

**HILBERT-SMITH CONJECTURE FOR  
 $K$ -QUASICONFORMAL GROUPS**

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**Abstract**

A more general version of Hilbert's fifth problem, called the Hilbert-Smith conjecture, asserts that among all locally compact topological groups only Lie groups can act effectively on finite-dimensional manifolds. We give a solution of the Hilbert-Smith Conjecture for  $K$ -quasiconformal groups acting on domains in the extended  $n$ -dimensional Euclidean space.

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**1. Introduction**

A more general version of Hilbert's fifth problem, called the Hilbert-Smith conjecture, asserts that among all locally compact topological groups only Lie groups can act effectively on finite dimensional manifolds. The Hilbert-Smith conjecture is still an open problem. Bochner and Montgomery [3] solved for diffeomorphisms in 1943; Repovš and Ščepin [12] solved for actions by Lipschitz mappings in 1997. Notice that diffeomorphisms and Lipschitz mappings are locally quasiconformal homeomorphisms, they are hence quasiconformal homeomorphisms on precompact subdomains. However, quasiconformal homeomorphisms are neither diffeomorphisms nor Lipschitz mappings. Martin [9] solved for quasiconformal category in 1999, see the following theorem.

**THEOREM 1.** (Martin 1999) *Let  $G$  be a locally compact group acting effectively by quasiconformal homeomorphisms on a Riemannian manifold. Then  $G$  is a Lie group.*

Here groups of quasiconformal homeomorphisms are required to be a locally compact group acting effectively on a Riemannian manifold.

A smooth manifold ( $C^\infty$  differentiable manifold) is a Riemannian manifold if there exists a Riemannian metric on it. For example, Proposition 1 [6] states that every domain  $\Omega$  in the extended  $n$ -dimensional Euclidean space  $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$  is a Riemannian manifold.

A topological group  $G$  is a topological transformation group of the topological space  $X$  if the following two conditions are satisfied: (1) There exists a homomorphism  $\phi : G \rightarrow \text{Homeo}(X)$ , where  $\text{Homeo}(X)$  is the group of homeomorphisms of  $X$ ; (2) The mapping  $f : G \times X \rightarrow X$  given by  $(g, x) \mapsto \phi(g)x$  is continuous. A topological transformation group  $G$  is acting effectively on a topological space  $X$  if for each non-trivial  $g \in G$  there exists  $x \in X$  such that  $x$  is not fixed by  $g$ . For example, Theorem 2.3.5 [7] states that each  $K$ -quasiconformal group acting on a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a topological transformation group, and Proposition 2 [6] gives that each  $K$ -quasiconformal group of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is acting effectively on  $\Omega$  in  $\overline{\mathbb{R}^n}$ .

Recently Gong [6] applied the above Martin's Theorem 1 and solved the Hilbert-Smith Conjecture for non-elementary  $K$ -quasiconformal groups acting on domains in  $\overline{\mathbb{R}^n}$ , see the following theorem.

**THEOREM 2.** (Gong 2008) *Suppose that  $\Omega$  is a domain in  $\overline{\mathbb{R}^n}$  and  $G$  is a non-elementary  $K$ -quasiconformal group acting on  $\Omega$ . Then  $G$  is a Lie group.*

A group  $G$  of self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is said to be discontinuous at a point  $x \in \Omega$  if there exists a neighborhood  $U$  of  $x$  such that  $g(U) \cap U = \emptyset$  for all but finite many  $g \in G$ . The ordinary set of  $G$  is the set of all  $x \in \Omega$  at which  $G$  is discontinuous. The complement of the ordinary set is called the limit set of  $G$ . We say that  $G$  is an elementary group if the limit set contains at most two points. Otherwise we say that  $G$  is non-elementary.

In this paper, we want to remove the condition of non-elementary group and give the following theorem to solve Hilbert-Smith Conjecture for the  $K$ -quasiconformal groups acting on domains in  $\overline{\mathbb{R}^n}$ .

**THEOREM 3.** *Suppose that  $\Omega$  is a domain in  $\overline{\mathbb{R}^n}$  and  $G$  is a  $K$ -quasiconformal group acting on  $\Omega$ . Then  $G$  is a Lie group.*

Let  $\Omega$  and  $\Omega'$  be subdomains of  $\overline{\mathbb{R}^n}$ , where  $n \geq 2$ . Let  $f : \Omega \rightarrow \Omega'$  be a homeomorphism,  $x \in \Omega$  and  $r < d(x, \partial\Omega)$ , where  $d$  is the Euclidean metric for  $\mathbb{R}^n$ . The infinitesimal distortion of  $f$  at  $x \in \Omega \setminus \{\infty, f^{-1}(\infty)\}$  is

$$H_f(x) = \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)},$$

where

$$L_f(x, r) = \max_{|h|=r} |f(x+h) - f(x)|, \quad l_f(x, r) = \min_{|h|=r} |f(x+h) - f(x)|.$$

The distortion function of  $f$  is the essential supremum

$$K(f) = \text{ess sup}_{x \in \Omega} H_f(x) = \|H_f(x)\|_\infty.$$

A homeomorphism  $f : \Omega \rightarrow \Omega'$  is called a  $K$ -quasiconformal, if

$$K(f) \leq K.$$

Clearly,  $1 \leq K < \infty$ . A homeomorphism  $f : \Omega \rightarrow \Omega'$  is called *quasiconformal* if it is a  $K$ -quasiconformal for some  $K$ . Thus a quasiconformal homeomorphism is a homeomorphism with uniformly bounded distortion; it distorts the shape of an infinitesimal sphere about each point by at most a uniformly bounded factor.

Let  $\Gamma(\Omega)$  be the family of all quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ , and let  $\Gamma_K(\Omega)$  be the family of all  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ . It is easy to see that

$$\Gamma(\Omega) = \cup_{K \geq 1} \Gamma_K(\Omega). \tag{1}$$

If  $f$  is a  $K$ -quasiconformal homeomorphism, the inverse  $f^{-1}$  is a  $K$ -quasiconformal homeomorphism; if  $f_j$  is a  $K_j$ -quasiconformal homeomorphism ( $j = 1, 2$ ), then  $f_1 \circ f_2$  is a  $K_1 K_2$ -quasiconformal homeomorphism [1]. Thus,  $\Gamma(\Omega)$  forms a group under composition. By contrast,  $\Gamma_K(\Omega)$  is not a group if  $K > 1$ . However, when  $K = 1$ , the family  $\Gamma_1(\Omega)$  of all 1-quasiconformal self homeomorphisms of  $\Omega$  in  $\overline{\mathbb{R}^n}$  is the conformal group of  $\Omega$ . Indeed,  $\Gamma_1(\Omega)$  is a subgroup of the Möbius transformation group if  $n > 2$  or if  $n = 2$  with  $\Omega = \overline{\mathbb{R}^n}$ . In the latter case when  $n = 2$  with  $\Omega = \overline{\mathbb{R}^2}$ ,  $\Gamma_1(\overline{\mathbb{R}^2})$  is just the classical Möbius transformation group, that is, the group of linear fractional transformations of  $\overline{\mathbb{C}}$ .

A subfamily  $G$  of  $\Gamma_K(\Omega)$  is called a  $K$ -quasiconformal group if it constitutes a subgroup of  $\Gamma(\Omega)$  under composition. For example, the quasiconformal conjugate

$$G = f^{-1} \circ \Gamma_1(\Omega) \circ f$$

of the subgroup  $\Gamma_1(\Omega)$  of Möbius transformations by a  $K$ -quasiconformal homeomorphism  $f : \Omega \rightarrow \Omega$  is a  $K^2$ -quasiconformal group acting on  $\Omega$ . For subdomains of the plane Sullivan and Tukia showed in [13, 14], using a result of Maskit regarding groups of conformal transformations, that the quasiconformal conjugate is in fact the only construction. That is, a  $K$ -quasiconformal group of a domain  $\Omega \subset \overline{\mathbb{R}^2}$  must be quasiconformally conjugate to a subgroup of Möbius transformations of a domain  $\Omega \subset \overline{\mathbb{R}^2}$ . However, the situation in higher dimensions is different; not every  $K$ -quasiconformal group is obtained by quasiconformally conjugates [10, 15].

## 2. Metric Space

In this section, we introduce a metric  $\rho$  for the topological space  $\Gamma(\Omega)$ , so a compact subset in the metric space  $\Gamma(\Omega)$  coincides with a sequentially compact subset. We give a characterization of an open set in the metric space  $(\Gamma(\Omega), \rho)$  in Theorem 4. As a consequence the topology induced from the metric  $\rho$  agrees with the topology induced from locally uniform convergence. Thus, all three topologies of  $\Gamma(\Omega)$  induced from the compact-open topology, from locally uniform convergence, and from the metric  $\rho$  are equivalent.

Consider the topological space  $\Gamma(\Omega)$  of all quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  equipped with the compact-open topology. We know from [4] that the compact-open topology is equivalent to the topology induced from locally uniform convergence. Notice that the topology of  $\overline{\mathbb{R}^n}$  and all notions of convergence will be taken with respect to the following spherical metric  $q$  :

$$q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad \text{for } x, y \in \mathbb{R}^n;$$

$$q(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad \text{for } x \in \mathbb{R}^n.$$

The virtue of the spherical metric  $q$  is that it allows  $\infty$  to be treated like any other finite point, and  $q(x, y) \leq 1$  for  $x, y \in \overline{\mathbb{R}^n}$ .

Since a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a second countable, locally compact Hausdorff topological space, there exists a sequence  $\{K_j\}$  of compact sets in  $\Omega$  such that  $\Omega = \cup_{j=1}^\infty K_j$  [17]. Moreover, the compact sets  $K_j$  can be chosen to satisfy the conditions:  $K_j \subset \text{int } K_{j+1}$  and if  $K \subset \Omega$  is a compact set then  $K \subset K_j$  for some  $j$ . Indeed, for each  $j \in \mathbb{N}$ , let

$$K_j = \{x \in \Omega : q(x, \partial\Omega) \geq \frac{1}{j}\}.$$

Then  $K_j$  is closed in the compact space  $\overline{\mathbb{R}^n}$ , hence  $K_j$  is compact in  $\overline{\mathbb{R}^n}$ . Since the compactness is independent of whether the topology of a space or its subspace is considered,  $K_j$  is compact in  $\Omega$ . Notice that for each  $j \in \mathbb{N}$  the interior of  $K_j$  is

$$\text{int } K_j = \{x \in \Omega : q(x, \partial\Omega) > \frac{1}{j}\}.$$

Thus  $K_j \subset \text{int } K_{j+1} \subset K_{j+1}$ ,  $\Omega = \cup_{j=1}^\infty K_j = \cup_{j=1}^\infty \text{int } K_j$ . It follows that if  $K \subset \Omega$  is a compact set, then  $K \subset K_j$  for some  $j$ .

PROPOSITION 1. *The topological space  $\Gamma(\Omega)$  of all quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a metric space, where the metric is defined by*

$$\rho(f, g) = \sum_{j=1}^\infty 2^{-j} \max_{x \in K_j} q(f(x), g(x)), \quad \text{for } f, g \in \Gamma(\Omega).$$

P r o o f. Notice that  $\max_{x \in K_j} q(f(x), g(x)) \leq 1$  for  $f, g \in \Gamma(\Omega)$  and the series  $\sum_{j=1}^\infty 2^{-j}$  is convergent. Thus, the series  $\sum_{j=1}^\infty 2^{-j} \max_{x \in K_j} q(f(x), g(x))$  is convergent for all  $f$  and  $g \in \Gamma(\Omega)$ . It is clear that  $\rho(f, g) \geq 0, \rho(f, g) = \rho(g, f)$  and  $\rho$  satisfies the triangle inequality.

Also notice that  $\Omega = \cup_{j=1}^\infty K_j$ , then it is easy to verify that  $\rho(f, g) = 0$  implies  $f = g$ . ■

In the next theorem, we give a characterization of open sets in the metric space  $\Gamma(\Omega)$ , which gives that open sets does not depend on the choice of the compact sets  $K_j$ . That is, the metric  $\rho$  for  $\Gamma(\Omega)$  is independent of the choice of the compact sets  $K_j$ .

THEOREM 4. *A subset  $U$  in the metric space  $(\Gamma(\Omega), \rho)$  is open if and only if for each  $f$  in  $U$  there exists a compact set  $K$  and a  $\delta > 0$  such that*

$$V_f := \{g \in \Gamma(\Omega) : \max_{x \in K} q(f(x), g(x)) < \delta\} \subset U.$$

**P r o o f.** If  $U$  is open in the metric space  $\Gamma(\Omega)$ , then each  $f \in U$ , there exists an open ball

$$B := \{g \in \Gamma(\Omega) : \rho(f, g) < \epsilon\}, \text{ for some } \epsilon > 0$$

such that  $B \subset U$ . We claim that there exists a compact set  $K$  and a  $\delta > 0$  such that  $V_f \subset B$ , hence  $V_f \subset U$ .

For some  $\epsilon > 0$ , there exists a  $m \in \mathbb{N}$  such that  $\sum_{j=m+1}^{\infty} 2^{-j} < \frac{\epsilon}{2}$ . Let  $K = K_m$  and let  $\delta = \frac{\epsilon}{2}$ . If  $g \in V_f$ , then  $\max_{x \in K} q(f(x), g(x)) < \delta$ . Since  $K_j \subset K_m = K$  for  $1 \leq j \leq m$ ,  $\max_{x \in K_j} q(f(x), g(x)) < \delta$ . For  $j \geq m+1$ ,  $\max_{x \in K_j} q(f(x), g(x)) \leq 1$ . Thus,

$$\begin{aligned} \rho(f, g) &= \sum_{j=1}^m 2^{-j} \max_{x \in K_j} q(f(x), g(x)) + \sum_{j=m+1}^{\infty} 2^{-j} \max_{x \in K_j} q(f(x), g(x)) \\ &< \sum_{j=1}^m 2^{-j} \frac{\epsilon}{2} + \sum_{j=m+1}^{\infty} 2^{-j} < \frac{\epsilon}{2} \sum_{j=1}^{\infty} 2^{-j} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That is  $g \in B$ . Conversely, if  $\delta > 0$  and a compact set  $K$  are given such that  $V_f \subset U$ . Then there exists a  $m \in \mathbb{N}$  such that  $K \subset K_m$ . Thus

$$\max_{x \in K} q(f(x), g(x)) \leq \max_{x \in K_m} q(f(x), g(x)).$$

Let  $\epsilon = 2^{-m}\delta$ . If  $g \in B$  then  $\rho(f, g) < \epsilon$ . We have

$$\begin{aligned} \epsilon &> \rho(f, g) = \sum_{j=1}^{\infty} 2^{-j} \max_{x \in K_j} q(f(x), g(x)) \\ &= 2^{-m} \max_{x \in K_m} q(f(x), g(x)) + \sum_{j \neq m}^{\infty} 2^{-j} \max_{x \in K_j} q(f(x), g(x)) \\ &> 2^{-m} \max_{x \in K_m} q(f(x), g(x)) \end{aligned}$$

Thus  $2^{-m} \max_{x \in K_m} q(f(x), g(x)) < \epsilon$  which gives  $\max_{x \in K_m} q(f(x), g(x)) < 2^m \epsilon = \delta$ . Therefore,  $\max_{x \in K} q(f(x), g(x)) \leq \max_{x \in K_m} q(f(x), g(x)) < \delta$  which gives  $g \in V_f$ . So  $B \subset V_f \subset U$ , it is that  $U$  is open.  $\blacksquare$

Notice that defining a topology for the metric space  $\Gamma(\Omega)$  is the same thing as defining the convergence of sequences in  $\Gamma(\Omega)$ . Thus, the next corollary is a consequence of Theorem 4.

**COROLLARY 1.** *The topology of  $\Gamma(\Omega)$  induced from the metric  $\rho$  agrees with the topology induced from locally uniform convergence. Thus, all three topologies of  $\Gamma(\Omega)$  induced from the compact-open topology, from locally uniform convergence, and from the metric  $\rho$  are equivalent.*

### 3. Locally compactness

As normal families play an important role in the property of compactness, we discuss normality at beginning of this section. For a family  $F$  of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  in  $\overline{\mathbb{R}^n}$ , we prove that the closure  $\overline{F}$  of a family of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  in  $\overline{\mathbb{R}^n}$  is compact in Theorem 6. For a  $K$ -quasiconformal group  $G$ , we prove that  $G$  is a locally compact group in Theorem 7. At the end of this section, we complete the proof of the main theorem, Theorem 3.

A family  $F$  of quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a normal family in  $\Omega$  if every sequence in  $F$  has a subsequence which converges locally uniformly in  $\Omega$ . Thus, every locally uniformly convergent sequence in  $\Gamma(\Omega)$  is a normal family in  $\Omega$ . Notice that each finite family can be regarded as a normal family and an infinite family  $F$  of quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  may not be a normal family in  $\Omega$ . However, next theorem states that every infinite family of  $K$ -quasiconformal homeomorphisms always contains a normal subfamily.

**THEOREM 5.** *Suppose that  $F$  is an infinite family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ . Then there exists an infinite normal subfamily  $F_0$  of  $F$  in  $\Omega$ .*

**P r o o f.** Let  $F$  be an infinite family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ .

If a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  with at least two boundary points, by Theorem 20.5 [16], every family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  with at least two boundary points is a normal family in  $\Omega$ , hence  $F$  is a normal family in  $\Omega$ .

If a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  with at most one boundary point, that is,  $\Omega = \overline{\mathbb{R}^n}$  or  $\overline{\mathbb{R}^n} \setminus \{x_1\}$ , by Theorem 3.1 [5], every family  $F$  of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  is either a normal family in  $\Omega$  or there exists a point  $x_0 \in \Omega$  and an infinite normal subfamily  $F_1$  of  $F$  in  $\Omega_0 = \Omega \setminus \{x_0\}$ . For the latter case,  $\Omega_0 = \overline{\mathbb{R}^n} \setminus \{x_0\}$  or  $\overline{\mathbb{R}^n} \setminus \{x_0, x_1\}$ , from Theorem 3.3 [5], there exists an infinite normal subfamily extension  $F_0$  of  $F_1 \subset F$  in  $\Omega$ . The proof is complete. ■

For a given normal family  $F$ , limit mappings of subsequences are not required to be in the normal family  $F$ . By contrast, limit mappings of subsequences could be in the closure of a normal family  $\overline{F}$ . Now we turn our attention to the closure  $\overline{F}$  of family in the topological space  $\Gamma(\Omega)$ .

**PROPOSITION 2.** *Let  $F$  be a family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ , then  $\overline{F}$  is also a family of  $K$ -quasiconformal self homeomorphisms of  $\Omega$ .*

**P r o o f.** Let  $F$  be a family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ , then the closure  $\overline{F}$  is formed by the family  $F$  together with its limit mappings of subsequences. By Corollaries 21.3 and 37.4 of [16], if a sequence of  $K$ -quasiconformal homeomorphisms is convergent locally uniformly to a mapping  $f$ , then  $f$  is either a  $K$ -quasiconformal self homeomorphism of  $\Omega$  or a constant. Since  $\overline{F} \subset \Gamma(\Omega)$  and constant limit mappings are not contained in  $\Gamma(\Omega)$ , the closure  $\overline{F}$  is a family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ . ■

Furthermore, we are going to show the next theorem that the closure  $\overline{F}$  of a family of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is compact in  $\Gamma(\Omega)$ .

**THEOREM 6.** *Let  $F$  be a family of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  in  $\overline{\mathbb{R}^n}$ , then the closure  $\overline{F}$  is compact in the space  $\Gamma(\Omega)$ .*

**P r o o f.** Let  $F$  be a family of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  in  $\overline{\mathbb{R}^n}$ , then  $\overline{F}$  is also a family of  $K$ -quasiconformal self homeomorphisms of  $\Omega$  by Proposition 2. Since  $\Gamma(\Omega)$  is a metric space by Proposition 1, the closure  $\overline{F}$  is compact if and only if  $\overline{F}$  is sequential compact. We need to show that every sequence in  $\overline{F}$  has a subsequence which converges locally uniformly in  $\Omega$  and the limit mapping is contained in  $\overline{F}$ . By Corollaries 21.3 and 37.4 of [16], the mapping  $f$  is either a  $K$ -quasiconformal self homeomorphism of  $\Omega$  or a constant.

Let  $\{f_j\}$  be a sequence in  $\overline{F}$ , then  $\{f_j\}$  is a sequence of  $K$ -quasiconformal self homeomorphisms of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ . By Theorem 5, there exists a subsequence  $\{f_{j_k}\}$  which converges locally uniformly to a mapping  $f$  in  $\Omega$ .

If  $\Omega = \overline{\mathbb{R}^n}$ . According to Theorem 21.5 and Corollary 37.4 [16], the limit mapping  $f$  is also a  $K$ -quasiconformal self homeomorphism of  $\overline{\mathbb{R}^n}$ , so  $f \in \overline{F}$ . Therefore,  $\overline{F}$  is compact in  $\Gamma(\overline{\mathbb{R}^n})$ .

If  $\Omega \neq \overline{\mathbb{R}^n}$ , then the boundary  $\partial\Omega \neq \emptyset$ . If the limit mapping  $f$  is a constant  $c$ , by Theorems 21.7 and 21.11 [16], then  $c \in \partial\Omega$ . It is followed that the limit mapping  $f$  is a  $K$ -quasiconformal self homeomorphism of  $\Omega$ , hence  $f \in \overline{F}$ , and  $\overline{F}$  is compact in  $\Gamma(\Omega)$ . ■

Now consider a  $K$ -quasiconformal group  $G$  acting on  $\Omega$ , we have the next important theorem.

**THEOREM 7.** *Suppose that  $\Omega$  is a domain in  $\overline{\mathbb{R}^n}$ , and that  $G$  is a  $K$ -quasiconformal group acting on  $\Omega$ . Then  $G$  is a locally compact topological transformation group in  $\Gamma_K(\Omega)$ .*

**P r o o f.** First of all, by Theorem 2.3.5 [7], each  $K$ -quasiconformal group acting on a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a topological transformation group. Let  $g \in G$  and let  $U$  be a neighborhood of  $g$ . Since  $G$  is a topological group,  $G$  is a regular topological space [11]. Thus, there exists a neighborhood  $V$  of  $g$  such that

$$g \in V \subset \overline{V} \subset U \subset G \subset \Gamma_K(\Omega).$$

Applying the previous Theorem 6,  $\overline{V}$  is compact in  $\Gamma_K(\Omega)$ . Therefore  $G$  is locally compact in  $\Gamma_K(\Omega)$ , hence  $G$  is a locally compact topological transformation group in  $\Gamma_K(\Omega)$ . ■

Finally, we are ready to complete the proof of Theorem 3 and solve Hilbert-Smith Conjecture for  $K$ -quasiconformal groups acting on domains in  $\overline{\mathbb{R}^n}$ .

**P r o o f of Theorem 3.** First, by Proposition 1 [6], every smooth manifold is a Riemannian manifold. Since each domain in  $\overline{\mathbb{R}^n}$  is a smooth manifold, every domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a Riemannian manifold.

Second, by Proposition 2 [6], every  $K$ -quasiconformal group  $G$  of a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$  is a topological transformation group acting effectively on  $\Omega$ .

Third, by the previous Theorem 7, if a  $K$ -quasiconformal group  $G$  acting on a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ , then  $G$  is a locally compact topological transformation group.

Finally, applying Martin's Theorem 1, each  $K$ -quasiconformal group is a Lie group acting on a domain  $\Omega$  in  $\overline{\mathbb{R}^n}$ . ■

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