Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 5 (2010) ISSN 1311-0454

SOME GROWTH AND DISTORTION THEOREMS FOR CLOSE-TO-CONVEX HARMONIC FUNCTIONS IN THE UNIT DISC

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Dedicated to Professor Gheorghe Oros
on the occasion of his 60th *birthday*

Abstract

Harmonic function in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ can be written as a sum of an analytic and an anti-analytic function, $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, and are called the analytic part and co-analytic part of f, respectively.

One of the most important questions in the study of the classes of such functions is related to bounds on the modulus of functions (growth) or modulus of the derivative (distortion). The aim of this paper is to give the growth and distortion theorems for the close-to-convex harmonic functions in the open unit disc $\mathbb{D}$.

MSC 2010: 30C45, 30C55
Key Words and Phrases: harmonic functions, growth theorem, distortion theorem

1. Introduction

Let $\mathcal{U}$ be a simply connected domain in the complex plane $\mathbb{C}$. A harmonic function f has the representation $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathcal{U}$ and are called the analytic and co-analytic part of f, respectively. Let $h(z) = z^m + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$ and $g(z) = b_m z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots$ $(m \in \mathbb{N})$ be analytic functions in the open unit disc $\mathbb{D}$. The Jacobian of the mapping $f = h(z) + \overline{g(z)}$, denoted by $J_{f(z)}$, and can be computed by $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2$. If $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0$ or $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 < 0$, then f is called a sense-preserving multivalent harmonic function. The class of all sense-preserving multivalent harmonic functions with $|b_m| < 1$ is denoted by $\mathcal{S}_{\mathcal{H}}(m)$, and the class of all sense-preserving multivalent harmonic functions with $b_m = 0$ is denoted by $\mathcal{S}^0_{\mathcal{H}}(m)$. For convenience, we will investigate sense-preserving harmonic functions, that is, functions for which $J_{f(z)} > 0$. If $J_{f(z)} < 0$, then $\overline{f}$ is sense-preserving. The second analytic dilatation of a harmonic function is given by $w(z) = g'(z)/h'(z)$. We also note that if f is locally univalent and sense-preserving, then $|w(z)| < 1$ for every $z \in \mathbb{D}$.

In this paper we examine the class of functions that are convex in the direction of real axis. The shear construction is essential to the present work as it allows one to study harmonic functions through their related analytic functions ([2], Hengartner and Shober). The shear construction produces a univalent harmonic function that maps $\mathbb{D}$ to a region that is convex in the direction of the real axis. This construction relies on the following theorem of Clunie and Sheil-Small.

Theorem 1.1. ([1]) *A harmonic function $f = h(z) + \overline{g(z)}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h(z) - g(z)$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

Hengartner and Shober [2] studied analytic functions $\psi(z)$ that are convex in the direction of the imaginary axis. They used a normalization which requires, in essence, that the right and left extremes of $\psi(\mathbb{D})$ be the images of 1 and -1. This normalization is as follows: there exist points z'_n converging to $z = 1$ and z''_n converging to $z = -1$ such that

$$\begin{aligned}\lim_{n\to\infty} \operatorname{Re}\{\psi(z'_n)\} &= \sup_{|z|<1} \operatorname{Re}\{\psi(z)\},\\ \lim_{n\to\infty} \operatorname{Re}\{\psi(z''_n)\} &= \inf_{|z|<1} \operatorname{Re}\{\psi(z)\}.\end{aligned} \tag{1}$$

If $\mathcal{CIA}$ is the class of functions on the domains, D, that are convex in the direction of the imaginary axis and admit a mapping $\psi(z)$ so that $\psi(\mathbb{D}) = D$ and $\psi(z)$ satisfies the normalization (1), then we have the following result:

THEOREM 1.2. ([2]) *Suppose that* $\psi(z)$ *is analytic and non-constant for* $|z| < 1$. *Then we have* $\mathrm{Re}[(1-z^2)\psi'(z)] \geq 0$ *for* $|z| < 1$ *if and only if*

(i) $\psi(z)$ *is univalent on* $\mathbb{D}$,

(ii) $\psi(\mathbb{D}) \in \mathcal{CIA}$, *and*

(iii) $\psi(z)$ *is normalized by* (1).

We note that in the light of Theorem 1.2 and the normalization (1), the region that is convex in the direction of the real and the region in the direction of the imaginary axis are determined by the following three remarks:

REMARK 1.3. ([2]) The condition $Re[(1-z^2)\psi'(z)] \geq 0$ (for $|z| < 1$), has an elementary geometric interpretation. If we parametrize the line segment and circular arcs γ_t, $-\pi/2 < t < \pi/2$, joining $z = -1$ to $z = 1$ in the unit disc by

$$\gamma_t : z = z(s) = \frac{e^{s+it} - 1}{e^{s+it} + 1}, \quad -\infty < s < \infty,$$

then one easily verifies that

$$\frac{d}{ds} Re[\psi(z(s))] = 2Re[(1 - z^2(s))\psi'(z(s))].$$

Consequently, the condition $Re[(1-z^2)\psi'(z)] \geq 0$ (for $|z| < 1$) is equivalent to the property that the circular arcs γ_t are mapped onto analytic arcs which may be represented as functions $v = v(u)$. It follows that $\psi(z)$ has the normalization (1). Furthermore, since the region bounded by $\psi(\gamma_t) \cup \psi(1) \cup \psi(-1)$ is convex in the v-direction for every $-\pi/2 < t < \pi/2$, we find that $\psi(|z| < 1)$ is also convex in the v-direction.

REMARK 1.4. ([2]) An analytic function $\psi(z)$ is close-to-convex if there exists a convex mapping $s(z)$ such that $Re[\psi'(z)/s'(z)] > 0$ for $|z| < 1$. Functions satisfying $Re[(1 - z^2)\psi'(z)] > 0$ are special close-to-convex functions associated with $s(z) = \frac{1}{2}\log[(1+z)/(1-z)]$. W. Kaplan [3] has shown that close-to-convex functions, hence functions satisfying $Re[(1 - z^2)\psi'(z)] > 0$, are univalent. The geometric interpretation of Remark 3 could also be used to show that functions satisfying $Re[(1 - z^2)\psi'(z)] > 0$ are univalent.

REMARK 1.5. ([4]) We also note that Theorem 1.1 has a natural generalization when f is convex in the direction α. In that situation $e^{-i\alpha}f$ and $\varphi(z) = e^{-i\alpha}h(z) - e^{i\alpha}g(z)$ are convex in the direction of the real axis, hence the function $h(z)-e^{i2\alpha}g(z)$ is convex in the direction α. In particular, we can use this construction when $\alpha=\pi/2$ to construct functions that are convex in the direction of the imaginary axis. At the same time, to be able to use this result for functions that are convex in the direction of the real axis, let us consider the following situation: suppose that $\varphi(z)$ is a function that is analytic and convex in the direction of the real axis. Furthermore, suppose that φ is normalized by the following. Let there exist points z_n' converging to $z=e^{i\alpha}$ and z_n'' converging to $z=e^{i(\alpha+\pi)}$ such that

$$\begin{aligned}\lim_{n\to\infty} Im\{\varphi(z_n')\} &= \sup_{|z|<1} Im\{\varphi(z)\},\\ \lim_{n\to\infty} Im\{\varphi(z_n'')\} &= \inf_{|z|<1} Im\{\varphi(z)\}.\end{aligned} \tag{2}$$

Consequently, if $\psi(z)$ satisfies (1), then $\varphi(z)=i\psi(e^{-i\alpha}z)$ satisfies (2).

Finally, let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0)=0$, $|\phi(z)|<1$ for every $z\in\mathbb{D}$. Denote by $\mathcal{P}(m)$ (with m being a positive integer) the family of functions $p(z)=m+p_1z+p_2z^2+\cdots$ which are regular in $\mathbb{D}$ and satisfying the conditions $p(0)=m$, $\text{Re}p(z)>0$ for all $z\in\mathbb{D}$, and such that $p(z)$ is in $\mathcal{P}(m)$ if and only if

$$p(z)=m\frac{1+\phi(z)}{1-\phi(z)} \tag{3}$$

for some function $\phi(z)\in\Omega$ and every $z\in\mathbb{D}$. Let $F(z)=z+\alpha_2z^2+\cdots$ and $G(z)=z+\beta_2z^2+\cdots$ be analytic functions in $\mathbb{D}$, if there exist a function $\phi(z)\in\Omega$ such that $F(z)=G(\phi(z))$ for all $z\in\mathbb{D}$, then we say that $F(z)$ subordinate to $G(z)$ and we write $F(z)\prec G(z)$. We also note that if $F(z)\prec G(z)$, then $F(\mathbb{D})\subset G(\mathbb{D})$.

Denote by $\mathcal{S}^0_{\mathcal{H}}\mathcal{C}(m)$ the class of all m-valent harmonic functions in the direction of real axis. In this paper we will give growth and distortion theorems for the class $\mathcal{S}^0_{\mathcal{H}}\mathcal{C}(m)$.

2. Main Results

LEMMA 2.1. *Let $\varphi(z)=z^m+c_{m+1}z^{m+1}+c_{m+2}z^{m+2}+\cdots$ be analytic in $\mathbb{D}$. If $\varphi(z)$ satisfies the condition* $\text{Re}\left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right]>0$, *then $\varphi(z)$ is a m-valent close-to-convex function and*

$$\frac{mr^{m-1}(1-r)}{(1+r^{2m})(1+r)}\le|\varphi'(z)|\le\frac{mr^{m-1}(1+r)}{(1-r^{2m})(1-r)} \tag{4}$$

for all $|z| = r < 1$.

P r o o f. Let consider the function $s(z) = \int_0^z \frac{\zeta^{m-1}}{1-\zeta^{2m}} d\zeta$. Since

$$1 + z\frac{s''(z)}{s'(z)} = m\frac{1+z^{2m}}{1-z^{2m}} \Rightarrow \operatorname{Re}\left(1 + z\frac{s''(z)}{s'(z)}\right) > 0$$

and

$$\operatorname{Re}\left(\frac{\varphi'(z)}{s'(z)}\right) = \operatorname{Re}\left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right] > 0,$$

then $\varphi(z)$ is a m-valent close-to-convex function for all $z \in \mathbb{D}$. On the other hand, since

$$\operatorname{Re}\left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right] > 0, \left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right]\Bigg|_{z=0} = m,$$

$$p(z) \in \mathcal{P}(m) \Leftrightarrow p(z) = m\frac{1+\phi(z)}{1-\phi(z)} \Leftrightarrow \phi(z) = \frac{p(z)-m}{p(z)+m},$$

for some $\phi(z) \in \Omega$, the function

$$\phi(z) = \frac{(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} - m}{(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} + m}$$

satisfies the conditions of Schwarz lemma, whence $|\phi(z)| \leq r$. Therefore we have

$$\left|\frac{(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} - m}{(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} + m}\right| \leq r \Rightarrow \left|(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} - m\frac{1+r^2}{1-r^2}\right| \leq \frac{2mr}{1-r^2}. \quad (5)$$

After straightforward calculations we obtain (4). ∎

COROLLARY 2.2. *If we take* $m = 1$ *in Lemma 2.1, we have*

$$\frac{1-r}{(1+r^2)(1+r)} \leq |\varphi'(z)| \leq \frac{1}{(1-r)^2}.$$

This inequalities were obtained by Hengartner and Shober [2].

THEOREM 2.3. *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}^0_{\mathcal{H}}(m)$, *and let* $\varphi(z) = h(z) - g(z)$, $w(z) = \frac{g'(z)}{h'(z)}$. *Furthermore, let* $\varphi(z)$ *satisfies the condition* $\operatorname{Re}\left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right] > 0$ *for all* $z \in \mathbb{D}$, *then for* $|z| < r$

$$\frac{mr^{m-1}(1-r)}{(1+r^{2m})(1+r)^2} \le |f_z| \le \frac{mr^{m-1}(1+r)}{(1-r)^2(1-r^{2m})} \tag{6}$$

and

$$\frac{|w(z)|mr^{m-1}(1-r)}{(1+r^{2m})(1+r)^2} \le |f_{\bar{z}}| \le \frac{mr^{m-1}(1-r)}{(1-r)^2(1-r^{2m})} \tag{7}$$

for all $|z| = r < 1$, *where* $w = g'/h'$ *is second analytic dilatation of* f.

P r o o f. Since $\varphi(z) = h(z) - g(z)$, then we have

$$h'(z) = f_z = \frac{\varphi'(z)}{1-w(z)}, \quad g'(z) = \overline{f_{\bar{z}}} = \frac{\varphi'(z)w(z)}{1-w(z)}, \quad |w(z)| < 1.$$

Therefore, we have

$$\frac{|\varphi'(z)|}{1+|w(z)|} \le |f_z| \le \frac{|\varphi'(z)|}{1-|w(z)|}, \tag{8}$$

and

$$\frac{|w(z)||\varphi'(z)|}{1+|w(z)|} \le |f_{\bar{z}}| \le \frac{|w(z)||\varphi'(z)|}{1-|w(z)|}. \tag{9}$$

Using Lemma 2.1 and the Schwarz lemma in (8) and (9), we get (6) and (7), respectively. We note that the inequalities are sharp because the extremal functions can be found as

$$(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}} = m\frac{1+z}{1-z}, (1-z^{2m})\frac{\varphi'(z)}{z^m} = m\frac{1-z}{1+z}$$

(i.e., $p(z) \in \mathcal{P}(m)$ then $\frac{1}{p(z)} \in \mathcal{P}(m)$)

$$h'(z) = \int_0^z \frac{\varphi'(\zeta)}{1-w(\zeta)}d\zeta, \quad g'(z) = \int_0^z \frac{\varphi'(\zeta)w(\zeta)}{1-w(\zeta)}d\zeta,$$

the solution of $h(z)$ and $g(z)$ must be found under the conditions $h(0) = g(0) = 0$.

$$\begin{aligned} f = h(z) + \overline{g(z)} &= \int_0^z \frac{\varphi'(\zeta)}{1-w(\zeta)}d\zeta + \overline{\int_0^z \frac{\varphi'(\zeta)w(\zeta)}{1-w(\zeta)}d\zeta} \\ &= \int_0^z \frac{\varphi'(\zeta)}{1-w(\zeta)}d\zeta + \overline{\int_0^z \frac{\varphi'(\zeta)}{1-w(\zeta)}d\zeta - \int_0^z \varphi'(\zeta)d\zeta} \\ &= \operatorname{Re}\left(\int_0^z \frac{2\varphi'(\zeta)}{1-w(\zeta)}d\zeta\right) - \overline{\varphi(z)}. \end{aligned}$$

We also note that the second dilatation must be chosen suitably. ∎

COROLLARY 2.4. *If we take $m = 1$ in Theorem 2.3, we have*

$$\frac{1-r}{(1+r^2)(1+r)^2} \le |f_z| \le \frac{1}{(1-r)^3},$$

$$\frac{|w(z)|(1-r)}{(1+r^2)(1+r)^2} \le |f_{\bar{z}}| \le \frac{r}{(1-r)^3}.$$

These inequalities were obtained by Schambroeck [4].

THEOREM 2.5. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^0_{\mathcal{H}}(m)$, and let $\varphi(z) = h(z) - g(z), w(z) = \frac{g'(z)}{h'(z)}, Re\left[(1-z^{2m})\frac{\varphi'(z)}{z^{m-1}}\right] > 0$, then*

$$|f| \le \int_0^r \frac{m\rho^{m-1}(1+\rho)^2}{(1-\rho)^2(1-\rho^{2m})} d\rho$$

P r o o f. We have

$$\begin{aligned} f = h(z) + \overline{g(z)} \Rightarrow |f| = |h(z) + \overline{g(z)}| &\le |h(z)| + |g(z)| \\ &\le \int_0^r |f_z(\rho e^{i\theta})e^{i\theta}| d\rho + \int_0^r |f_{\bar{z}}(\rho e^{i\theta})e^{-i\theta}| d\rho \\ &= \int_0^r |f_z(\rho e^{i\theta})| d\rho + \int_0^r |f_{\bar{z}}(\rho e^{i\theta})| d\rho \\ &= \int_0^r \frac{m\rho^{m-1}(1+\rho)}{(1-\rho)^2(1-\rho^{2m})} d\rho + \int_0^r \frac{m\rho^{m}(1+\rho)}{(1-\rho)^2(1-\rho^{2m})} d\rho. \end{aligned}$$

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COROLLARY 2.6. *If we take $m = 1$ in Theorem 2.5, then we obtain*

$$|f| \le \frac{r}{(1-r)^2}.$$

This inequality was obtained by Schambroeck [4].

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Received: GFTA, August 27-31, 2010