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COEFFICIENT ESTIMATES FOR ANALYTIC FUNCTIONS CONCERNED WITH HANKEL DETERMINANT

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We dedicate this paper to the 70th anniversary of Professor Srivastava

Abstract

For the coefficients of analytic functions $f(z)$ defined by the conditions $f(0) = f'(0) - 1 = 0$ and $\mathrm{Re}[\alpha(f(z)/z) + \beta f'(z)] > \gamma$ for some complex parameters α and β satisfying $\alpha + n\beta \neq 0$ $(n = 1, 2, 3, \cdots)$, and for some real number γ $(0 \leqq \gamma < 1)$, where $\mathrm{Re}(\alpha + \beta) > \gamma$ in the open unit disk $\mathbb{U}$, the upper bounds of the generalized functional $\left|a_n a_{n+2} - \mu a_{n+1}^2\right|$ concerned with the second Hankel determinant $H_2(n)$ for all n $(n = 1, 2, 3, \cdots)$ and some real number μ are discussed. Furthermore, by the help of these results, the same things for starlike functions of order γ $(\frac{1}{2} \leqq \gamma < 1)$ in $\mathbb{U}$ are also considered.

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions. Furthermore, let $\mathcal{P}(\gamma)$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in $\mathbb{U}$ and satisfy $\mathrm{Re}(p(z)) > \gamma$ $(z \in \mathbb{U})$ for some γ $(0 \leqq \gamma < 1)$. In particular, we say that $p(z) \in \mathcal{P} \equiv \mathcal{P}(0)$ is the Carathéodory function (cf. [1]).

We next introduce the following subclass

$$\mathcal{QR}(\alpha, \beta; \gamma) = \left\{ f(z) \in \mathcal{A} : \mathrm{Re}\left[\alpha\left(\frac{f(z)}{z}\right) + \beta f'(z)\right] > \gamma \quad (z \in \mathbb{U}) \right\}$$

of $\mathcal{A}$ for some complex parameters α and β satisfying $\alpha + n\beta \neq 0$ $(n = 1, 2, 3, \cdots)$, and for some real number γ $(0 \leqq \gamma < 1)$, where $\mathrm{Re}(\alpha + \beta) > \gamma$. Then, for the case $\alpha \leqq 1$, we write

$$\mathcal{QR}_\alpha(\gamma) \equiv \mathcal{QR}(\alpha, 1-\alpha; \gamma), \quad \mathcal{Q}(\gamma) \equiv \mathcal{QR}_1(\gamma) \quad \text{and} \quad \mathcal{R}(\gamma) \equiv \mathcal{QR}_0(\gamma).$$

Example 1.1. Let $A(\alpha, \beta, \gamma) = \dfrac{2\,(\mathrm{Re}(\alpha+\beta) - \gamma)}{\alpha + \beta}$. Then, the function

$$f_d(z) = \begin{cases} (1 - A(\alpha,\beta,\gamma))\, z + A(\alpha,\beta,\gamma) z\, {}_2F_1\left(\dfrac{\alpha+\beta}{d\beta}, 1; 1 + \dfrac{\alpha+\beta}{d\beta}; z^d\right) & (\text{if } \ \beta \neq 0) \\[2ex] \dfrac{z + \left(\dfrac{2\,(\mathrm{Re}(\alpha) - \gamma)}{\alpha} - 1\right) z^{d+1}}{1 - z^d} & (\text{if } \ \beta = 0) \end{cases} \tag{1.1}$$

belongs to the class $\mathcal{QR}(\alpha, \beta; \gamma)$.

Remark 1.2. The function $f_d(z)$ has the following Taylor's series expansion

$$f_d(z) = z + \sum_{k=1}^{\infty} \frac{2\,(\mathrm{Re}(\alpha+\beta) - \gamma)}{\alpha + (kd+1)\beta} z^{kd+1}.$$

The next lemma was given by Jack [5].

Lemma 1.3. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. If there is a point $z_0 \in \mathbb{U}$ such that*

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)|,$$

then it follows that

$$z_0 w'(z_0) = k w(z_0)$$

for some real number k $(k \geqq 1)$.

Using the above lemma, we obtain the following result.

Theorem 1.4. *For $\gamma^* = \dfrac{2\gamma + 1 - \alpha}{3 - \alpha}$, the inclusion relation*

$$\mathcal{QR}_\alpha(\gamma) \subset \mathcal{Q}(\gamma^*)$$

holds true.

P r o o f. For $f(z) \in \mathcal{QR}_\alpha(\gamma)$, we define $w(z)$ by

$$\frac{f(z)}{z} = \frac{1 + (1 - 2\gamma^*)w(z)}{1 - w(z)} \qquad \left(0 \leqq \gamma^* = \frac{2\gamma + 1 - \alpha}{3 - \alpha} < 1\right).$$

Then, we have that

$$f'(z) = \frac{1 + (1 - 2\gamma^*)w(z)}{1 - w(z)} + \frac{2(1 - \gamma^*)zw'(z)}{(1 - w(z))^2}.$$

Thus, we see that

$$\alpha\left(\frac{f(z)}{z}\right) + (1 - \alpha)f'(z) = \frac{1 + (1 - 2\gamma^*)w(z)}{1 - w(z)} + 2(1 - \alpha)\frac{(1 - \gamma^*)zw'(z)}{(1 - w(z))^2}.$$

Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, if there is a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then we can write

$$w(z_0) = e^{i\theta}, \quad z_0 w'(z_0) = k w(z_0) = k e^{i\theta} \qquad (k \geqq 1)$$

by Lemma 1.3. For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned}
&\mathrm{Re}\left[\alpha\left(\frac{f(z_0)}{z_0}\right) + (1 - \alpha)f'(z_0)\right] \\
&= \mathrm{Re}\left[\frac{1 + (1 - 2\gamma^*)w(z_0)}{1 - w(z_0)} + 2(1 - \alpha)\frac{(1 - \gamma^*)zw'(z_0)}{(1 - w(z_0))^2}\right] \\
&= \mathrm{Re}\left[\frac{1 + (1 - 2\gamma^*)e^{i\theta}}{1 - e^{i\theta}}\right] + 2(1 - \alpha)\mathrm{Re}\left[\frac{(1 - \gamma^*)ke^{i\theta}}{(1 - e^{i\theta})^2}\right] \\
&= \gamma^* - \frac{(1 - \alpha)(1 - \gamma^*)k}{1 - \cos\theta}
\end{aligned}$$

$$\leqq \frac{1}{2}\left\{(3-\alpha)\gamma^{*}-(1-\alpha)\right\}=\gamma$$

which contradicts our assumption. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Noting that

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)=\operatorname{Re}\left(\frac{1+(1-2\gamma^{*})w(z)}{1-w(z)}\right)>\gamma^{*}$$

for $|w(z)| < 1$, we see that $f(z) \in \mathcal{Q}(\gamma^*)$. ∎

Putting $\alpha = 0$ in Theorem 1.4, we obtain

Corollary 1.5.

$$\mathcal{R}(\gamma) \subset \mathcal{Q}(\gamma^{*}), \quad \textit{where } \gamma^{*}=\frac{2\gamma+1}{3}.$$

Noonan and Thomas [7] (see also [8]) have stated the q–th Hankel determinant as

$$H_q(n)=\det\begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \quad (n,q \in \mathbb{N}=\{1,2,3,...\}).$$

This determinant is investigated by many mathematicians with $q = 2$. For example, we can know that the functional $|H_2(1)| = \left|a_3 - a_2^2\right|$ is known as the Fekete-Szegö functional (see, [2]) and they found the sharp upper bounds of the further generalized functional $\left|a_3 - \mu a_2^2\right|$ for functions $f(z) \in \mathcal{S}$ where μ is some real number as follows:

Theorem 1.6. *If $f(z) \in \mathcal{S}$, then*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} 3-4\mu & (\mu \leqq 0) \\ 1+2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & (0 \leqq \mu \leqq 1) \\ 4\mu-3 & (\mu \geqq 1). \end{cases}$$

Recently, the functional $|H_2(2)| = \left|a_2a_4 - a_3^2\right|$ and its generalization functional $\left|a_2a_4 - \mu a_3^2\right|$ are also studied. Therefore, motivated by these functionals, we discuss the upper bounds of the functional $\left|a_na_{n+2} - \mu a_{n+1}^2\right|$ defined by using the coefficients of functions $f(z) \in \mathcal{QR}(\alpha, \beta; \gamma)$ for each n $(n = 1, 2, 3, \cdots)$ and real number μ in this article.

2. Preliminary results

In order to discuss our problems, we need some known or new results. The following lemma is well-known result.

LEMMA 2.1. *If a function* $p(z) \in \mathcal{P}(\gamma)$, *then*

$$|c_k| \leqq 2(1-\gamma) \quad (k = 1, 2, 3, \cdots).$$

The result is sharp for

$$p(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + \sum_{k=1}^{\infty} 2(1-\gamma)z^k.$$

Hayami and Owa [3] have given the following result.

LEMMA 2.2. *If a function* $p(z) \in \mathcal{P}(\gamma)$, *then the representations*

$$\begin{cases} 2(1-\gamma)c_2 = c_1^2 + \{4(1-\gamma)^2 - c_1^2\}\zeta \\ 4(1-\gamma)^2 c_3 = c_1^3 + 2\{4(1-\gamma)^2 - c_1^2\}c_1\zeta - \{4(1-\gamma)^2 - c_1^2\}c_1\zeta^2 \\ \qquad\qquad +2(1-\gamma)\{4(1-\gamma)^2 - c_1^2\}(1-|\zeta|^2)\eta \end{cases}$$

for some complex numbers ζ *and* η $(|\zeta| \leqq 1, |\eta| \leqq 1)$, *are obtained.*

THEOREM 2.3. *If a function* $p(z) \in \mathcal{P}$, *then*

$$|\nu c_n - c_{n-s}c_s| \leqq \begin{cases} 2|2-\nu| & (\mathrm{Re}(\nu) \leqq 1) \\ 2|\nu| & (\mathrm{Re}(\nu) \geqq 1)\,. \end{cases} \qquad (1 \leqq s < n)$$

Equality is attained for the function

$$p(z) = \frac{1+z^d}{1-z^d} \quad (\mathrm{Re}(\nu) \leqq 1) \quad \text{and} \quad p(z) = \frac{1+z^l}{1-z^l} \quad (\mathrm{Re}(\nu) \geqq 1)\,,$$

where d *is a positive common devisor of* $(n, s, n-s)$ *and* $l|n$, $l \not| s$, $l \not| (n-s)$ *and* l *is positive.*

P r o o f. Since any function $p(z)$ satisfying the condition of the theorem is the limit of a sequence of functions of the form

$$p_m(z) = \sum_{k=1}^{m} \lambda_k \frac{1+e^{it_k}z}{1-e^{it_k}z}, \tag{2.1}$$

where $\lambda_k \geqq 0$ and $\sum_{k=1}^{m} \lambda_k = 1$, it is sufficient that we only prove the theorem for functions of the form (2.1). For such functions, we can write that

$$c_n = \sum_{k=1}^{m} 2\lambda_k e^{int_k} \qquad (n \geqq 1).$$

Thus, applying the triangle inequality and the Schwarz inequality, we see that

$$|\nu c_n - c_{n-s}c_s| = 2\left|\sum_{k=1}^{m} \lambda_k(\nu e^{int_k} - 2e^{ist_k})Q\right| = 2\left|\sum_{k=1}^{m} \lambda_k B_k\right|$$

$$\leqq 2\sum_{k=1}^{m} \lambda_k |B_k| \; \leqq \; 2\left(\sum_{k=1}^{m} \lambda_k |B_k|^2\right)^{\frac{1}{2}},$$

where $Q = \sum_{k=1}^{m} \lambda_k e^{i(n-s)t_k}$ and $B_k = (\nu e^{int_k} - 2e^{ist_k})Q$. Then, letting $\nu = |\nu|e^{i\varphi}$ $(\varphi \in \mathbb{R})$ and noting that

$$\sum_{k=1}^{m} \lambda_k |B_k|^2 = |\nu|^2 + 4|Q|^2 - 4|\nu| \mathrm{Re}\left(e^{-i\varphi}\sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} Q\right),$$

$$\mathrm{Re}\left(e^{-i\varphi}\sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} Q\right) = \mathrm{Re}\left(e^{-i\varphi}\left|\sum_{k=1}^{m} \lambda_k e^{i(n-s)t_k}\right|^2\right) = |Q|^2 \cos\varphi$$

and

$$|Q|^2 = \left|\sum_{k=1}^{m} \lambda_k e^{i(n-s)t_k}\right|^2 \leqq \left(\sum_{k=1}^{m} \lambda_k |e^{i(n-s)t_k}|\right)^2 = 1,$$

we have

$$2\left(\sum_{k=1}^{m} \lambda_k |B_k|^2\right)^{\frac{1}{2}} = 2\sqrt{|\nu|^2 + 4|Q|^2(1 - |\nu|\cos\varphi)}$$

$$\leqq \begin{cases} 2\sqrt{|\nu|^2 + 4(1 - \mathrm{Re}(\nu))} = 2|2-\nu| & \big(\mathrm{Re}(\nu) \leqq 1;\ |Q|^2 = 1\big) \\ 2|\nu| & \big(\mathrm{Re}(\nu) \geqq 1;\ |Q|^2 = 0\big). \end{cases}$$

The proof of the theorem is completed. ∎

Remark 2.4. When ν is a real number in Theorem 2.3, we arrive at the result due to Hayami and Owa [4]. Espesially, if $\nu = 1$, we have the lemma proven by Livingston [6].

3. Main results

Our first result is contained in the following theorem.

THEOREM 3.1. *If $f(z) \in \mathcal{QR}(\alpha, \beta; \gamma)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)}{|\alpha+3\beta|}\,|B(\mu)| & (|B(\mu)| \geqq 1) \\[2ex] \dfrac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)}{|\alpha+3\beta|} & (|B(\mu)| \leqq 1) \end{cases}$$

with equality for $f_1(z)$ $(|B(\mu)| \geqq 1)$ and $f_2(z)$ $(|B(\mu)| \leqq 1)$ given by (1.1), where

$$B(\mu) = 1 - \frac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)(\alpha+3\beta)}{(\alpha+2\beta)^2}\mu.$$

P r o o f. For functions $f(z) \in \mathcal{QR}(\alpha, \beta; \gamma)$, it follows that

$$p(z) = \frac{\alpha\left(\dfrac{f(z)}{z}\right) + \beta f'(z) - \gamma - i\mathrm{Im}(\alpha+\beta)}{\mathrm{Re}(\alpha+\beta)-\gamma} = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}.$$

Moreover, we know that

$$a_n = \frac{\mathrm{Re}(\alpha+\beta)-\gamma}{\alpha+n\beta}c_{n-1} \qquad (n \geqq 2) \tag{3.1}$$

By the help of Lemma 2.1 and Lemma 2.2, without losing generality, we can suppose that $0 \leqq c_1 \leqq 2$ and we deduce

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{\mathrm{Re}(\alpha+\beta)-\gamma}{\alpha+3\beta}c_2 - \mu\left(\frac{\mathrm{Re}(\alpha+\beta)-\gamma}{\alpha+2\beta}\right)^2 c_1^2\right|$$

$$= \left(\mathrm{Re}(\alpha+\beta)-\gamma\right)\left|\left(\frac{1}{2(\alpha+3\beta)} - \frac{\mathrm{Re}(\alpha+\beta)-\gamma}{(\alpha+2\beta)^2}\mu\right)c_1^2 + \frac{4-c_1^2}{2(\alpha+3\beta)}\zeta\right|$$

$$\leqq \frac{\mathrm{Re}(\alpha+\beta)-\gamma}{|\alpha+3\beta|}\left[2 + \frac{c_1^2}{2}\left\{\left|1 - \frac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)(\alpha+3\beta)}{(\alpha+2\beta)^2}\mu\right| - 1\right\}\right]$$

$$\leqq \begin{cases} \dfrac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)}{|\alpha+3\beta|}\,|B(\mu)| & (|B(\mu)| \geqq 1;\ c_1 = 2, c_2 = 2) \\[2ex] \dfrac{2\left(\mathrm{Re}(\alpha+\beta)-\gamma\right)}{|\alpha+3\beta|} & (|B(\mu)| \leqq 1;\ c_1 = 0, c_2 = 2) \end{cases}$$

This completes the proof of the theorem. ∎

We next derive the following theorem.

THEOREM 3.2. *If* $f(z) \in \mathcal{QR}(\alpha,\beta;\gamma)$, *then* $\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq$

$$\begin{cases} 2\left|\dfrac{(\mathrm{Re}(\alpha+\beta)-\gamma)^2}{(\alpha+n\beta)(\alpha+(n+2)\beta)}\right|\left[|2-C(\mu)|+|C(\mu)|\right] & (\mathrm{Re}\,[C(\mu)] \leqq 1)\,, \\ 4\left|\left(\dfrac{\mathrm{Re}(\alpha+\beta)-\gamma}{\alpha+(n+1)\beta}\right)^2\mu\right| & (\mathrm{Re}\,[C(\mu)] \geqq 1)\,, \end{cases}$$

for $n \geqq 2$, *with equality for* $f_d(z)$ $(\mathrm{Re}\,[C(\mu)] \geqq 1;\ d \geqq 2$ *and* $d|n)$ *given by (1.1), where*

$$C(\mu) = \frac{(\alpha+n\beta)(\alpha+(n+2)\beta)}{(\alpha+(n+1)\beta)^2}\mu.$$

P r o o f. By virtue of the relation (3.1) and Theorem 2.3, we see that

$$\begin{aligned} \left|a_n a_{n+2} - \mu a_{n+1}^2\right| &= \left|\frac{(\mathrm{Re}(\alpha+\beta)-\gamma)^2}{(\alpha+n\beta)(\alpha+(n+2)\beta)}\right| \\ &\quad \times \left|c_{n-1}c_{n+1} - \frac{(\alpha+n\beta)(\alpha+(n+2)\beta)}{(\alpha+(n+1)\beta)^2}\mu c_n^2\right| \end{aligned}$$

$$\leqq \left|\frac{(\mathrm{Re}(\alpha+\beta)-\gamma)^2}{(\alpha+n\beta)(\alpha+(n+1)\beta)}\right|\left[|c_{n-1}c_{n+1} - C(\mu)c_{2n}| + |C(\mu)|\left|c_{2n}-c_n^2\right|\right]$$

$$\leqq \left|\frac{(\mathrm{Re}(\alpha+\beta)-\gamma)^2}{(\alpha+n\beta)(\alpha+(n+1)\beta)}\right|\left[|c_{n-1}c_{n+1} - C(\mu)c_{2n}| + 2\,|C(\mu)|\right]$$

$$\leqq \begin{cases} 2\left|\dfrac{(\mathrm{Re}(\alpha+\beta)-\gamma)^2}{(\alpha+n\beta)(\alpha+(n+2)\beta)}\right|\left[|2-C(\mu)|+|C(\mu)|\right] & (\mathrm{Re}\,[C(\mu)] \leqq 1)\,, \\ 4\left|\left(\dfrac{\mathrm{Re}(\alpha+\beta)-\gamma}{\alpha+(n+1)\beta}\right)^2\mu\right| & (\mathrm{Re}\,[C(\mu)] \geqq 1)\,. \end{cases}$$

∎

Putting $\alpha \leqq 1$ and $\beta = 1-\alpha$ in Theorem 3.1 and Theorem 3.2, we obtain the following corollaries.

Corollary 3.3. *If $f(z) \in \mathcal{QR}_\alpha(\gamma)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} 2(1-\gamma)\left\{\dfrac{1}{3-2\alpha} - \dfrac{2(1-\gamma)}{(2-\alpha)^2}\mu\right\} & (\mu \leqq 0) \\ \dfrac{2(1-\gamma)}{3-2\alpha} & \left(0 \leqq \mu \leqq \dfrac{(2-\alpha)^2}{(3-2\alpha)(1-\gamma)}\right) \\ 2(1-\gamma)\left\{\dfrac{2(1-\gamma)}{(2-\alpha)^2}\mu - \dfrac{1}{3-2\alpha}\right\} & \left(\mu \geqq \dfrac{(2-\alpha)^2}{(3-2\alpha)(1-\gamma)}\right) \end{cases}$$

with equality for $f_1(z)$ $\left(\mu \leqq 0 \text{ or } \mu \geqq \dfrac{(2-\alpha)^2}{(3-2\alpha)(1-\gamma)}\right)$
and $f_2(z)$ $\left(0 \leqq \mu \leqq \dfrac{(2-\alpha)^2}{(3-2\alpha)(1-\gamma)}\right)$ *given by* (1.1).

Corollary 3.4. *If $f(z) \in \mathcal{QR}_\alpha(\gamma)$, then* $\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq$

$$\begin{cases} 4(1-\gamma)^2\left\{\dfrac{1}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)} - \dfrac{1}{(n+1-n\alpha)^2}\mu\right\} \\ \qquad\qquad (\mu \leqq 0) \\ \dfrac{4(1-\gamma)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)} \\ \qquad\qquad \left(0 \leqq \mu \leqq \dfrac{(n+1-n\alpha)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)}\right) \\ \dfrac{4(1-\gamma)^2}{(n+1-n\alpha)^2}\mu \\ \qquad\qquad \left(\mu \geqq \dfrac{(n+1-n\alpha)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)}\right) \end{cases}$$

with equality for $f_1(z)$ $(\mu \leqq 0)$, $f_2(z)$

$$\left(0 \leqq \mu \leqq \frac{(n+1-n\alpha)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)};\ n = 3, 5, 7, \cdots\right)$$

and $f_d(z)$ $\left(\mu \geqq \dfrac{(n+1-n\alpha)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)}\right)$ *where* $d \geqq 2$ *and* $d|n$ *given by* (1.1).

In particular, setting $\alpha = 1$ or $\alpha = 0$ in Corollary 3.3 and Corollary 3.4, we know the following results for the class $\mathcal{Q}(\gamma)$ or $\mathcal{R}(\gamma)$ by Hayami and Owa [4].

Corollary 3.5. *If $f(z) \in \mathcal{Q}(\gamma)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} 2(1-\gamma)\{1-2(1-\gamma)\mu\} & (\mu \leqq 0) \\ 2(1-\gamma) & \left(0 \leqq \mu \leqq \dfrac{1}{1-\gamma}\right) \\ 2(1-\gamma)\{2(1-\gamma)\mu - 1\} & \left(\mu \geqq \dfrac{1}{1-\gamma}\right) \end{cases}$$

with equality for $f_1(z)$ $\left(\mu \leqq 0 \text{ or } \mu \geqq \dfrac{1}{1-\gamma}\right)$ and $f_2(z)$ $\left(0 \leqq \mu \leqq \dfrac{1}{1-\gamma}\right)$ given by (1.1).

Corollary 3.6. *If $f(z) \in \mathcal{Q}(\gamma)$, then*

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq \begin{cases} 4(1-\gamma)^2(1-\mu) & (\mu \leqq 0) \\ 4(1-\gamma)^2 & (0 \leqq \mu \leqq 1) \quad (n = 2, 3, 4, \cdots) \\ 4(1-\gamma)^2\mu & (\mu \geqq 1) \end{cases}$$

with equality for $f_1(z)$ $(\mu \leqq 0)$, $f_2(z)$ $(0 \leqq \mu \leqq 1\,;\, n = 3, 5, 7, \cdots)$ and $f_d(z)$ $(\mu \geqq 1)$ with $d|n$ and $d \geqq 2$ given by (1.1).

Corollary 3.7. *If $f(z) \in \mathcal{R}(\gamma)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{2(1-\gamma)}{3}\left\{1 - \dfrac{3}{2}(1-\gamma)\mu\right\} & (\mu \leqq 0) \\ \dfrac{2(1-\gamma)}{3} & \left(0 \leqq \mu \leqq \dfrac{4}{3(1-\gamma)}\right) \\ \dfrac{2(1-\gamma)}{3}\left\{\dfrac{3}{2}(1-\gamma)\mu - 1\right\} & \left(\mu \geqq \dfrac{4}{3(1-\gamma)}\right) \end{cases}$$

with equality for $f_1(z)$ $\left(\mu \leqq 0 \text{ or } \mu \geqq \dfrac{4}{3(1-\gamma)}\right)$ and $f_2(z)$ $\left(0 \leqq \mu \leqq \dfrac{4}{3(1-\gamma)}\right)$ given by (1.1).

COROLLARY 3.8. *If $f(z) \in \mathcal{R}(\gamma)$, then*

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq \begin{cases} \dfrac{4(1-\gamma)^2}{n(n+2)}\left\{1 - \dfrac{n(n+2)}{(n+1)^2}\mu\right\} & (\mu \leqq 0) \\[2ex] \dfrac{4(1-\gamma)^2}{n(n+2)} \quad \left(0 \leqq \mu \leqq \dfrac{(n+1)^2}{n(n+2)}\right) & (n = 2,3,4,\cdots) \\[2ex] \dfrac{4(1-\gamma)^2}{(n+1)^2}\mu & \left(\mu \geqq \dfrac{(n+1)^2}{n(n+2)}\right) \end{cases}$$

with equality for $f_1(z)$ $(\mu \leqq 0)$, $f_2(z)$ $\left(0 \leqq \mu \leqq \dfrac{(n+1)^2}{n(n+2)}\,;\, n = 3,5,7,\cdots\right)$ and $f_d(z)$ $\left(\mu \geqq \dfrac{(n+1)^2}{n(n+2)}\right)$ with $d|n$ and $d \geqq 2$ given by (1.1).

4. Strictly estimates for $n = 2$

We note that the function satisfying the equality of Corollary 3.4 is unknown for the case n is even number with

$$0 \leqq \mu \leqq \frac{(n+1-n\alpha)^2}{(n-(n-1)\alpha)(n+2-(n+1)\alpha)}.$$

The next result is better than the part of Corollary 3.4 for $n = 2$.

THEOREM 4.1. *If $f(z) \in \mathcal{QR}_\alpha(\gamma)$, then*

$$\left|a_2 a_4 - \mu a_3^2\right| \leqq \begin{cases} D(\alpha,\gamma,\mu) & \left(\dfrac{(2-\sqrt{2})(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{3(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)}\right) \\[2ex] \dfrac{4(1-\gamma)^2}{(3-2\alpha)^2}\mu & \left(\dfrac{3(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{(2-\alpha)(4-3\alpha)}\right) \end{cases},$$

where $D(\alpha,\gamma,\mu) =$

$$\frac{\left\{9(3-2\alpha)^4 - 16(2-\alpha)(4-3\alpha)(3-2\alpha)^2\mu + 8(2-\alpha)^2(4-3\alpha)^2\mu^2\right\}(1-\gamma)^2}{2(2-\alpha)(4-3\alpha)(3-2\alpha)^2\left\{(3-2\alpha)^2 - (2-\alpha)(4-3\alpha)\mu\right\}}.$$

Equality is attained for $f_2(z)$ $\left(\dfrac{3(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{(2-\alpha)(4-3\alpha)}\right)$ given by (1.1).

P r o o f. Let $f(z) \in \mathcal{QR}_\alpha(\gamma)$. Then, there is a function $p(z) \in \mathcal{P}(\gamma)$ such that

$$\alpha\left(\frac{f(z)}{z}\right) + (1-\alpha)f'(z) = p(z)$$

which implies that

$$a_n = \frac{c_{n-1}}{\alpha + n(1-\alpha)} \qquad (n \geqq 2).$$

Therefore, applying Lemma 2.1 and Lemma 2.2, we can suppose that $0 \leqq c_1 = c \leqq 2(1-\gamma)$ and obtain that

$$\left|a_2 a_4 - \mu a_3^2\right| = \left|\frac{c_1 c_3}{(2-\alpha)(4-3\alpha)} - \mu\frac{c_2^2}{(3-2\alpha)^2}\right|$$

$$= \frac{1}{4(1-\gamma)^2}\left|\left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)c^4\right.$$

$$+2\left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)(4(1-\gamma)^2 - c^2)c^2\zeta$$

$$-(4(1-\gamma)^2 - c^2)\left(\frac{c^2}{(2-\alpha)(4-3\alpha)} + \frac{(4(1-\gamma)^2 - c^2)\mu}{(3-2\alpha)^2}\right)\zeta^2$$

$$\left.+\frac{2(1-\gamma)(4(1-\gamma)^2 - c^2)(1-|\zeta|^2)c}{(2-\alpha)(4-3\alpha)}\eta\right|$$

$$\leqq \frac{1}{4(1-\gamma)^2}\left[(4(1-\gamma)^2 - c^2)\left\{\frac{c^2 - 2(1-\gamma)c}{(2-\alpha)(4-3\alpha)} + \frac{(4(1-\gamma)^2 - c^2)\mu}{(3-2\alpha)^2}\right\}\rho^2\right.$$

$$+2(4(1-\gamma)^2 - c^2)\left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)c^2\rho + \left(\frac{1}{(2-\alpha)(4-3\alpha)}\right)c^4$$

$$\left.+\frac{2(1-\gamma)(4(1-\gamma)^2 - c^2)c}{(2-\alpha)(4-3\alpha)}\right] \equiv \frac{F(\rho)}{4(1-\gamma)^2},$$

where $\rho = |\zeta| \leqq 1$. We discuss the following two cases.

For the case (i), $\mu \geqq \dfrac{(3-2\alpha)^2}{2(2-\alpha)(4-3\alpha)} \geqq \dfrac{(3-2\alpha)^2 c}{(2-\alpha)(4-3\alpha)\{2(1-\gamma)+c\}}$ $(0 \leqq c \leqq 2(1-\gamma))$, since

$$\frac{c^2 - 2(1-\gamma)c}{(2-\alpha)(4-3\alpha)} + \frac{4(1-\gamma)^2 - c^2}{(3-2\alpha)^2}\mu \geqq \frac{c^2 - 2(1-\gamma)c + 2(1-\gamma)c - c^2}{(2-\alpha)(4-3\alpha)} = 0,$$

we derive that

$$F'(\rho) = 2(4(1-\gamma)^2 - c^2)\left\{\left(\frac{c^2 - 2(1-\gamma)c}{(2-\alpha)(4-3\alpha)} + \frac{4(1-\gamma)^2 - c^2}{(3-2\alpha)^2}\mu\right)\rho + \left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)c^2\right\} \geqq 0$$

for all ρ $(0 \leqq \rho \leqq 1)$.

For the case (ii), $0 \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{2(2-\alpha)(4-3\alpha)}$, by the help of the case (i), we readily know that $F'(\rho) \geqq 0$ for any ρ in the case (ii–1), $c \leqq \dfrac{2(1-\gamma)(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2 - (2-\alpha)(4-3\alpha)\mu} \leqq 2(1-\gamma)$.

Further, the coefficient of ρ in $F'(\rho)$ is negative in the case (ii–2), $c \geqq \dfrac{2(1-\gamma)(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2 - (2-\alpha)(4-3\alpha)\mu}$.
Thus,

$$F'(\rho) \geqq F'(1) = 2(4(1-\gamma)^2 - c^2)\left\{2\left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)c^2 - \frac{2(1-\gamma)c}{(2-\alpha)(4-3\alpha)} + \frac{4(1-\gamma)^2\mu}{(3-2\alpha)^2}\right\} \equiv 2(4(1-\gamma)^2 - c^2)G(c)$$

Then, $G(c)$ can be represented by

$$G(c) = 2\left(\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right)\left(c - \frac{1-\gamma}{2\left(1 - \frac{(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2}\right)}\right)^2$$

$$+\frac{\left[8(2-\alpha)^2(4-3\alpha)^2\mu\left\{\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right\} - (3-2\alpha)^2\right](1-\gamma)^2}{2(2-\alpha)^2(4-3\alpha)^2(3-2\alpha)^2\left\{\frac{1}{(2-\alpha)(4-3\alpha)} - \frac{\mu}{(3-2\alpha)^2}\right\}}$$

Therefore, the case (ii–2) is also separated into two parts below.

For the case (ii–2–1) $\dfrac{(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{2(2-\alpha)(4-3\alpha)}$, a simple computation gives us that

$$\frac{2(1-\gamma)(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2 - (2-\alpha)(4-3\alpha)\mu} \geqq \frac{1-\gamma}{2\left(1 - \frac{(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2}\right)}$$

which implies that

$$G(c) \geqq G\left(\frac{2(1-\gamma)(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2-(2-\alpha)(4-3\alpha)\mu}\right)$$
$$= \frac{4(1-\gamma)^2(2-\alpha)(4-3\alpha)\mu^2}{(3-2\alpha)^2\{(3-2\alpha)^2-(2-\alpha)(4-3\alpha)\mu\}} > 0.$$

Therefore, $F'(\rho) \geqq 0$ in this case.

Next, for the case (ii–2–2) $\dfrac{(2-\sqrt{2})(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)}$, it follows that $F'(\rho) \geqq 0$ because

$$G(c) \geqq G\left(\frac{1-\gamma}{2\left(1-\dfrac{(2-\alpha)(4-3\alpha)\mu}{(3-2\alpha)^2}\right)}\right) \geqq 0.$$

Consequently, bringing the two cases (i) and (ii) together, we derive that $F(\rho)$ is increasing for $\dfrac{(2-\sqrt{2})(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \dfrac{(3-2\alpha)^2}{(2-\alpha)(4-3\alpha)}$, and therefore

$$\frac{F(\rho)}{4(1-\gamma)^2} \leqq \frac{F(1)}{4(1-\gamma)^2}$$
$$= \frac{1}{4(1-\gamma)^2}\left[-2\left(\frac{1}{(2-\alpha)(4-3\alpha)}-\frac{\mu}{(3-2\alpha)^2}\right)C^2\right.$$
$$\left.+4(1-\gamma)^2\left(\frac{3}{(2-\alpha)(4-3\alpha)}-\frac{4\mu}{(3-2\alpha)^2}\right)C+\frac{16(1-\gamma)^4\mu}{(3-2\alpha)^2}\right] \equiv H(C),$$

where $0 \leqq C = c^2 \leqq 4(1-\gamma)^2$. Noting that the coefficient of C^2 is negative and $H'(C_1) = 0$ for $C_1 = \dfrac{\left\{3(3-2\alpha)^2-4(2-\alpha)(4-3\alpha)\mu\right\}(1-\gamma)^2}{(3-2\alpha)^2-(2-\alpha)(4-3\alpha)\mu} < 4(1-\gamma)^2$, we obtain that

$$H(C) \leqq H(C_1) = D(\alpha,\gamma,\mu) \quad \left(\frac{(2-\sqrt{2})(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \frac{3(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)}\right)$$

and

$$H(C) \leqq H(0) = \frac{4(1-\gamma)^2}{(3-2\alpha)^2}\mu \quad \left(\frac{3(3-2\alpha)^2}{4(2-\alpha)(4-3\alpha)} \leqq \mu \leqq \frac{(3-2\alpha)^2}{(2-\alpha)(4-3\alpha)}\right).$$

The proof of the theorem is completed. ∎

Letting $\alpha = 1$ or $\alpha = 0$ in Theorem 4.1, we derive the next corollaries due to Hayami and Owa [4].

COROLLARY 4.2. *If $f(z) \in \mathcal{Q}(\gamma)$, then*

$$\left|a_2 a_4 - \mu a_3^2\right| \leqq \begin{cases} \dfrac{(9-16\mu+8\mu^2)(1-\gamma)^2}{2(1-\mu)} & \left(\dfrac{2-\sqrt{2}}{4} \leqq \mu \leqq \dfrac{3}{4}\right) \\[2ex] 4(1-\gamma)^2\mu & \left(\dfrac{3}{4} \leqq \mu \leqq 1\right) \end{cases}$$

with equality for $f(z) = \dfrac{z+(1-2\gamma)z^3}{1-z^2}$ $\left(\dfrac{3}{4} \leqq \mu \leqq 1\right)$.

COROLLARY 4.3. *If $f(z) \in \mathcal{R}(\gamma)$, then*

$$\left|a_2 a_4 - \mu a_3^2\right| \leqq \begin{cases} \dfrac{(729-1152\mu+512\mu^2)(1-\gamma)^2}{144(9-8\mu)} & \left(\dfrac{9(2-\sqrt{2})}{32} \leqq \mu \leqq \dfrac{27}{32}\right) \\[2ex] \dfrac{4(1-\gamma)^2}{9}\mu & \left(\dfrac{27}{32} \leqq \mu \leqq \dfrac{9}{8}\right) \end{cases}$$

with equality for $f(z) = -(1-2\gamma)z + (1-\gamma)\log\left(\dfrac{1+z}{1-z}\right)$ $\left(\dfrac{27}{32} \leqq \mu \leqq \dfrac{9}{8}\right)$.

5. Applications to starlike functions

In this section, we consider the upper bounds of the functional $\left|a_n a_{n+2} - \mu a_{n+1}^2\right|$ for starlike functions of order γ by applying the previous results for the class $\mathcal{Q}(\gamma)$. Then, for some real γ $(0 \leqq \gamma < 1)$, we define the subclass

$$\mathcal{S}^*(\gamma) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{U}) \right\}$$

of $\mathcal{A}$ consisting of starlike functions of order γ. By Lemma 1.3, we deduce

THEOREM 5.1. *It follows that*

$$\mathcal{S}^*(\gamma) \subset \mathcal{Q}(\gamma^*) \qquad \left(\frac{1}{2} \leqq \gamma < 1\right)$$

where $\gamma^* = \dfrac{1}{3-2\gamma}$.

P r o o f. For $f(z) \in \mathcal{S}^*(\gamma)$ $(\frac{1}{2} \leqq \gamma < 1)$, we define $w(z)$ by

$$\frac{f(z)}{z} = \frac{1+(1-2\gamma^*)w(z)}{1-w(z)} \qquad \left(0 \leqq \gamma^* = \frac{1}{3-2\gamma} < 1\right).$$

Then, we see that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2(1-\gamma^*)zw'(z)}{(1+(1-2\gamma^*)w(z))(1-w(z))}.$$

Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, if there is a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then we can write

$$w(z_0) = e^{i\theta}, \quad z_0 w'(z_0) = kw(z_0) = ke^{i\theta} \qquad (k \geqq 1)$$

by Lemma 1.3. For such a point $z_0 \in \mathbb{U}$, we have that

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) &= 1 + \operatorname{Re}\left(\frac{2(1-\gamma^*)ke^{i\theta}}{1-2\gamma^* e^{i\theta} - (1-2\gamma^*)e^{i2\theta}}\right) \\ &= 1 - \operatorname{Re}\left(\frac{(1-\gamma^*)k}{\gamma^*(1-\cos\theta) + i(1-\gamma^*)\sin\theta}\right) \end{aligned}$$

$$= 1 - \frac{\gamma^*(1-\gamma^*)k}{2\gamma^{*2} + (1-2\gamma^*)(1+\cos\theta)} \leqq 1 - \frac{(1-\gamma^*)}{2\gamma^*} = \gamma,$$

which contradicts our assumption. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$, that is, $f(z) \in \mathcal{Q}(\gamma^*)$. ∎

In consideration of the above theorem, applying Corollary 3.5 and Corollary 3.6, we immediately know the following result.

THEOREM 5.2. *If* $f(z) \in \mathcal{S}^*(\gamma)$ $\left(\frac{1}{2} \leqq \gamma < 1\right)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{4(1-\gamma)}{3-2\gamma}\left\{1 - \dfrac{4(1-\gamma)}{3-2\gamma}\mu\right\} & (\mu \leqq 0) \\[2ex] \dfrac{4(1-\gamma)}{3-2\gamma} & \left(0 \leqq \mu \leqq \dfrac{3-2\gamma}{2(1-\gamma)}\right) \\[2ex] \dfrac{4(1-\gamma)}{3-2\gamma}\left\{\dfrac{4(1-\gamma)}{3-2\gamma}\mu - 1\right\} & \left(\mu \geqq \dfrac{3-2\gamma}{2(1-\gamma)}\right) \end{cases}$$

and

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq \begin{cases} \dfrac{16(1-\gamma)^2}{(3-2\gamma)^2}(1-\mu) & (\mu \leqq 0) \\[2ex] \dfrac{16(1-\gamma)^2}{(3-2\gamma)^2} & (0 \leqq \mu \leqq 1) \\[2ex] \dfrac{16(1-\gamma)^2}{(3-2\gamma)^2}\mu & (\mu \geqq 1)\,. \end{cases}$$

Taking $\gamma = \dfrac{1}{2}$ in Theorem 5.2, we derive

COROLLARY 5.3. *If* $f(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} 1-\mu & (\mu \leqq 0) \\[2ex] 1 & (0 \leqq \mu \leqq 2) \\[2ex] \mu - 1 & (\mu \geqq 2) \end{cases}$$

and

$$\left|a_n a_{n+2} - \mu a_{n+1}^2\right| \leqq \begin{cases} 1-\mu & (\mu \leqq 0) \\[2ex] 1 & (0 \leqq \mu \leqq 1) \\[2ex] \mu & (\mu \geqq 1)\,. \end{cases}$$

Then, we remark the following result due to Hayami and Owa [3, Corollary 2].

REMARK 5.4. If $f(z) \in \mathcal{S}^*\left(\dfrac{1}{2}\right)$, then

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} 1-\mu & \left(\mu \leqq \dfrac{1}{2}\right) \\[2ex] \dfrac{1}{2} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{3}{2}\right) \\[2ex] \mu - 1 & \left(\mu \geqq \dfrac{3}{2}\right) \end{cases}$$

with equality for $f(z) = \dfrac{z}{1-z}$ $\left(\mu \leqq \dfrac{1}{2} \ or \ \mu \geqq \dfrac{3}{2}\right)$ and $f(z) = \dfrac{z}{\sqrt{1-z^2}}$ $\left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{3}{2}\right)$.

This result shows us that Theorem 5.2 may be a rough result. Thus, we have the next problem.

Problem 5.5. Can we find the sharp upper bounds of the functional $\left|a_n a_{n+2} - \mu a_{n+1}^2\right|$ for functions $f(z) \in \mathcal{S}^*(\gamma)$ $(0 \leqq \gamma < 1)$, and for any n $(n = 1, 2, 3, \cdots)$ and real number μ?

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