Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 4 (2010) ISSN 1311-0454

APPLICATION OF SUBORDINATION PRINCIPLE TO LOG-HARMONIC α-SPIRALLIKE MAPPINGS

Melike Aydoğan [1], **Yaşar Polatoğlu** [2]

We dedicate this paper to the 70^{th} anniversary of Professor Srivastava

Abstract

Let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving logharmonic mapping is the solution of the non-linear elliptic partial differential equation $\overline{f_{\overline{z}}} = w(z) f_z (\frac{\overline{f}}{f})$ where $w(z) \in H(D)$ is the second dilatation of f such that $|w(z)| < 1$ for all $z \in D$. It has been shown that if f is a non-vanishing logharmonic mapping, then f can be expressed as $f(z) = h(z).\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation $f = z.|z|^{2\beta} h(z)\overline{g(z)}$, where $Re\beta > -\frac{1}{2}$ and $h(z)$, $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. [1], [2], [3]. The class of all logharmonic mappings is denoted by S^*_{LH}.

The aim of this paper is to give an application of the subordination principle to the class of spirallike logharmonic mappings which was introduced by Abdulhadi and Hengartner [1].

MSC 2010: 30C45, 30C55

Key Words and Phrases: analytic function, log-harmonic mapping, spirallike function

---

1. Introduction

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined in the open unit disc $D = \{z \in C : |z| < 1\}$. A sense preserving log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{1}$$

where $w(z)$ is the second dilatation of f and $w(z) \in H(D)$, $|w(z)| < 1$ for every $z \in D$. It has been shown that if f is non vanishing logharmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{2}$$

where $h(z)$ and $g(z)$ are analytic in D with the normalization $h(0) \neq 0$, $g(0) = 1$. On the other hand if f vanishes at $z = 0$, but it is not identically zero, then f admits the following representation

$$f = z.\,|z|^{2\beta}\, h(z)\overline{g(z)}, \tag{3}$$

where $Re\beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in the open disc D with the normalization $h(0) \neq 0$, $g(0) = 1$. Also we note that the univalent logharmonic mapping have been studied extensively (see [1], [2], [3]) and the class of univalent logharmonic mappings is denoted by S_{LH}.

Let $f = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping. We say that f is a starlike logharmonic mapping if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0$$

for all $z \in D$, and the class of all starlike logharmonic mappings is denoted by ST^*_{LH}.

Let $\varphi(z)$ be analytic in D and let α be a real number such that $|\alpha| < \frac{\pi}{2}$. If $\varphi = 0$, $\varphi'(0) \neq 0$ and if

$$Re(e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)}) > 0, \tag{4}$$

then $\varphi(z)$ is univalent (see [5]) and is said to be spirallike. Under these conditions we have

$$e^{i\alpha}z\frac{\varphi'(z)}{\varphi(z)} = Q(z), \tag{5}$$

where $ReQ(z) > 0$ and $Q(0) = e^{i\alpha}$. Defining $P(z) = Q(z)\sec\alpha - i\tan\alpha$, we may write

$$z\frac{\varphi'(z)}{\varphi(z)} = e^{-i\alpha}[P(z)\cos\alpha + i\sin\alpha], \tag{6}$$

where $ReP(z) > 0$, $P(0) = 1$. The class of spirallike functions is denoted by S^*_α. In particular with $\alpha = 0$, S^*_0 coincides with the class of normalized starlike functions. The relationship between S^*_α and S^*_0 is indicated in the following lemma.

LEMMA 1.1. *$f(z) \in S_{0,p}$ if and only if there is a $g(z) \in S_{0,p}$ such that*

$$[\frac{f(z)}{z}]^{\exp(i\alpha)} = [\frac{g(z)}{z}]^{\cos\alpha}, \tag{7}$$

where the branches are chosen so that each side of the equation has the value 1, when $z = 0$.

On the other hand, in the paper by Abdulhadi and Muhanna [3], the following theorem was proved.

THEOREM 1.2. *Let $f(z) = z.h(z).\overline{g(z)}$ be a logharmonic mapping in D, $0 \notin hg(D)$. Then $f \in ST^*_{LH}$ if and only if $\varphi(z) = z\frac{h(z)}{g(z)} \in ST^*$.*

Finally, let Ω be the family of functions $\phi(z)$ which are analytic in D and satisfying the conditions $\phi(0) = 0$ $|\phi(z)| < 1$ for every $z \in D$ and let $s_1(z) = z + a_2z^2 + a_3z^3 + ...$, $s_2(z) = z + b_2z^2 + b_3z^3 + ...$ be analytic functions in D. We say that $s_1(z)$ is subordinate to $s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in D$ and denote by $s_1(z) \prec s_2(z)$.

2. Main results

Considering Lemma 1.1 and Theorem 1.2 together, we obtain the following lemma.

LEMMA 2.1. *$\phi(z) \in S^*_\alpha$ if and only if there is a $f(z) = zh(z)\overline{g(z)} \in ST^*_{LH}$ such that*

$$(\frac{\phi(z)}{z})^{e^{i\alpha}} = (\frac{h(z)}{g(z)})^{\cos\alpha}, \tag{8}$$

where the branches are chosen so that both sides of the equation has the value 1, when $z = 0$.

THEOREM 2.2. *Using Lemma 2.1, we have the following equality*

$$e^{i\alpha}z.\frac{\phi'(z)}{\phi(z)} = \cos\alpha[1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}] + i\sin\alpha. \tag{9}$$

P r o o f. We have:

$$f = z.\,|z|^{2\beta}\,h(z)\overline{g(z)} \Rightarrow \frac{zf_z}{f} = \beta + 1 + z\frac{h'(z)}{h(z)};\ \frac{\overline{z}f_{\overline{z}}}{f} = \beta + \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}; \tag{10}$$

$$w(z) = \frac{\overline{f}_{\overline{z}}}{\overline{f}}\frac{f}{f_z} = \frac{\overline{\beta} + z\frac{g'(z)}{g(z)}}{1 + \beta + z\frac{h'(z)}{h(z)}}. \tag{11}$$

In the equality (11) if we take $\beta = 0$, then we obtain:

$$w(z) = \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}}. \tag{12}$$

Therefore, we have $w(0) = 0$, $|w(z)| < 1$, and then we can say that $w(z)$ satisfies the conditions of the Schwarz lemma, and

$$1 - w(z) = \frac{1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}} \tag{13}$$

$$\frac{w(z)}{1 - w(z)} = \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}}. \tag{14}$$

Using equalities (9), (10), equalities (13) and (14) can be written in the following form

$$1 - w(z) = \frac{\frac{1}{\cos\alpha}[z\frac{\phi'(z)}{\phi(z)} - i\sin\alpha]}{z\frac{f_z}{f}}, \tag{15}$$

$$\frac{w(z)}{1 - w(z)} = \frac{z\frac{\overline{f}_{\overline{z}}}{\overline{f}}}{\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)} - i\sin\alpha]}. \tag{16}$$

Using the subordination principle, these equalities can be written as

$$\frac{\frac{1}{\cos\alpha}[z\frac{\phi'(z)}{\phi(z)} - i\sin\alpha]}{z\frac{f_z}{f}} \prec 1 - z, \tag{17}$$

$$\frac{z\frac{\overline{f}_{\overline{z}}}{\overline{f}}}{\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)} - i\sin\alpha]} \prec \frac{z}{1 - z}. \tag{18}$$

On the other hand, since the transformations $1-z$ and $\frac{z}{1-z}$ map $|z| = r$ onto the discs with centers $c_1(r) = (1, 0)$, $c_2(r) = (\frac{r^2}{1-r^2})$ and radius $\rho_1(r) = r$ $\rho_2(r) = \frac{r}{1-r^2}$ respectively, using the subordination principle then we have:

$$\left|\frac{\frac{1}{\cos\alpha}[z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]}{z\frac{f_z}{z}}-1\right|\le r;\left|\frac{z\frac{\overline{f_{\overline{z}}}}{f}}{\frac{1}{\cos\alpha}[e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha]}-\frac{r^2}{1-r^2}\right|\le\frac{r}{1-r^2}. \tag{19}$$

After simple calculations, from equalities (19) we get the following theorem. The inequalities (19) can be written in the form, respectively:

$$\frac{\left|\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)\right|}{1-r}\le\left|\frac{zf_z}{f}\right|\le\frac{\left|\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)\right|}{1+r}, \tag{20}$$

$$\left|\frac{zf_{\overline{z}}}{f}\right|\le\frac{r\left|\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)\right|}{1-r}. \tag{21}$$

On the other hand we have:

$$p(z)=\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)\Rightarrow\left|p(z)-\frac{1+r^2}{1-r^2}\right|\le\frac{2r}{1-r^2}\Rightarrow$$

$$\left|\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)-\frac{1+r^2}{1-r^2}\right|\le\frac{2r}{1-r^2}\Rightarrow$$

$$\frac{(\cos\alpha-|\sin\alpha|)-(\cos\alpha+|\sin\alpha|)r}{1+r}\le\left|z\frac{\phi'(z)}{\phi(z)}\right|\le\frac{(\cos\alpha+|\sin\alpha|)-(\cos\alpha-|\sin\alpha|)r}{1-r} \tag{22}$$

$$\frac{1}{\cos\alpha}\left|z\frac{\phi'(z)}{\phi(z)}\right|-|\tan\alpha|\le\left|\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{\phi'(z)}{\phi(z)}-i\sin\alpha)\right|\le\frac{1}{\cos\alpha}\left|z\frac{\phi'(z)}{\phi(z)}\right|+|\tan\alpha|. \tag{23}$$

Using the inequalities (22) and (23) in the inequalities (20) and (21), and after the simple calculations, we get

$$F_1(r,\alpha)\le\left|\frac{zf_z}{f}\right|\le F_2(r,\alpha),$$

$$F_1(r,\alpha)=\frac{(\cos\alpha-|\sin\alpha|-\cos\alpha\,|\tan\alpha|)-(\cos\alpha+|\sin\alpha|+\cos\alpha\,|\tan\alpha|)r}{(1-r^2)\cos\alpha},$$

$$F_2(r,\alpha)=\frac{(\cos\alpha+|\sin\alpha|+\cos\alpha\,|\tan\alpha|)+(\cos\alpha-|\sin\alpha|-\cos\alpha\,|\tan\alpha|)r}{(1-r^2)\cos\alpha},$$

$$\left|\frac{zf_{\overline{z}}}{f}\right|\le F_3(r,\alpha), \tag{24}$$

$$F_3(r,\alpha)=\frac{r[(\cos\alpha+|\sin\alpha|+\cos\alpha\,|\tan\alpha|)+(\cos\alpha-|\sin\alpha|-\cos\alpha\,|\tan\alpha|)r]}{(1-r)^2\cos\alpha}.$$

■

So, we have the following theorem.

THEOREM 2.3 *Let $f = zh(z)\overline{g(z)}$ be logharmonic spirallike function then*

$$F_1(r,\alpha) \leq \left|\frac{zf_z}{f}\right| \leq F_2(r,\alpha),$$

$$\left|\frac{zf_{\overline{z}}}{f}\right| \leq F_3(r,\alpha).$$

P r o o f. Using Theorem 2.2, we get the result. ■

References

[1] Z. Abdulhadi and W.Hengartner, Spirallike logharmonic mappings. *Complex Variables Theory Appl.* **9** (1987), 121-130.

[2] Z. Abdulhadi and W.Hengartner, One pointed univalent logharmonic mappings. *J. Math. Anal. Appl.* **203**, No 2 (1996), 333-351.

[3] Z. Abdulhadi and Y.A.Muhanna, Starlike log-harmonic mappings of order α . *J. Inequl. Pure Appl. Math.* **7**, No 4 (2006), Article 123.

[4] Z. Abdulhadi, Close to starlike logharmonic mappings. *Internat J.Math and Math. Sci.* **19**, No 3 (1996), 563-574.

[5] Z. Abdulhadi, Typically real logharmonic mappings. *Internat J.Math and Math. Sci.* **31**, No 1 (2002), 1-9.

[6] Z. Abdulhadi and D. Bshouty, Univalent functions in H.$\overline{H}$. *Trans. Amer. Math. Soc.* **305**, No 2 (1988), 841-849.

[7] T. Basgoze and F.R. Keogh, The Hardy class of a spirallike function and its derivative. *Proc. of the Amer. Math. Soc.* **26**, No 2 (Oct. 1970), 266-269.

[1] *Department of Computer Engineering*
Faculty of Computer Engineering and Architecture
Yeni Yuzyil University
Ayazma Road No 26, Topkapı, İstanbul, TURKEY
e-mail: melike.aydogan@yeniyuzyil.edu.tr

Received: GFTA, August 27-31, 2010

[2] *Department of Mathematics and Computer Sciences*
İstanbul Kültür University
34156, Bakirköy, İstanbul, TURKEY
e-mail: y.polatoglu@iku.edu.tr