SOME APPLICATIONS OF THE GRAPHIC METHOD

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Abstract. *There are discussed three groups of problems which are solved with the help of the graphic method.*

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The study of the working textbooks ([1], [2], [3], etc.) for the secondary school, reveals that there is not paid the necessary attention on the applications of the graphic method for the solving of different types of problems from the secondary school course in Mathematics. On the other hand, the practice and our own experience [4] show that the solving of problems through the graphic method helps the development of important qualities of thinking, such as: creativity, quickness of mind, combinativeness, etc., and on the same time it brings to the improvement of the skills of students for solving of problems.

The essence of the graphic method is explained in [4] and the accent is put on its use for solving equations and inequalities, containing modules, and on some parametric equations and systems of such equations.

In the present paper we analyze different groups of problems, which can contribute to the enrichment and development of the skills of the students for the solving of these problems applying the graphic method.

I group. Solving some types of equations or inequalities through the graphic method

A. Equations and inequalities with one unknown quantity without parameter.

Problem 1. Solve the equation $x+2^x-3=0$.

Solution. If we look at the function $y = f(x)= x+2^x–3$, we don't know how its graphic looks like. If we write the given equation in the following kind $2^x=3–x$ and having in mind that we know the graphic of the function $y=2^x$, which is on the left side of the equation, and we also know the graphic of the function $y=3–x$, so we can use the graphic method for the solution of the given equation. For this purpose let's draw the graphics of these functions on one and same coordinate system (fig. 1). From the figure we see that the two graphics have only one common point M_0 (because one of the functions is just increasing, while the other is just decreasing one). Therefore the abscissa x_0 of the point M_0 is the only solution of the given equation. Also is evident that $x_0\in(0;3)$. Since the number $x_0=1$ satisfies the equation $2^x=3-x$, so it is a solution of the equation.

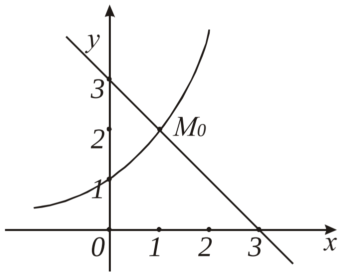

Fig. 1.

Problem 2. Solve the inequality $x+2^x-3>0$.

Solution. Let's write the inequality in the following kind $2^x>3-x$ and from the graphics on fig. 1 it's seen that the graphic of the indicative function $y=2^x$ is above the line $y=3-x$ only when $x>1$. Therefore the solutions of the given inequality are the numbers $x\in(1;\ +\infty)$.

The graphic method is very useful, when we have to solve some equations or inequalities which have one and same expression containing the unknown quantity, while the free article or some of the addends are varied. Then, once we have drawn the graphic of function determined by this repeated expression, we can use it some times to solve the different problems.

Problem 3. Solve the inequalities:

a) $|x+2|-2|x-1|\geq 0$; b) $|x+2|-2|x-1|<3$; c) $|x+2|-2|x-1|>4$;

d) $|x+2|-2|x-1|<4-x$; e) $-3\leq|x+2|-2|x-1|<0$.

Solution. Let's look at the function $y_1=f(x)=|x+2|-2|x-1|$. In order to draw its graphic; we will present it "with some formulae". For this purpose we will beat its definitional area $(-\infty;+\infty)$ into intervals by the numbers that equivalent the different modules to 0. Thus we receive the following expression of the function y_1: $f(x)=\begin{cases}x-4; x\leq-2\\ 3x; -2\leq x\leq 1\\ -x+4; x\geq 1\end{cases}$.

The graphic of this function is formed of the rays $\overrightarrow{AM}, \overrightarrow{BN}$ and the segment *AB* (fig. 2).

In order to solve the different inequalities in the problem, we will also discuss the function $y_2=c$ (*const*), which graphic is a line, parallel to the axis *Ox*.

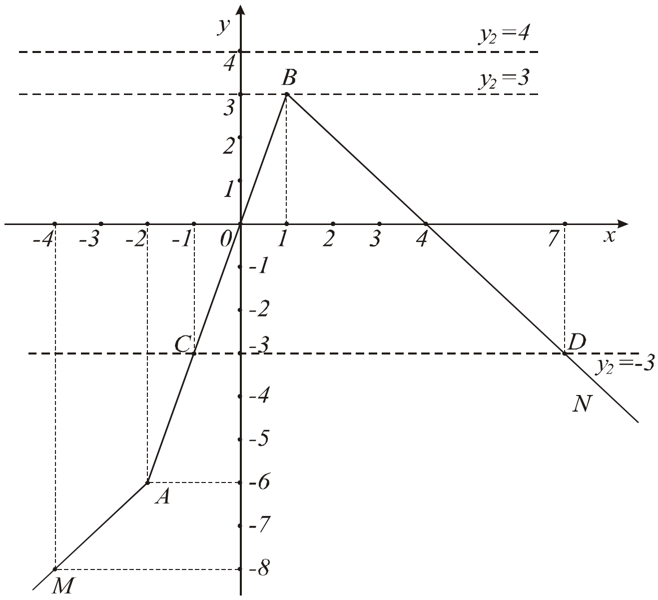

Fig. 2.

Thus giving to c specific values and recording the direction of the corresponding inequality, we receive the following results:

a) When $c=0, y_2 \equiv Ox$, therefore the graphic of the function y_1 is „above” the axis Ox when $x \in (1;4)$, and since the inequality $f(x) \geq 0$ is not strict, then the solutions are all the numbers $x \in [1;4]$.

b) When $c=3$ the line $y_2=3$ passes through the point B and so all the points from the graphic of the function y_1 (except of the point B) are „under” the line $y_2=3$. Then the inequality $f(x) < 3$ is satisfied for any $x \in (-\infty;1) \cup (1;+\infty)$.

c) The inequality $f(x) > 4$ hasn’t got a solution (why?).

d) Here the right side of the inequality is the linear function $y_3 = 4 - x$, which graphic is a line, including the ray $BN^{\rightarrow}$. This means that the graphics of $f(x)$ is „under” the line y_3 when $x < 1$. Therefore the inequality in this case is satisfied for any $x \in (-\infty;1)$.

e) The double inequality $-3 \leq f(x) < 0$ is satisfied for those values of x, for which the graphic of the function $f(x)$ is between the parallel lines $y_2 = -3$ and $y_2 = 0$, i.e. between the line CD and the axis Ox /C is the crossing point of the line $y_2 = -3$ with the segment AB, that’s why it has coordinates (-1;-3), while D is the crossing point of the line $y_2 = -3$ with the ray $BN^{\rightarrow}$ and has coordinates (7;-3)/. Therefore the solutions of this double inequality are all numbers x of the semi-closed from the left interval $x \in [-1;7)$.

B. Equations with a parameter

Problem 4. Find out for which values of the parameter *a* the equation $\sqrt{x+a}=x$ has exactly two solutions?

Solution. The equation is equivalent to the system

$$\begin{cases} x \geq 0 \\ a = x^2 - x. \end{cases}$$

In the coordinate plane *Oxa* this system defines the part of the parabola which is represented with the continuous line on fig. 3, i.e. the graphic is an arc of parabola. All the points of this arc and only they have coordinates (*x; a*), which satisfied the given system.

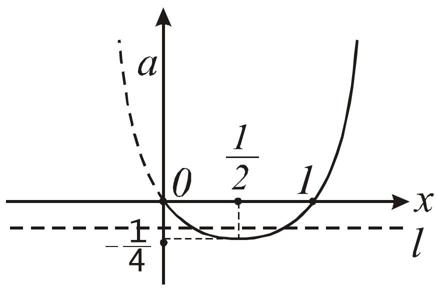

Fig. 3.

That's why the number of the solutions of the equation (and the inequality as well), in every fixed value of the parameter *a*, is equal to the number of points in which the graphic is crossed by the horizontal line corresponding to this certain value of *a*. Since the pick of the parabola has an ordinate

–0,25, then when $-0{,}25 < a \leq 0$ the lines have exactly two common points with the graphic and therefore the given equation has exactly two different solutions.

II group. Systems of equations with two unknown quantities

Problem 5. Find out the values of the parameter *a*, for which the system has a solution.

$$\begin{cases} x = a + \sqrt{y} \\ y^2 - x^2 - 2x + 4y + 3 = 0 \end{cases}$$

Solution. From the first equation of the system when $x \geq a$ follows $y=(x-a)^2$. Therefore this equation sets a family of "semi-parabolas" (the parabolas $y=(x-a)^2$ „slide“ with their pick through the x-axis, moreover we discuss only the right sector). Let's expand into factors the left side of the second equation:

$$y^2 - x^2 - 2x + 4y + 3 = (y^2 + 4y + 4) - (x^2 + 2x + 1) = (y + x + 3)(y - x + 1).$$

Then the graphic of the second equation is the union of the two lines y = –x–3 and y = x–1 (fig. 4).

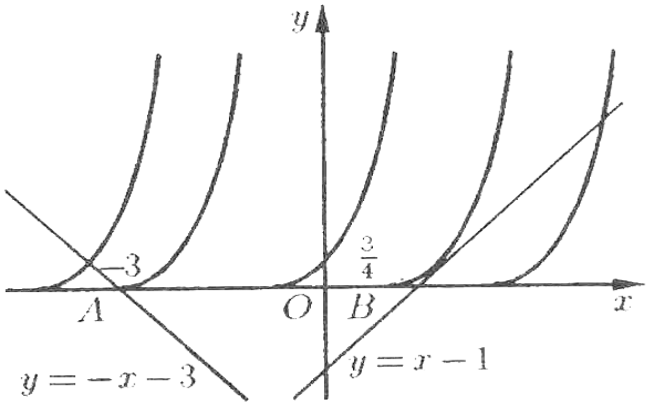

Fig. 4.

Let's see for which values of the parameter *a* the family of "semi- parabolas" has even one common point together with one of the two lines.

If the picks of the "semi-parabolas" are situated to the right of point *A*(–3;0), but to the left of point *B*(0,75;0) – then it corresponds to the situation of the pick at the moment when the "semi-parabola" contacts the line $y = x - 1$, so it's obvious that the graphics haven't got common points.

If the pick is situated in point *A*, then it's obvious that $a = -3$. When $a < -3$, the line y = –x–3 crosses the „semi-parabolas".

We will "catch" the case of contact as we examine when the system $\begin{cases} y = x - 1 \\ y = (x-a)^2 \end{cases}$ has one solution – when the equation $x - 1 = (x-a)^2$ has a single solution. This is satisfied when *a* = 0,75. When *a* > 0,75 the line, y = x–1 crosses the „semi-parabolas".

Consequently the system has a solution, if $a \leq -3$ or $a \geq 0,75$, and it hasn't got a solution, when –3<*a*<0,75.

Problem 6. How many solutions has the system

$\begin{cases} y - x^2 = c \\ x - y^2 = c \end{cases}$ according to the values of the parameter *c*?

Solution. Let's write the system in the following way $\begin{cases} y = x^2 + c \\ x - (x^2 + c)^2 = c \end{cases}$

The number of the solutions of the second equation is equal to the number of solutions of the whole system. It's normal form is $x^4 + 2cx^2 - x + c^2 + c = 0$.

With regard to x the equation is from fourth rate, but according to c it's from second rate. That's why we will treat it as a quadratic with regard to c and receive the set of equations

$$\left[\begin{array}{l} c=-x^2+x \\ c=-x^2-x-1 \end{array}\right.$$

Their graphics in the coordinate plane Oxc are shown on fig.5.

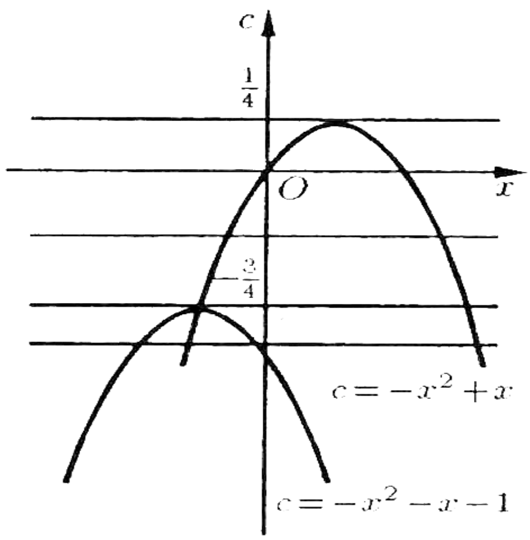

Fig. 5.

The coordinates of the crossing points of the parabolas can be found out, as we solve the equation $-x^2-x-1=-x^2+x$, where is obtained $x=-0{,}5$.

In order to write down the answer, we have to note that the common point of these parabolas is the pick of the parabola $c=-x^2-x-1$.

Finally we receive the following result: if c<0,75, the system has four solutions; if $-0{,}75\le c<0{,}25$, the solutions are two; if $c=0{,}25$, the solution is one; if $c>0{,}25$, the system has no solutions.

III group. Inequalities of first rate with two unknown quantities

While solving such inequalities, there must be given a geometric interpretation of their solutions and they have to be written in a system, which is practically not performed in the secondary school. We will present this solving the following problems.

Problem 7. Give a geometric interpretation of the solutions of the inequality $\boldsymbol{x+y-1>0}$ and write down the two kinds of records of its solutions.

Solution. Let's write down the inequality in the form $y > -x + 1$. Let's draw the graphic of the function $y = -x + 1$ - the line *AB* on fig. 6. Because of $y > -x + 1$, the solutions of the given inequality are the coordinates of all the points that are "above" the line *AB*, i.e. the geometric interpretation of the solutions is the shaded part on fig 6.

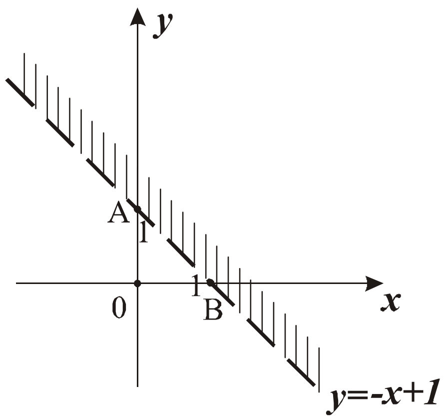

Fig. 6.

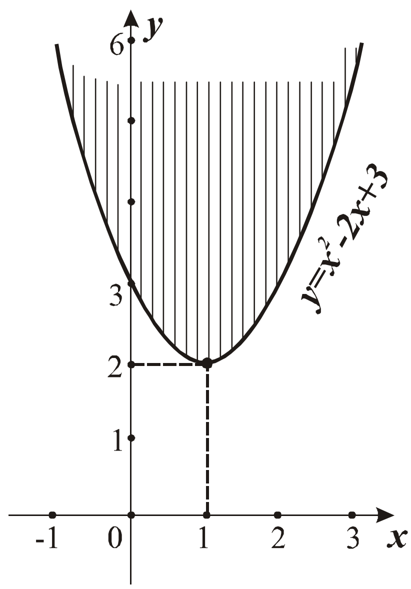

Fig. 7

These solutions can be written in two ways:

$$\begin{cases} -\infty < x < +\infty \\ 1 - x < y < +\infty \end{cases} \quad \text{or} \quad \begin{cases} -\infty < y < +\infty \\ 1 - y < x < +\infty \end{cases}.$$

Problem 8. Give a geometric interpretation of the solutions of the inequality $x(x-2) \leq y-3$ and write down the two records of its solutions.

Solution. Let's present the given inequality in the form $y \geq x^2 - 2x + 3$. Let's draw the graphic of the function $y = x^2 - 2x + 3$ - the parabola on fig. 7. Since $y \geq x^2 - 2x + 3$ (the inequality is not strict), then the solutions are all the ordered pairs of numbers, whose images on the coordinative plane are the points of the parabola $y = x^2 - 2x + 3$ and the points that are "above" it.

These solutions can be written in the following way:

$$\left|\begin{array}{l} -\infty < x < +\infty \\ x^2 - 2x + 3 < y < +\infty \end{array}\right. \quad \text{or} \quad \left|\begin{array}{l} 2 \leq y < +\infty \\ 1 - \sqrt{y-2} \leq y < 1 + \sqrt{y-2} \end{array}\right.,$$

where $y = 2$ is the minimum of the quadratic function $y = x^2 - 2x + 3$.

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