{\beta}}) d\sigma \,, \quad (44)$$

namely:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{\beta_1}^{\gamma_1,\delta_1} \; D_{\beta_2}^{\gamma_2,\delta_2} \; \ldots \; D_{\beta_m}^{\gamma_m,\delta_m}. \quad (45)$$

Remark. Very recently, Luchko and Trujillo [21] have introduced and studied the so-called E-K fractional derivatives of Caputo type, as extensions of the Caputo modification of the R-L fractional derivatives. About the Caputo derivatives, see e.g. the survey papers in [5], vol. **10**, No 3 (2007).

Combination of Theorems 6 and 7, leads to the *next step in clarifying the structure of variety of known operators*: generalized or classical, fractional or integer order integrations, differentiations or differ-integrations. Namely, in [17] we introduce an *unified theory based on the common notion "generalized fractional differ-integrals"*. By now, operators $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ with all $\delta_k \geq 0, k = 1, ..., m$ have been considered as (fractional) integrals while those with all $\delta_k < 0, k = 1, ..., m$ have been undertaken as formal denotations for the generalized fractional derivatives (cf. (38) and (41)): $I_{(\beta_k),m}^{(\gamma_k'+\delta_k'),(-\delta_k')} =$

$D_{(\beta_k),m}^{(\gamma_k'),(\delta_k')}$, i.e. $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)}$. Now, having a decomposition theorem in mind, we may consider both symbols $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}, D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ as *generalized fractional differ-integrals.* If not all of the components of multi-order of "differ-integration" $\delta = (\delta_1, ..., \delta_m)$ are of the same sign, we simply interpret them as "mixed" products of E.-K. fractional integrals and derivatives. For example, if $\delta_1 < 0, ..., \delta_s < 0,\ \delta_{s+1} = ... = \delta_{s+j} = 0,\ \delta_{s+j+1} > 0, ..., \delta_m > 0$, then

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} := D_{(\beta_1,...,\beta_s),s}^{(\gamma_1+\delta_1,...,\gamma_s+\delta_s),(-\delta_1,...,-\delta_s)}\ I_{(\beta_{s+j+1},...,\beta_m),m-s-j}^{(\gamma_{s+j+1},...,\gamma_m),(\delta_{s+j+1},...,\delta_m)} \tag{46}$$

$$= \prod_{i=1}^{s} D_{\beta_i}^{\gamma_i+\delta_i,-\delta_i} \prod_{k=s+j+1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}\ ,\ \text{a}\ (m-j)\text{-tuple fractional differ-integral.}$$

Remark. A statement more general than Theorem 7, can be found for example in Kalla and Kiryakova [9, 10], and [17, Th.5.2.3]. It deals with products of commuting E-K fractional integrals both of forms (4) (R-L type) and their right-hand sided analogues (so-called Weyl type). Then the result is GFC operator involving kernel-functions of form $H_{m+n,m+n}^{m,n}$ instead (13).

Theorem 7 is the key to the numerous applications of the GFC operators. Some of them can be seen in the monograph [17] and articles [18, 20, 19, 1]. For other properties of these operators, images of elementary and special functions and details of the GFC sketched here, see [17], Chs. 1 and 5.

5. Examples

1) In the case $m = 1$, the "multiple" E-K operators (14),(16) and (21),(45) reduce to the *classical ("single") E-K operators* (4), (44).

2) For $m = 2$ the operators of our GFC reduce to the *hypergeometric operators* (5), since $G_{2,2}^{2,0}$ is expressed via the Gauss hypergeometric function.

3) For $m = 3$ the kernel-function $G_{3,3}^{3,0}$ gives the so-called *Horn's (Appell's) F_3-function.* Such operators of form (14) have been considered by Marichev [25], Saigo et al. [31].

4) Let $m > 1$ be arbitrary, but all $\delta_k = 1, k = 1, \ldots, m$. Then, the operators of form (14),(22): $L = cz^{\beta} I_{\beta,m}^{(\gamma_k),(1,...,1)}$, $B = (1/c) D_{\beta,m}^{(\gamma_k),(1,...,1)} z^{-\beta}$ are the *hyper-Bessel integral operators*, resp. *hyper-Bessel differential operators*

$$B = z^{\alpha_0} \frac{d}{dz} z^{\alpha_1}\ \ldots\ \frac{d}{dz} z^{\alpha_m} = z^{-\beta} \prod_{k=1}^{m} \left(z\frac{d}{dz} + \beta\gamma_k \right), \quad \beta > 0. \tag{47}$$

5) A more general case than 4) gives a fractional indices analogue of the hyper-Bessel operators. The operators $L = zI_{(\rho_k),m}^{(\mu_k-1),(1/\rho_k)}$ of form (16)

happen to be analytical extension of the *Gel'fond-Leontiev operators of generalized integration (resp. differentiation)* with respect to the *multi-index Mittag-Leffler functions*, introduced in Kiryakova [19].

6) Many linear integration and differentiation operators used in geometric functions theory, *in studies on classes of univalent functions*, are GFC operators, see e.g. [20], also Kiryakova in [5], vol. 9, No 2 (2006), 159-176.

For more extensive list of other particular cases of the GFC operators, including *transmutation operators*, see [17, 1] and other recent papers.

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