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A REMARK ON A PAPER ON M-SERIES

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In this note, we show that the M-Series, recently discussed by M. Sharma [*Fract. Calc. Appl. Anal.* **11**, No 2 (2008), 187-192] is not a new special function. As a matter of fact, it is a special case of the generalized Wright function, introduced by E.M. Wright [*J. London Math. Soc.* **10** (1935), 287-293], since

$$\kappa\, {}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(a_1,1),\dots,(a_p,1),(1,1)\\(b_1,1),\dots,(b_q,1),(1,\alpha)\end{array}\middle| z\right]=\sum_{r=0}^{\infty}\frac{(a_1)_r\dots(a_p)_r(1)_r}{(b_1)_r\dots(b_q)_r\Gamma(\alpha r+1)r!}\,z^r$$

$$=\sum_{r=0}^{\infty}\frac{(a_1)_r\dots(a_p)_r}{(b_1)_r\dots(b_q)_r\Gamma(\alpha r+1)}\,z^r={}_p\overset{\alpha}{M}_q\,(z):={}_p\overset{\alpha}{M}_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z)\,, \tag{1}$$

where ${}_p\overset{\alpha}{M}_q\,(z)$ is the so-called M-series, and

$$(a)_r:=\Gamma(a+r)/\Gamma(a)\quad,\quad \kappa=\prod_{j=1}^{q}\Gamma(b_j)_r/\prod_{j=1}^{p}(a_j)_r\quad,\quad \alpha\in\mathbb{C}\quad,\quad p\le q+1. \tag{2}$$

By virtue of (1), it is interesting to observe that the results for fractional integration and differentiation of the M-Series in Sharma's paper are special cases of the results given earlier by A.A. Kilbas [*Fract. Calc. Appl. Anal.* **8**, No 2 (2005), p.117, eq.(11) and p.119, eq.(14)] in the same journal.

Further, the relation connecting the M-series and the $\bar{H}$-function due to A.A. Inayat-Hussain [*J. Phys. A: Gen.* **20** (1987), 4119-4128], as given by M. Sharma in his paper is erroneous. Its corrected version is given by

$$\kappa\, H^{1,p+1}_{p+1,q+1}\left[-z\,\middle|\begin{array}{c}(1-a_1,1),\dots,(1-a_p,1),(0,1)\\(0,1),(1-b_1,1),\dots,(1-b_q,1),(0,\alpha)\end{array}\right]$$

$$={}_p\overset{\alpha}{M}_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z)\,, \tag{3}$$

where the H-function is as defined in the monograph by A.M. Mathai and R.K. Saxena [*The H-Function with Applications in Statistics and Other Disciplines*, Halsted Press & J. Wiley and Sons, 1978]. Finally, in view of (3), the definition of the $\bar{H}$-function given in equations (1) and (2) of Sharma's paper are redundant.

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Editorial Note. This is to thank Prof. R.K. Saxena for his pointing out that the mentioned M-series is more correct to be referred to as a special case of the Fox H-function (as in (3) above), rather than as an Inayat-Hussain $\bar{H}$-function (although the $\bar{H}$-function includes the H-function, but then M. Sharma's eq. (6) needs to be corrected). Of course, it is evident that the M-series is a case of the Wright ${}_{p+1}\Psi_{q+1}$-function, as discussed in the mentioned paper, at top of p.190 (”... being generalized hypergeometric functions...”), but to avoid misunderstandings like this, the corresponding representation needed to be written explicitly!

It is well known fact that (almost) all special functions studied in the literature, more or less extensively, are special cases of the Wright generalized hypergeometric function, thus - of the Fox H-function, and therefore - also of the Inayat-Hussain $\bar{H}$-function... Also well-known (from many papers after Fox's ones in 60's and all the books on special functions involving the H-function), are results for the Riemann-Liouville (R-L) and even for more general fractional integration and differentiation operators of the $H^{m,n}_{p,q}$-function as a $H^{m,n+1}_{p+1,q+1}$-function. All of them can be derived as an easy exercise on using the formula for the integral of product of two H-functions over $(0,\infty)$, say $H^{s,t}_{u,v}$ and $H^{m,n}_{p,q}$, as a $H^{m+t,n+s}_{p+v,q+u}$-function, and knowledge on the basic properties and special cases of the H-function.

In papers like this, the interest is to separate *specific classes of special functions that are in a sense ”close” with respect to fractional integration and differentiation*, as formulas (8) and (10) in Sharma's paper point out (the ${}_p\overset{\alpha}{M}_q$-function has R-L integrals and derivatives being again M-series, but of increased orders $(p+1,q+1)$). As noted additionally by Prof. Saxena, all special functions which are special cases of the H-function and obey the closeness property and the integral addition property are more useful in applications to physical problems than those with merely closeness property. On the other side, the M-series is interesting *as an intermediate case* between the ${}_pF_q$-function (as a ”classical” special function) and the Mittag-Leffler function (as the simplest one of the recently studied ”special functions of fractional calculus”).

V. Kiryakova, Managing Editor “FCAA” Journal