

**SERIES IN MITTAG-LEFFLER FUNCTIONS:  
INEQUALITIES AND CONVERGENT THEOREMS**

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*This paper is dedicated to the 70th anniversary of Professor Srivastava*

**Abstract**

In studying the behaviour of series, defined by means of the Mittag-Leffler functions, on the boundary of its domain of convergence in the complex plane, we prove Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems. Asymptotic formulae are also provided for the Mittag-Leffler functions in the case of “large” values of indices that are used in the proofs of the convergence theorems for the considered series.

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**1. Introduction**

The Mittag-Leffler functions  $E_\alpha$  (Mittag-Leffler, 1902-1905) and  $E_{\alpha,\beta}$  (Agarwal 1953, see also [5]), are defined in the whole complex plane  $\mathbb{C}$  by the power series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0. \quad (1.1)$$

The Mittag-Leffler functions (1.1) are examples of entire functions of a given order  $\rho = 1/\alpha$  and a type  $\sigma = 1$ . They have been studied in details by Dzrbashjan [1], [2]: asymptotic formulae in different parts of the complex plane, distribution of the zeros, kernel functions of inverse Borel type integral transforms, various relations and representations. The detailed properties of these functions can be found in the contemporary monographs of Kilbas et al. [4] and Podlubny [15].

In our previous papers ([9] - [12]) we studied series in systems of some other representatives of the SF of FC, which are fractional indices analogues of the Bessel functions and also multi-index Mittag-Leffler functions (in the sense of [5],[6],[7]), and proved Cauchy-Hadamard, Abel and Tauberian type theorems in the complex domain.

In this paper we prove some inequalities in the complex plane  $\mathbb{C}$  and on its compact subsets, asymptotic formulae for "large" values of indices of the functions (1.1) and study the convergence of series in such kind of functions.

## 2. Inequalities and asymptotic formulae

In this point we prove some asymptotic formulae for "large" values of indices. Denote

$$\theta_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+1)}, \quad \theta_{n,\beta}(z) = \Gamma(\beta) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+\beta)}, \quad (2.1)$$

$$\theta_{\alpha,n}(z) = \Gamma(n) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+n)}. \quad (2.2)$$

LEMMA 2.1. *Let  $n \in \mathbb{N}$ ,  $z \in \mathbb{C}$  and  $K \subset \mathbb{C}$  be a nonempty compact set. Then the following inequalities hold*

$$|\theta_n(z)| \leq \frac{1}{n!} (\exp(|z|) - 1), \quad |\theta_{n,\beta}(z)| \leq \frac{1}{(n-1)!} \frac{|z| \exp(|z|)}{\beta}, \quad (2.3)$$

$$|\theta_{\alpha,n}(z)| \leq \frac{\Gamma(n)}{\Gamma(\alpha+n)} \Gamma(\alpha+1) (E_\alpha(|z|) - 1), \quad (2.4)$$

and moreover there exists a constant  $C$ ,  $0 < C < \infty$ , such that

$$|\theta_n(z)| \leq C/n!, \quad |\theta_{n,\beta}(z)| \leq C/(n-1)!, \quad |\theta_{\alpha,n}(z)| \leq C \frac{\Gamma(n)}{\Gamma(\alpha+n)}, \quad (2.5)$$

for all the natural numbers  $n$  and each  $z \in K$ .

P r o o f. First, let  $z \in \mathbb{C}$ . Then we can write

$$\theta_n(z) = \frac{1}{\Gamma(n+1)} \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(kn+1)} z^k = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{n!}{(kn)!} z^k.$$

Denoting  $u_k(z) = \frac{n!}{(kn)!} z^k$ , we obtain the estimate  $|u_k(z)| \leq \frac{|z|^k}{k!}$  for the absolute value of  $u_k(z)$ . Since the series  $\sum_{k=1}^{\infty} \frac{|z|^k}{k!}$  converges for each  $z \in \mathbb{C}$  and its sum is  $\exp(|z|) - 1$ , then the first of the estimates (2.3) hold on the whole complex plane.

The second of the estimates (2.3) are proved in [13].

To prove (2.4), we write

$$\theta_{\alpha,n}(z) = \frac{\Gamma(n)}{\Gamma(\alpha+n)} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha k+n)} z^k$$

and denoting

$$\tilde{\gamma}_{n,k} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha k+n)}, \quad \tilde{u}_{n,k}(z) = \tilde{\gamma}_{n,k} z^k,$$

we obtain consecutively

$$\tilde{\gamma}_{n,1} = 1; \quad 0 < \tilde{\gamma}_{n,k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha k+1)} \prod_{s=1}^{n-1} \frac{(\alpha+s)}{(\alpha k+s)} \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha k+1)}, \quad \text{for } k \in \mathbb{N}, k \neq 1,$$

$$|\tilde{u}_{n,k}(z)| = \tilde{\gamma}_{n,k} |z|^k \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha k+1)} |z|^k, \quad \text{for } k \in \mathbb{N},$$

and therefore

$$|\theta_{\alpha,n}(z)| \leq \frac{\Gamma(n)\Gamma(\alpha+1)}{\Gamma(\alpha+n)} \left( \sum_{k=0}^{\infty} \frac{|z|^k}{\Gamma(\alpha k+1)} - 1 \right)$$

which proves (2.4).

Further, for all  $z$  on the compact set  $K$ , the inequalities (2.5) follow immediately from the inequalities (2.3) and (2.4). ■

**THEOREM 2.1.** *For the Mittag-Leffler functions  $E_n, E_{n,\beta}, E_{\alpha,n}$  ( $n \in \mathbb{N}$ ), the following asymptotic formulae*

$$E_n(z) = 1 + \theta_n(z), \quad z \in \mathbb{C}, \quad \theta_n(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.6)$$

$$E_{n,\beta}(z) = \frac{1}{\Gamma(\beta)} (1 + \theta_{n,\beta}(z)), \quad z \in \mathbb{C}, \quad \theta_{n,\beta}(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.7)$$

$$E_{\alpha,n}(z) = \frac{1}{\Gamma(n)} (1 + \theta_{\alpha,n}(z)), \quad z \in \mathbb{C}, \quad \theta_{\alpha,n}(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.8)$$

are valid. The functions  $\theta_n(z)$ ,  $\theta_{n,\beta}(z)$ ,  $\theta_{\alpha,n}(z)$  are holomorphic for  $z \in \mathbb{C}$ . The convergence is uniform on the compact subsets of  $\mathbb{C}$ .

**P r o o f.** The identities (2.6)-(2.8) obtain due to (1.1), (2.1) and (2.2) automatically. The holomorphy of  $\theta_n(z)$ ,  $\theta_{n,\beta}(z)$ ,  $\theta_{\alpha,n}(z)$  follows from the holomorphy of  $E_n(z)$ ,  $E_{n,\beta}(z)$ ,  $E_{\alpha,n}(z)$  on the whole complex plane and the equalities (2.6) - (2.8). The rest follows immediately from Lemma 2.1. ■

**NOTE 2.1.** According to the asymptotic formulae (2.6) - (2.8), it follows there exists a natural number  $M$  such that the functions  $E_n$ ,  $\Gamma(n)E_{\alpha,n}$ ,  $\Gamma(\beta)E_{n,\beta}$  have not any zeros at all for  $n > M$ .

### 3. Series in Mittag-Leffler functions

We introduce the following auxiliary functions, related to Mittag-Leffler's functions, adding  $\tilde{E}_0(z)$ ,  $\tilde{E}_{0,\beta}(z)$  and  $\tilde{E}_{\alpha,0}(z)$  just for completeness, namely:

$$\begin{aligned} \tilde{E}_0(z) &= 1; \quad \tilde{E}_n(z) = z^n E_n(z), \quad n \in \mathbb{N}, \\ \tilde{E}_{0,\beta}(z) &= 1; \quad \tilde{E}_{n,\beta}(z) = \Gamma(\beta) z^n E_{n,\beta}(z), \quad n \in \mathbb{N}; \quad \beta > 0, \\ \tilde{E}_{\alpha,0}(z) &= 1; \quad \tilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z), \quad n \in \mathbb{N}; \quad \alpha > 0, \end{aligned}$$

and consider the series in these functions, respectively:

$$\sum_{n=0}^{\infty} a_n \tilde{E}_n(z), \quad \sum_{n=0}^{\infty} a_n \tilde{E}_{n,\beta}(z), \quad \sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha,n}(z), \quad (3.1)$$

with complex coefficients  $a_n$  ( $n = 0, 1, 2, \dots$ ).

Our main objective is to study the convergence of the series (3.1) in the complex plane. We prove theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler functions (as for example in Saqabi and Kiryakova [19]). Convergence theorems are obtained also for series in other special functions, for example, for series in Laguerre and Hermite polynomials [16] - [18], and resp. by the author for series in Bessel functions and their Wright's 2, 3, and 4-index generalizations in the previous papers [9] - [12].

#### 4. Cauchy-Hadamard and Abel type theorems

In the beginning we give a theorem of Cauchy-Hadamard type for every one of the above series.

**THEOREM 4.1. (OF CAUCHY-HADAMARD TYPE).** *The domain of convergence of each one of the series (3.1) with complex coefficients  $a_n$  is the disk  $|z| < R$  with a radius of convergence  $R = 1/\Lambda$ , where*

$$\Lambda = \limsup_{n \rightarrow \infty} (|a_n|)^{1/n}. \tag{4.1}$$

*The cases  $\Lambda = 0$  and  $\Lambda = \infty$  can be included in the general case, provided  $1/\Lambda$  means  $\infty$ , respectively  $0$ .*

**I d e a o f P r o o f.** Using the asymptotic formulae (2.6) - (2.8), we evaluate the absolute value of the general term of each of the series (3.1). Further the proof goes separately in the three cases:  $\Lambda = 0$ ,  $0 < \Lambda < \infty$ ,  $\Lambda = \infty$ . We show the absolute convergence of the series (3.1) in the circular domain  $\{z : z \in \mathbb{C}, |z| < R\}$ . In the second case we prove that the series is divergent for  $|z| > R$  and in the third case - divergent for all complex  $z \neq 0$ . ■

Let  $z_0 \in \mathbb{C}$ ,  $0 < R < \infty$ ,  $|z_0| = R$  and  $g_\varphi$  be an arbitrary angular domain with size  $2\varphi < \pi$  and with vertex at the point  $z = z_0$ , which is symmetric with respect to the straight line defined by the points  $0$  and  $z_0$ . The following theorem is valid.

**THEOREM 4.2. (OF ABEL TYPE).** *Let  $\{a_n\}_{n=0}^\infty$  be a sequence of complex numbers,  $\Lambda$  be the real number defined by (4.1),  $0 < \Lambda < \infty$ . Let  $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$ . If  $f(z)$ ,  $g(z; \beta)$ ,  $h(z; \alpha)$  are the sums respectively of the first, second and third of the series (3.1) on the domain  $K$ , and these series converge at the point  $z_0$  of the boundary of  $K$ , then:*

$$\lim_{z \rightarrow z_0} f(z) = \sum_{n=0}^\infty a_n \tilde{E}_n(z_0), \quad \lim_{z \rightarrow z_0} g(z; \beta) = \sum_{n=0}^\infty a_n \tilde{E}_{n, \beta}(z_0), \tag{4.2}$$

$$\lim_{z \rightarrow z_0} h(z; \alpha) = \sum_{n=0}^\infty a_n \tilde{E}_{\alpha, n}(z_0), \tag{4.3}$$

*provided  $|z| < R$  and  $z \in g_\varphi$ .*

P r o o f. Let us consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n \tilde{E}_n(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) \quad (4.4)$$

and represent it in the form

$$\Delta(z) = \sum_{n=0}^k a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) + \sum_{n=k+1}^{\infty} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)).$$

Let  $p > 0$ . By using the notations

$$\beta_m = \sum_{n=k+1}^m a_n \tilde{E}_n(z_0), \quad m > k, \quad \beta_k = 0, \quad \gamma_n(z) = 1 - \tilde{E}_n(z)/\tilde{E}_n(z_0),$$

and the Abel transformation (see in ([8], vol.1, ch.1, p.32, 3.4:7), we obtain consecutively:

$$\begin{aligned} \sum_{n=k+1}^{k+p} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) &= \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1}) \gamma_n(z) \\ &= \beta_{k+p} \gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n (\gamma_{n+1}(z) - \gamma_n(z)), \end{aligned}$$

i.e.

$$\begin{aligned} \sum_{n=k+1}^{k+p} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) &= (1 - \tilde{E}_{k+p}(z)/\tilde{E}_{k+p}(z_0)) \sum_{n=k+1}^{k+p} a_n \tilde{E}_n(z_0) \\ &\quad - \sum_{n=k+1}^{k+p-1} \left( \sum_{s=k+1}^n a_s \tilde{E}_s(z_0) \right) \left( \frac{\tilde{E}_n(z)}{\tilde{E}_n(z_0)} - \frac{\tilde{E}_{n+1}(z)}{\tilde{E}_{n+1}(z_0)} \right). \end{aligned}$$

According to Note 2.1, there exists a natural number  $M$  such that  $\tilde{E}_n(z_0) \neq 0$  when  $n > M$ . Let  $k > M$ . Then, for every natural  $n > k$ :

$$\begin{aligned} &\tilde{E}_n(z)/\tilde{E}_n(z_0) - \tilde{E}_{n+1}(z)/\tilde{E}_{n+1}(z_0) \quad (4.5) \\ &= (z/z_0)^n \frac{(1+\theta_n(z))(1+\theta_{n+1}(z_0)) - (z/z_0)(1+\theta_{n+1}(z))(1+\theta_n(z_0))}{(1+\theta_n(z_0))(1+\theta_{n+1}(z_0))}. \end{aligned}$$

For the right hand side of (4.5) we apply the Schwartz lemma. Then we get that there exists a constant  $C$ :

$$|\tilde{E}_n(z)/\tilde{E}_n(z_0) - \tilde{E}_{n+1}(z)/\tilde{E}_{n+1}(z_0)| \leq C|z - z_0||z/z_0|^n.$$

Analogously, there exists a constant  $B$ :

$$|1 - \tilde{E}_{k+p}(z)/\tilde{E}_{k+p}(z_0)| \leq B|z - z_0| \leq 2B|z_0|.$$

Let  $\varepsilon$  be an arbitrary positive number and choose  $N(\varepsilon)$  so large that for  $k > N(\varepsilon)$  the inequality

$$\left| \sum_{s=k+1}^n a_s \tilde{E}_s(z_0) \right| < \min(\varepsilon \cos \varphi / (12B|z_0|), \varepsilon \cos \varphi / (6C|z_0|))$$

holds for every natural  $n > k$ . Therefore, for  $k > \max(M, N(\varepsilon))$ :

$$\left| \sum_{s=k+1}^{\infty} a_s \tilde{E}_s(z_0) \right| \leq \min(\varepsilon \cos \varphi / (12B|z_0|), \varepsilon \cos \varphi / (6C|z_0|)),$$

and

$$\begin{aligned} \left| \sum_{n=k+1}^{\infty} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) \right| &\leq (\varepsilon \cos \varphi / 6) \left( 1 + \sum_{n=k+1}^{\infty} |z_0|^{-1} |z - z_0| |z/z_0|^n \right) \\ &\leq (\varepsilon \cos \varphi / 6) (1 + |z - z_0| / (|z_0| - |z|)). \end{aligned}$$

But near the vertex of the angular domain  $g_\varphi$  in the part  $d_\varphi$  closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle, we have  $|z - z_0| / (|z_0| - |z|) < 2 / \cos \varphi$ , i.e.  $|z - z_0| \cos \varphi < 2(|z_0| - |z|)$ . That is why the inequality

$$\left| \sum_{n=k+1}^{\infty} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) \right| < (\varepsilon \cos \varphi) / 6 + \varepsilon / 3 \leq \varepsilon / 2 \tag{4.6}$$

holds for  $z \in d_\varphi$  and  $k > \max(M, N(\varepsilon))$ . Fix some  $k > \max(M, N(\varepsilon))$  and after that choose  $\delta(\varepsilon)$  such that if  $|z - z_0| < \delta(\varepsilon)$  then the inequality

$$\left| \sum_{n=0}^k a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) \right| < \varepsilon / 2 \tag{4.7}$$

holds inside  $d_\varphi$ . We get

$$|\Delta(z)| = \left| \sum_{n=0}^{\infty} a_n (\tilde{E}_n(z_0) - \tilde{E}_n(z)) \right|$$

for the module of the difference (4.4). From (4.6) and (4.7) it follows that the first of the equalities (4.2) is satisfied.

The proofs of the second of the equalities (4.2) and (4.3) go by analogy. ■

### 5. $(E, z_0)$ - summations

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \dots \tag{5.1}$$

To define its Abel summability ([3], p.20, 1.3 (2)), we consider also the power series  $\sum_{n=0}^{\infty} a_n z^n$ .

DEFINITION 5.1. The series (5.1) is called A - summable, if the series  $\sum_{n=0}^{\infty} a_n z^n$  converges in the disk  $D = \{z : z \in \mathbb{C}, |z| < 1\}$  and moreover there exists  $\lim_{z \rightarrow 1-0} \sum_{n=0}^{\infty} a_n z^n = S$ .

The complex number  $S$  is called A-sum of the series (5.1) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S \quad (A).$$

NOTE 5.1. The A-summation is regular. It means that if the series (5.1) converges, then it is A-summable, and its A-sum is equal to its usual sum.

NOTE 5.2. The A-summability of the series (5.1) does not imply in general its convergence. But, with additional conditions on the growth of the general term of the series (5.1), the convergence can be ensured.

Note that each of the functions  $\tilde{E}_n(z), \tilde{E}_{n,\beta}(z), \tilde{E}_{\alpha,n}(z)$ , ( $n \in \mathbb{N}$ ), being an entire function, not identically zero, has no more than finite number of zeros in the closed and bounded set  $|z| \leq R$  ([8], Vol. 1, Ch. 3, §6, 6.1, p.305). Moreover, because of Note 2.1, no more than finite number of these functions have some zeros.

Let  $z_0 \in \mathbb{C}$ ,  $|z_0| = R$ ,  $0 < R < \infty$ ,  $\tilde{E}_n(z_0) \neq 0$ ,  $\tilde{E}_{n,\beta}(z_0) \neq 0$ , and  $\tilde{E}_{\alpha,n}(z_0) \neq 0$ . For the sake of brevity, denote

$$E_n^*(z; z_0) = \frac{\tilde{E}_n(z)}{\tilde{E}_n(z_0)}, \quad E_{n,\beta}^*(z; z_0) = \frac{\tilde{E}_{n,\beta}(z)}{\tilde{E}_{n,\beta}(z_0)}, \quad E_{\alpha,n}^*(z; z_0) = \frac{\tilde{E}_{\alpha,n}(z)}{\tilde{E}_{\alpha,n}(z_0)}.$$

DEFINITION 5.2. The series (5.1) is said to be  $(E, z_0)$  - summable, if the series

$$\sum_{n=0}^{\infty} a_n E_n^*(z; z_0), \quad (5.2)$$

converges in the disk  $|z| < R$  and, moreover, there exists the limit

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_n^*(z; z_0), \quad (5.3)$$



provided  $z$  remains on the segment  $[0, z_0]$ .

DEFINITION 5.3. The series (5.1) is said to be  $(E_\beta, z_0)$  - summable (respectively  $(E_\alpha, z_0)$  - summable), if the series

$$\sum_{n=0}^{\infty} a_n E_{n,\beta}^*(z; z_0) \quad (\text{respectively } \sum_{n=0}^{\infty} a_n E_{\alpha,n}^*(z; z_0))$$

converges in the disk  $|z| < R$  and, moreover, there exists the limit

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_{n,\beta}^*(z; z_0) \quad (\text{respectively } \lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_{\alpha,n}^*(z; z_0)),$$

provided  $z$  remains on the segment  $[0, z_0]$ .

NOTE 5.3. Every  $(E, z_0)-$ ,  $(E_\beta, z_0)-$ ,  $(E_\alpha, z_0)-$  summation is regular, and this property is just a particular case of Theorem 4.2.

### 6. Tauberian type theorems

A Tauberian theorem is a statement that relates the Abel summability and the standard convergency of a numerical series by means of some assumptions imposed on the general term of the series under question. A classical result in this direction is given by Theorem 85 ([3]).

In this paper we extend the validity of such type of assertion to series in Mittag-Leffler functions. Tauber type theorems are given also for summations by means of Laguerre polynomials [16], and Bessel type functions by the author [9] - [12].

THEOREM 6.1. (OF TAUBER TYPE). *If the series (5.1) is  $(E, z_0)$  - summable, (or  $(E_\beta, z_0)$  respectively  $(E_\alpha, z_0)$  - summable), and*

$$\lim_{n \rightarrow \infty} n a_n = 0, \tag{6.1}$$

*then it is convergent.*

P r o o f. Let  $z$  belong to the segment  $[0, z_0]$ . Taking into account the asymptotic formula (2.6) for the Mittag-Leffler functions, we obtain:

$$a_n E_n^*(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta_n(z)}{1 + \theta_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \tilde{\theta}_n(z; z_0)\right),$$

where  $\tilde{\theta}_n(z; z_0) = \frac{\theta_n(z) - \theta_n(z_0)}{1 + \theta_n(z_0)}$ . Then, due to (2.5):

$$\tilde{\theta}_n(z; z_0) = O(1/n!). \tag{6.2}$$

Let us write (5.2) in the form

$$\sum_{n=0}^{\infty} a_n E_n^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1 + \tilde{\theta}_n(z; z_0)\right). \quad (6.3)$$

Denoting  $w_n(z) = a_n \left(\frac{z}{z_0}\right)^n \tilde{\theta}_n(z; z_0)$  we consider the series  $\sum_{n=0}^{\infty} w_n(z)$ . Since  $|w_n(z)| \leq |a_n| |\tilde{\theta}_n(z; z_0)|$  and according to condition (6.1) and the relationship (6.2), there exists a constant  $C$ , such that  $|w_n(z)| \leq C/n^2$ . Since  $\sum_{n=1}^{\infty} 1/n^2$  converges, the series  $\sum_{n=0}^{\infty} w_n(z)$  is also convergent, even absolutely and uniformly on the segment  $[0, z_0]$ . Therefore (since  $\lim_{z \rightarrow z_0} w_n(z) = 0$ )

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z \rightarrow z_0} w_n(z) = 0.$$

Obviously, the assumption that the series (5.1) is  $(E, z_0)$ -summable implies the existence of the limit (5.3). Then, having in mind that (6.3) can be written in the form

$$\sum_{n=0}^{\infty} a_n E_n^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \tilde{\theta}_n(z; z_0),$$

we conclude that there exists the limit

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \quad (6.4)$$

and, moreover,

$$\lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n E_n^*(z; z_0) = \lim_{z \rightarrow z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n.$$

From the existence of the limit (6.4) it follows that the series (5.1) is A-summable. Then according to Theorem 85 from [3], the series (5.1) converges.

The proofs of the theorem for the other two cases  $(E_\beta, z_0)$ - and  $(E_\alpha, z_0)$ -summations go in the same way. ■

At first sight it seems that the condition  $a_n = o(1/n)$  is essential. Nevertheless, Littlewood succeeds to weaken it and obtain the strengthened version of the Tauber theorem ([3], Theorem 90).

A Littlewood generalization of the  $o(n)$  version of a Tauber type theorem (Theorem 6.1) is also given in this part. Similar theorem for series in Bessel functions series is also proved, see [12].

**THEOREM 6.2 (OF LITTLEWOOD TYPE).** *If the series (5.1) is  $(E, z_0)$ -summable, (or  $(E_\beta, z_0)$  respectively  $(E_\alpha, z_0)$ -summable), and*

$$a_n = O(1/n) \quad (6.5)$$

then the series (5.1) converges.

**I d e a o f t h e P r o o f.** Using Theorem 90 in the place of Theorem 85, [3], the proof of the  $(E_\beta, z_0)$  summability follows the line of that of Theorem 6.1 and the ideas of the proofs of the cases of  $(E, z_0)$ - and  $(E_\alpha, z_0)$ -summabilities, given in [13], respectively [14]. ■

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