Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 14, NUMBER 1 (2011)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

SURVEY PAPER

FRACTIONAL FOKKER-PLANCK-KOLMOGOROV TYPE EQUATIONS AND THEIR ASSOCIATED STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

Dedicated to 80-th anniversary of Professor Rudolf Gorenflo

There is a well-known relationship between the Itô stochastic differential equations (SDEs) and the associated partial differential equations called Fokker-Planck equations, also called Kolmogorov equations. The Brownian motion plays the role of the basic driving process for SDEs. This paper provides fractional generalizations of the triple relationship between the driving process, corresponding SDEs and deterministic fractional order Fokker-Planck-Kolmogorov type equations.

MSC 2010: 26A33, 35R11, 35R60, 35Q84, 60H10

Key Words and Phrases: fractional differential equation (FDE), Lévy process, time-change, stable subordinator, stochastic differential equation (SDE), Fokker-Planck equation, Kolmogorov equations

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pp. 56–79 , DOI: 10.2478/s13540-011-0005-9

Springer

1. Introduction

One of Albert Einstein's *Annus Mirabilis* 1905 papers[1] was devoted to the theoretical explanation of the Brownian motion. A little earlier (in 1900) Bachelier published his doctoral dissertation[2] modeling Brownian motion from the economics point of view. In 1908 Langevin published his work with a stochastic differential equation which was "understood mathematically" only after a stochastic calculus was introduced by Itô in 1944-48. The Fokker-Planck equation, a deterministic form of describing the dynamics of a random process in terms of transition probabilities, was invented in 1913-17. The Fokker-Planck equation uses an initial condition with Dirac's delta where complete "mathematical understanding" became available only after the appearance of the distribution (generalized function) theory (Sobolev (1938), Schwartz (1951)) and was embodied in Kolmogorov's backward and forward equations. Today the relationship between the Itô stochastic differential equations driven by Brownian motion and their associated Fokker-Planck-Kolmogorov partial differential equations is well understood (41).

The goal of this paper is twofold. The paper provides a brief survey of the authors' recent work and other closely related results on fractional generalizations of this triple relationship between the driving process, corresponding SDEs, and associated deterministic fractional order pseudo-differential equations. Due to the rapid development of this theory we found that such a brief survey would be useful for readers focused on fractional calculus. In the last few decades, fractional Fokker-Planck-Kolmogorov (FPK) type equations have been used to model the dynamics of complex processes in many fields, including physics (6; 33; 34; 48), finance (11; 40), hydrology (4), cell biology (8; 39), etc. Complexity includes phenomena such as weak or strong correlations, different sub- or super-diffusive modes, memory and jump effects. For example, experimental studies of the motion of proteins or other macromolecules in a cell membrane show apparent sub-diffusive motion with several simultaneous diffusive modes (see, e.g. (39)). The second goal of the paper is to present new methods developed recently and their corresponding results.

Fractional generalizations of the classical Fokker-Planck, or forward and backward Kolmogorov equations, in the sense that the first order time derivative on the left side of equation (2) (Section **2**) is replaced by a

[1] "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen" ("On the Motion of Small Particles Suspended in a Stationary Liquid, as Required by the Molecular Kinetic Theory of Heat")

[2] "Théorie de la spéculation" ("The Theory of Speculation")

time-fractional order derivative, has appeared in the framework of continuous time random walks (CTRWs) and fractional kinetic theory (9; 33; 35; 36; 48). A CTRW is a random walk subordinated to a renewal process. Namely, a CTRW can be defined using two independent sequences of i.i.d. random variables (see Section **5**). Time-fractional versions of FPK type equations with a single fractional derivative are connected with time-changed Lévy processes, where the time-change arises as the inverse, or equivalently the first hitting time of level t, for a single stable subordinator (3; 13; 24; 43; 46). More general distributed order time fractional FPK equations correspond to a time-change which is the inverse to mixtures of stable subordinators with some mixing measure (19). The term "fractional generalization" in this paper is used in a wider sense including not only equations with a single fractional derivative, but also FPK type equations with distributed and variable fractional orders. The papers (19; 20) study relationships between fractional FPK type equations and their associated SDEs, which were not known in the case of fractional FPK type equations. Two other recent papers are closely related with the research discussed there. The paper (22) derives a specialized form of the Itô Formula for stochastic integrals driven by time-changed semi-martingales and applies it in a variety of examples connecting them to fractional order equations. The paper (18) establishes FPK type equations associated with time-changed Gaussian processes and ongoing work is directed at connecting them with their associated stochastic differential equations.

In Section **3** we provide a brief survey of these results. Section **4** proves an abstract theorem which can be applied to establish a connection between FPK type equations and their associated SDEs in many particular cases.

The driving process of a stochastic differential equation plays a key role in the dynamics and future evolution of the solution to that SDE. The processes associated with fractional order FPK equations are usually driven by complex processes. Even in the simplest case of the fractional equation $\partial^\beta u = \kappa_\beta \, \Delta u$, where κ_β is the diffusion coefficient, Δ is the Laplace operator, and ∂^β is a fractional derivative (in some sense) of order $0 < \beta < 1$, the driving process is not even a Lévy process. Therefore, understanding the properties of the driving process elucidates many properties of the process itself. In fact, the driving processes of SDEs corresponding to fractional FPK type equations are not Brownian motion, or even Lévy processes. In Section **5** we briefly discuss driving processes related to fractional FPK equations, connecting them to CTRWs as well as the importance of Duhamel's principle for fractional FPK type equations.

2. Auxiliaries

Let the stochastic process Y_t solve an Itô stochastic differential equation

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t,\ Y_0 = x, \tag{1}$$

where B_t is m-dimensional Brownian motion defined on a probability space with an appropriate filtration; $x \in \mathbb{R}^d$ $(d \geq 1)$, is a fixed point; the mappings $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfy a Lipschitz condition. Recall that the Brownian motion is a Gaussian process with independent and stationary increments, and a.s. continuous sample paths (see details, e.g. (2; 41)). In the general setting we consider, the Cauchy problem for the associated FPK equation is

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{A}u(t,x),\ u(0,x) = \varphi(x),\ t > 0,\ x \in \mathbb{R}^d, \tag{2}$$

where $\mathcal{A}$ is the differential operator

$$\mathcal{A} = \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{3}$$

with coefficients $b_j(x)$ and $\sigma_{i,j}(x)$ connected with the coefficients of SDE (1) as follows: $b(x) = (b_1(x), \ldots, b_d(x))$ and $\sigma_{i,j}(x)$ is the (i,j)-th entry of the product of the $d \times m$ matrix $\sigma(x)$ with its transpose $\sigma^T(x)$. The solution $u(t,x)$ of the Cauchy problem (2) represents the conditional transition probabilities of the process Y_t, which solves SDE (1). Namely, $u(t,x) = E[\varphi(Y_t)|Y_0 = x]$. Moreover, the operators $T_t : \varphi(x) \to u(t,x)$ for $t > 0$, defined as

$$T_t\varphi(x) = E[\varphi(Y_t)|Y_0 = x], \tag{4}$$

form a strongly continuous semigroup on the Banach space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at infinity with sup-norm (or on $L_p(\mathbb{R}^d)$).

The Lévy processes are a broad class of driving processes for which stochastic differential equations are well defined. A Lévy process is a stochastically continuous stochastic process with independent stationary increments. The Brownian motion is the only continuous Lévy process. All other Lévy processes allow jumps, but are right continuous with left limits. Processes with such jumps are named *càdlàg*, the French abbreviation of "continu à droite, limite à gauche". Lévy processes are entirely specified with three parameters (b, Σ, ν), where $b \in \mathbb{R}^d$ is a vector responsible for the drift component, Σ is a nonnegative definite (covariance) matrix corresponding to the Brownian component, and ν is a Lévy measure which specifies weights allocated to jump sizes. The Lévy measure ν is defined so that it satisfies the condition

$$\int_{\mathbb{R}^d\setminus 0} \min(1, x^2) d\nu < \infty.$$

A general characterization of the Lévy processes is given by the celebrated Lévy-Khintchine formula through its characteristic function $\Phi_t(\xi)$, $\xi \in \mathbb{R}^d$, or Lévy symbol $\Psi(\xi)$, $\xi \in \mathbb{R}^d$, which are related as $\Phi_t(\xi) = e^{t\Psi(\xi)}$. The Lévy symbol $\Psi(\xi)$ is

$$\Psi(\xi) = i(b,\xi) - \frac{1}{2}(\Sigma\xi,\xi) + \int_{\mathbb{R}^d\setminus\{0\}} (e^{i(w,\xi)} - 1 - i(w,\xi)\chi_{(|w|\leq 1)}(w))\nu(dw). \quad (5)$$

For any Lévy process, its Lévy symbol is a continuous, hermitian, conditionally positive definite function with $\Psi(0) = 0$. Two subclasses of Lévy processes are of particular interest in this paper:

(1) symmetric α-stable Lévy processes which consist of processes with parameters $b = 0$, $\Sigma = 0$, and a Lévy measure ν such that the Lévy symbol has the form

$$\Psi(\xi) = -c|\xi|^{\alpha}, (0 < \alpha \leq 2);$$

and

(2) 1-dimensional β-stable subordinators which are nonnegative increasing stable Lévy processes with stability index $0 < \beta < 1$. The Lévy symbol of a β-stable subordinator is $\Psi(s) = s^{\beta}$, $s > 0$.

For further details related to the Lévy processes and the stochastic differential equations driven by Lévy process, we refer the reader to (2; 38).

Let L_t be a Lévy process with parameters (b, B_t, ν). If the SDE is driven by the process L_t, i.e. is given in the form

$$Y_t = x + \int_0^t b(Y_{s-})ds + \int_0^t \sigma(Y_{s-})dB_s + \int_0^t \int_{R^d\setminus\{0\}} G(Y_{s-}, w)N(ds, dw), \quad (6)$$

with Lipschitz continuous mappings $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying some appropriate growth conditions, then the operator $\mathcal{A}$ in FPK equation (2) becomes a pseudo-differential operator and has the form

$$\begin{aligned}\mathcal{A}\varphi(x) &= [b(x)\cdot\nabla]\varphi(x) - \frac{1}{2}\Big[\nabla\cdot\Sigma(x)\nabla\Big]\varphi(x) \\ &+ \int (\varphi(x + G(x,w)) - \varphi(x) - \chi_{(|w|<1)}(w)\Big[G(x,w)\cdot\nabla\Big]\varphi(x))\nu(dw), \quad (7)\end{aligned}$$

with a domain which contains $C_0^2(\mathbb{R}^d)$. The integral in equation (7) is taken over $\mathbb{R}^d \setminus \{0\}$ and the symbol ∇ stands for the gradient $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$. The measure N in equation (6) is a Poisson random measure (2) with intensity ν. Again, like the previous case, solutions of the Cauchy problem (2) with $\mathcal{A}$ in (7) and SDE (6) are connected through formula (4) which expresses the Markovian character of both processes.

The widest class of processes for which the Itô calculus can be extended is the class of semimartingales. A semimartingale is a càdlàg process which has the representation

$$Z_t = Z_0 + V_t + M_t$$

where V_t is an adapted finite variation process and M_t is a local martingale, see details in (37). It can be shown that any Lévy process is a semimartingale. The composition of two Lévy processes is again a Lévy process. In particular, for a stable Lévy process L_t, the time-changed process L_{D_t} with a stable Lévy subordinator D_t is again a Lévy process. Transition probabilities of such a time-changed process satisfy FPK type equation (2) with a space-fractional order pseudo-differential operator $\mathcal{A}$ on the right hand side. However, the composition L_{W_t}, where $W_t = \inf\{\tau \geq 0 : D_\tau > t\}$, the inverse to the stable subordinator D_t is not a Lévy process, but still is a semimartingale. This drastically changes the associated FPK equation: now it is a time- and space-fractional differential equation (14; 29), implying non-Markovian behaviour of the process. To establish the result one needs to know properties of the process W_t, which is continuous and nondecreasing; hence it is a time-change process. W_t is self-similar. Moreover, the density $f_{W_1}(\tau)$ of W_1 is infinitely differentiable with power-law decay at infinity and vanishes at an exponential rate when $\tau \to 0$. More precisely, (27; 42):

$$f_{W_1}(\tau) \sim \frac{(\frac{\beta}{\tau})^{\frac{2-\beta}{2(1-\beta)}}}{\sqrt{2\pi\beta(1-\beta)}}\, e^{-(1-\beta)(\frac{\tau}{\beta})^{-\frac{\beta}{1-\beta}}},\ \tau \to 0; \tag{8}$$

$$f_{W_1}(\tau) \sim \frac{\beta}{\Gamma(1-\beta)\tau^{1+\beta}},\ \tau \to \infty. \tag{9}$$

It is known (see e.g. (38)) that mixtures of independent stable processes of different indices are no longer stable. However, density functions of mixtures can be effectively described. Below we give such a description for a mixture of two independent stable subordinators with different indices.

LEMMA **2.1**. *Let $D_t = c_1 D_t^{(1)} + c_2 D_t^{(2)}$, where c_1, c_2 are positive constants and $D_t^{(1)}$ and $D_t^{(2)}$ are independent stable subordinators with respective densities $f_1^{(1)}$, $f_1^{(2)}$ at $t = 1$, and indices β_1 and β_2 in $(0, 1)$. Then the* *inverse W_t of D_t has density*

$$f_{W_t}(\tau) = -\frac{\partial}{\partial\tau}\left\{\frac{1}{c_2\tau^{\frac{1}{\beta}_2}}\left[\left(Jf_1^{(1)}\right)\left(\frac{\cdot}{c_1\tau^{\frac{1}{\beta}_1}}\right) * f_1^{(2)}\left(\frac{\cdot}{c_2\tau^{\frac{1}{\beta}_2}}\right)\right](t)\right\}, \quad (10)$$

where J is the integration operator and $*$ stands for convolution of densities. Moreover, there exist a number $\beta \in (0,1)$ and positive constants C, k, not depending on τ, such that for all $t < \infty$ the estimate

$$f_{W_t}(\tau) \le C\exp\left(-k\tau^{\frac{1}{1-\beta}}\right) \quad (11)$$

holds.

P r o o f. Due to self-similarity and independence of $D_t^{(1)}$ and $D_t^{(2)}$, the cumulative distribution function of W_t is

$$F_{W_t}(\tau) = 1 - \left[F_1^{(1)}\left(\frac{\cdot}{c_1\tau^{\frac{1}{\beta}_1}}\right) * F_1^{(2)}\left(\frac{\cdot}{c_2\tau^{\frac{1}{\beta}_2}}\right)\right](t),$$

where $F_1^{(1)}$ and $F_1^{(2)}$ are respective cumulative distribution functions of $D_t^{(1)}$ and $D_t^{(2)}$ at $t = 1$. The representation (10) follows immediately upon differentiating the distribution function $F_{W_t}(\tau)$ of the process W_t with respect to τ.

In order to prove estimate (11), we suppose for clarity that $0 < \beta_1 < \beta_2 < 1$ in representation (10). It follows that $f_{W_t}(\tau) = I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{\beta_2 c_2 \tau^{1+\frac{1}{\beta_2}}}\int_0^t \left(Jf_1^{(1)}\right)\left(\frac{s}{c_1\tau^{\frac{1}{\beta}_1}}\right) f_1^{(2)}\left(\frac{t-s}{c_2\tau^{\frac{1}{\beta}_2}}\right) ds, \quad (12)$$

$$I_2 = \frac{1}{\beta_1 c_1 c_2 \tau^{1+\frac{1}{\beta_1}+\frac{1}{\beta_2}}}\int_0^t s\cdot f_1^{(1)}\left(\frac{s}{c_1\tau^{\frac{1}{\beta}_1}}\right) f_1^{(2)}\left(\frac{t-s}{c_2\tau^{\frac{1}{\beta}_2}}\right) ds, \quad (13)$$

and

$$I_3 = \frac{1}{c_2\tau^{\frac{1}{\beta_2}}}\int_0^t s\cdot \left(Jf_1^{(1)}\right)\left(\frac{s}{c_1\tau^{\frac{1}{\beta}_1}}\right) \left(f_1^{(2)}\right)'\left(\frac{t-s}{c_2\tau^{\frac{1}{\beta}_2}}\right) ds. \quad (14)$$

Integration by parts reduces I_3 to the sum of integrals of types I_1 and I_2, namely, $I_3 = \beta_2 c_2 \tau^{1+\frac{1}{\beta}_2} I_1 + \beta_1\tau I_2$. Therefore, it suffices to estimate I_1 and I_2. First notice that both functions $f_1^{(1)}$, $f_1^{(2)}$ are continuous on $[0,\infty)$, and $Jf_1^{(1)}(t) \le 1$. Consequently, in accordance with the mean value theorem, there exist numbers $s_*, s_{**} \in (0,t)$ such that

$$I_1 \le \frac{t}{\beta_2 c_2 \tau^{1+\frac{1}{\beta_2}}} f_1^{(2)}\left(\frac{s_*}{c_2 \tau^{\frac{1}{\beta}_2}}\right), \tag{15}$$

and

$$I_2 = \frac{t s_{**}}{\beta_1 c_1 c_2 \tau^{1+\frac{1}{\beta_1}+\frac{1}{\beta_2}}} f_1^{(1)}\left(\frac{s_{**}}{c_1 \tau^{\frac{1}{\beta}_1}}\right) f_1^{(2)}\left(\frac{t-s_{**}}{c_2 \tau^{\frac{1}{\beta}_2}}\right). \tag{16}$$

For τ small enough, (9) implies

$$I_1 \le C_1, \quad I_2 \le C_2 \tau \quad \text{and} \quad I_3 \le C_3 \tau^2,$$

where C_1, C_2 and C_3 are constants not depending on τ. These estimates and continuity of convolution imply boundedness of $f_{W_t}(\tau)$ for any $\tau < \infty$. Now suppose that τ is large enough. Then taking (8) into account in (15) and (16), it is not hard to verify that

$$I_1 \le \frac{C_3}{\tau^{\frac{1-2\beta_2}{2(1-\beta_2)}}} \exp\left(-k_1 \tau^{\frac{1}{1-\beta_2}}\right), \tag{17}$$

and

$$I_2 \le \frac{C_4}{\tau^{1-\frac{\beta_1}{2(1-\beta_1)}-\frac{\beta_2}{2(1-\beta_2)}}} \exp\left(-k_2(\tau^{\frac{1}{1-\beta_1}} + \tau^{\frac{1}{1-\beta_2}})\right), \tag{18}$$

where C_3, C_4 and k_1, k_2 are positive constants not depending on τ. Selecting $\beta = \beta_1 = \min(\beta_1, \beta_2)$, $C = \max(C_3, C_4)$, and $k = \min(k_1, 2k_2) - \varepsilon$, where $\varepsilon \in (0, \min(k_1, 2k_2))$, yields (11). □

REMARK **2.1**. Estimate (11) can be extended to processes W_t which are inverses of stochastic processes of the form $D_t = \sum_{k=1}^N c_k D_t^{(k)}$, where $D_t^{(k)}, k = 1, \ldots, N$, are independent stable subordinators of respective indices $\beta_k \in (0,1)$ and c_k are positive constants, or of arbitrary mixtures of stable subordinators with a mixing measure μ with *supp* $\mu \subset (0,1)$.

Further properties of the density of W_t are listed below (Lemmas **2.2**, **2.3**) in the more general case when W_t represents the inverse of a process which belongs to the class $\mathcal{S}$ of mixtures of an arbitrary number of stable subordinators with a mixing measure μ (see details in (19; 20)). The density in this case is denoted by $f_t^{\mu}(\tau)$. Note that such mixtures model complex diffusions and other types of stochastic processes with several simultaneous diffusion modes. Their associated FPK type equations are *distributed order differential equations* (see e.g. (23; 28; 43)).

By definition, the Caputo-Djerbashian derivative of order β is given by

$$ {}_{\tau}\mathbf{D}_{*}^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_{\tau}^{t} \frac{g'(s)ds}{(t-s)^{\beta}}, \ 0 < \beta < 1, \tag{19} $$

where $\Gamma(\cdot)$ stands for Euler's gamma function. We write $\mathbf{D}_{*}^{\beta}$ when $\tau = 0$. Using the fractional integration operator J^{α}, $\alpha > 0$, one can represent $\mathbf{D}_{*}^{\beta}$ in the form $\mathbf{D}_{*}^{\beta} = J^{1-\beta}\frac{d}{dt}$. The distributed order differential operator with the mixing measure μ is

$$ \mathbf{D}_{\mu,t} g(t) = \int_{0}^{1} \mathbf{D}_{*}^{\beta} g(t) d\mu(\beta). $$

An equivalent but slightly different representation is also possible through the Riemann-Liouville derivative

$$ \mathbf{D}_{RL}^{\beta} g(t) = \frac{d}{dt} J^{1-\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_{0}^{t} \frac{g(s)ds}{(t-s)^{\beta}}, \tag{20} $$

see e.g. (12). The next functions $\mathcal{K}_{\mu}(t)$ and $\Phi_{\mu}(s)$ are important for our further analysis:

$$ \mathcal{K}_{\mu}(t) = \int_{0}^{1} \frac{t^{-\beta}}{\Gamma(1-\beta)} d\mu(\beta), \ t > 0, \tag{21} $$

$$ \Phi_{\mu}(s) = \int_{0}^{1} s^{\beta} d\mu(\beta), \ s > 0. \tag{22} $$

Using the Laplace transform formula for fractional derivatives,

$$ \widetilde{[\mathbf{D}_{*}^{\beta} g]}(s) = s^{\beta}\tilde{g}(s) - s^{\beta-1} g(0+) \tag{23} $$

where $\tilde{g}(s) \equiv \mathcal{L}[g](s) = \int_0^{\infty} g(t)e^{-st}dt$, the Laplace transform of g, one can easily show that

$$ \widetilde{[\mathbf{D}_{\mu,t} g]}(s) = \Phi_{\mu}(s)\tilde{g}(s) - g(0+)\frac{\Phi_{\mu}(s)}{s}, \ s > 0. \tag{24} $$

Since $\widetilde{[t^{-\beta}]}(s) = \Gamma(1-\beta)s^{\beta-1}$ one can also verify the validity of the relation

$$ \widetilde{\mathcal{K}_{\mu}}(s) = \frac{\Phi_{\mu}(s)}{s}, \ s > 0. \tag{25} $$

LEMMA **2.2**. *The density function $f_t^{\mu}(\tau)$ possesses the following properties:*

(a) $\lim_{t \to +0} f_t^{\mu}(\tau) = \delta_0(\tau), \ \tau \geq 0;$
(b) $\lim_{\tau \to +0} f_t^{\mu}(\tau) = \mathcal{K}_{\mu}(t), \ t > 0;$
(c) $\lim_{\tau \to \infty} f_t^{\mu}(\tau) = 0, \ t \geq 0;$
(d) $\mathcal{L}_{t \to s}[f_t^{\mu}(\tau)](s) = \frac{\Phi_{\mu}(s)}{s} e^{-\tau \Phi_{\mu}(s)}, \ s > 0, \ \tau \geq 0.$

The proof of this lemma can be found in (20). It follows from part (d) of Lemma **2.2** that

$$-\frac{\partial}{\partial\tau}\widetilde{f_t^{\mu}(\tau)}(s)=\frac{\Phi_{\mu}^{2}(s)}{s}e^{-\tau\Phi_{\mu}(s)},\ s>0, \tag{26}$$

which is used in the proof of the following lemma.

LEMMA **2.3**. *The function $f_t^{\mu}(\tau)$ satisfies for each $t>0$ the equation*

$$\mathbf{D}_{\mu,t}f_t^{\mu}(\tau)=-\frac{\partial}{\partial\tau}f_t^{\mu}(\tau)-\delta_0(\tau)\mathcal{K}_{\mu}(t), \tag{27}$$

in the sense of distributions.

The proof follows by taking the Laplace transform of (27) and using formulas (24),(26), and parts (a) and (b) of Lemma **2.2**.

3. Time-changed Lévy driving process

Now we discuss the question: What stochastic differential equation is associated with the Cauchy problem for a distributed fractional order FPK type equation

$$\mathbf{D}_{\mu}v(t)=\mathcal{A}v(t),\ t>0,\ v(0)=\varphi, \tag{28}$$

where $\mathcal{A}$ is the operator in equation (7), and φ is a density function? In the description of this relationship and for its applications, the driving process plays a crucial role.

First we introduce a class of stochastic processes defined through the measure μ used in the definition of the left hand side of equation (28). Namely, let $\mathcal{S}$ be the class of strictly increasing semimartingales V_t whose Laplace transforms take the form $\exp\left(\int_0^1 s^{\beta}d\mu(\beta)\right),\ s\geq 0$. This class obviously contains stable subordinators (the case when μ is the Dirac delta function with mass on $\beta\in(0,1)$) and all mixtures of finitely many independent stable subordinators (the case when μ is a linear combination of Dirac delta functions with masses on $\beta_j\in(0,1)$). By construction, $V_0=0$ a.s., and V_t can be considered as a mixture of independent stable subordinators with mixing measure μ. Thus, the role of μ in the fractional FPK equation (28) is to indicate how stable subordinators are mixed. For the process $V_t\in\mathcal{S}$ we use the notation $V_t=D_t^{\mu}$ to indicate its relation to the mixing measure μ.

The following theorem on abstract semigroups defined on a Banach space is useful to answer the question posed above. Theorems in this section are proved in the paper (19).

THEOREM **3.1**. *Assume that $D_t^\mu \in \mathcal{S}$ where μ is a positive finite measure with supp $\mu \subset (0,1)$ and let W_t^μ be the inverse process to D_t^μ. Then the vector-function $v(t) = \int_0^\infty f_{W_t^\mu}(\tau) T_\tau \varphi \, d\tau$, where T_t form a strongly continuous semigroup with the infinitesimal generator $\mathcal{A}$ and $\varphi \in Dom(\mathcal{A})$, exists and satisfies the abstract Cauchy problem (28).*

Consider the SDE driven by a semimartingale formed with the help of the Lévy process which is the driving process of SDE (6) and the time-change process W_t^μ, which is the inverse to $D_t^\mu \in \mathcal{S}$:

$$X_t = x + \int_0^t b(X_{s-}) dW_s^\mu + \int_0^t \sigma(X_{s-}) dB_{W_s^\mu} + \int_0^t \int_{R^n \setminus \{0\}} G(Y_{s-}, w) N(dW_s^\mu, dw). \tag{29}$$

THEOREM **3.2**. *Assume that $D_t^\mu \in \mathcal{S}$, where μ is a positive finite mixing measure with supp $\mu \subset (0,1)$ and let W_t^μ be its inverse. Suppose that a stochastic process Y_τ satisfies SDE (6) and let $X_t = Y_{W_t^\mu}$. Then:*
1) X_t satisfies SDE (29);
2) if Y_τ is independent of W_t^μ, then the function $u(t,x) = E[\varphi(X_t)|X_0 = x]$ satisfies the Cauchy problem (28) with operator $\mathcal{A}$ in (7).

REMARK **3.1**. This theorem provides a complete answer to the above question in the case when the time-change process W_t^μ is independent of the driving process L_t of SDE (6).

In the particular case of the mixture of a finite number of stable subordinators, Theorem **3.2** is formulated in the following form.

THEOREM **3.3**. *Let $D_t^{(k)}$, $k = 1, \dots, N$ be independent stable subordinators of respective indices $\beta_k \in (0,1)$. Define $D_t = \sum_{k=1}^N c_k D_t^{(k)}$, with positive constants c_k, and let E_t be its inverse. Suppose that a stochastic process Y_τ satisfies the SDE (6) driven by a Lévy process where the continuous mappings b, σ, G, are bounded. Let $X_t = Y_{E_t}$. Then:*
1) X_t satisfies SDE (29) driven by the time-changed Lévy process;
2) if Y_τ is independent of E_t, then the function $u(t,x) = E[\varphi(X_t)|X_0 = x]$ satisfies the following Cauchy problem

$$\sum_{k=1}^N C_k \mathbf{D}_*^{\beta_k} u(t,x) = \mathcal{A} u(t,x), t > 0, \tag{30}$$

$$u(0,x) = \varphi(x),\ x \in \mathbb{R}^n, \tag{31}$$

where $\varphi \in C_0^2(\mathbb{R}^n)$, $C_k = c_k^{\beta_k}$, $k = 1, \ldots, N$, *and the pseudo-differential operator* $\mathcal{A}$ *is given in (7).*

Remark **3.2**. The particular case of Theorem **3.3** with one stable subordinator D_t and zero Lévy measure ν was studied recently in paper (26) and applied to the particle tracking problem. Theorem **3.1** can also be used to establish a connection between SDEs in a bounded domain with absorption or reflection conditions at the boundary and their associated FPK type equations with initial and boundary value conditions.

Additionally, Theorem **3.3** allows a fractional generalization of the celebrated Feynman-Kac formula. Recall, that if the right hand side of equation (7) is of the form $\mathcal{A}u - q(x)u$, where $q(x) \geq 0$ is continuous, then the semigroup T_t in equation (4) takes the form

$$(T_t^q \varphi)(x) = \mathbb{E}\left[\exp\left(-\int_0^t q(Y_s)ds\right)\varphi(Y_t)\Big|Y_0 = x\right], \tag{32}$$

which is called the Feynman-Kac formula. Replacing Y_t by $X_t = Y_{W_t}$ in (32), we obtain a fractional Feynman-Kac formula corresponding to equation (28) with the right hand side $\mathcal{A}u - q(x)u$. However, the family T_t in the fractional Feynman-Kac formula does not possess the semigroup property.

4. An abstract theorem and its applications

In this section we prove an abstract theorem on a connection between fractional FPK type equations and their associated SDEs, thereby generalizing theorems in Section **3**. This theorem can be applied to establish such a connection for other important driving processes, including fractional Brownian motion, linear fractional stable motion, as well as driving processes of SDEs in bounded domains, etc. Integrals in this section for vector-function integrands should be understood in the sense of Bochner. Assume that the operators A and B of this section are linear closed operators and the spectrum of A is located on the left side of the imaginary axis of the complex plane.

Theorem **4.1**. *Let a vector-function* $u(t) \in C^{(1)}(0,\infty)$ *be a solution to the abstract Cauchy problem*

$$u'(t) = Bu(t) + \frac{(\gamma+1)t^\gamma}{2}Au(t),\ \ t > 0,\ \ u(0) = \varphi, \tag{33}$$

where $\gamma \in (-1,1)$. *Suppose that* $D_t^\mu \in \mathcal{S}$ *and* W_t^μ *is the inverse process to* D_t^μ. *Then the vector-function* $v(t) = \int_0^\infty f_{W_t^\mu}(\tau)u(\tau)\,d\tau$ *satisfies the* *abstract Cauchy problem*

$$\mathbf{D}_\mu v(t) = Bv(t) + \frac{\gamma+1}{2} G^\mu_{\gamma,t} Av(t),\ t>0,\ v(0)=\varphi, \tag{34}$$

where the operator $G^\mu_{\gamma,t}$ *is defined as*

$$G^\mu_{\gamma,t} v(t,x) = \Phi_\mu(t) * \mathcal{L}^{-1}_{s\to t}\left[\frac{\Gamma(\gamma+1)}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{m_\mu(z)\tilde{v}(z,x)}{(\rho(s)-\rho(z))^{\gamma+1}}dz\right](t), \tag{35}$$

where $*$ *denotes the usual convolution of two functions,* $0<C<s$,

$$\rho(z) = \int_0^1 e^{\beta \mathrm{Ln}(z)} d\mu(\beta), \quad m_\mu(z) = \frac{\int_0^1 \beta z^\beta d\mu(\beta)}{\rho(z)},$$

and $\Phi_\mu(t)$ *is defined in (22). The family of operators* $\{G_{\gamma,t},\ -1<\gamma<1\}$ *possesses the semigroup property.*

P r o o f. In order to derive equation (34), we first compute

$$\mathbf{D}_{\mu,t} v(t) = \int_0^\infty D_{\mu,t} f^\mu_t(\tau) u(\tau) d\tau.$$

Here $\mathbf{D}_{\mu,t}$ is the same operator as $\mathbf{D}_\mu$, but we emphasize that it is acting on the variable t under the integral. We note also that changing the order of $\mathbf{D}_{\mu,t}$ and the integral is valid, since estimate (11) extends to a mixture having mixing measure μ with $supp\, \mu \subset (0,1)$. Due to Lemma **2.3**, we have

$$\mathbf{D}_{\mu,t} v(t,x) = -\int_0^\infty \frac{\partial f^\mu_t(\tau)}{\partial \tau} u(\tau) d\tau - \mathcal{K}_\mu(t)\int_0^\infty \delta_0(\tau) u(\tau) d\tau. \tag{36}$$

Integration by parts in the first integral in (36) gives $\int_0^\infty f^\mu_t(\tau)\frac{\partial u(\tau)}{\partial \tau} d\tau$ and the two limit terms $\lim_{\tau\to\infty} f^\mu_t(\tau)u(\tau)$ and $\lim_{\tau\to 0} f^\mu_t(\tau)u(\tau)$. The first limit is zero due to part (c) of Lemma **2.2** and the condition on the spectrum of the operator A. The second limit has the same value as the second integral on the right side of (36), but with the opposite sign, due to part (b) of Lemma **2.2**. Hence,

$$\mathbf{D}_{\mu,t} v(t,x) = \int_0^\infty f^\mu_t(\tau)\frac{\partial}{\partial \tau} u(\tau,x) d\tau.$$

Due to equation (33), we obtain

$$\begin{aligned}\mathbf{D}^\beta_{*,t} v(t) &= Bv(t) + \frac{\gamma+1}{2} A\int_0^\infty f_t(\tau)\tau^\gamma u(\tau) d\tau \\ &= Bv(t) + \frac{\gamma+1}{2} AG_{\gamma,t} v(t),\end{aligned}$$

where

$$G_{\gamma,t} v(t) = \int_0^\infty f_t(\tau)\tau^\gamma u(\tau) d\tau. \tag{37}$$

The initial condition for $v(t)$ immediately follows from part (a) of Lemma **2.2**. Representation (35) for the operator $G_{\gamma,t}$ is proved in paper (20), as well as the fact that the family $\{G_{\gamma,t},\ -1 < \gamma < 1\}$ possesses the semigroup property. Namely, the latter means that for any $\gamma, \delta \in (-1,1), \gamma + \delta \in (-1,1)$, one has $G^{\mu}_{\gamma,t} \circ G^{\mu}_{\delta,t} = G^{\mu}_{\gamma+\delta,t} = G^{\mu}_{\delta,t} \circ G^{\mu}_{\gamma,t}$, where "$\circ$" denotes the composition of two operators. □

REMARK **4.1**. Theorem **4.1** generalizes Theorem **3.1**. In fact, if $B = 0$ and $\gamma = 0$, then $G^{\mu}_{0,t} \equiv I$, where I is the identity operator, so Theorem **4.1** represents Theorem **3.1** in a slightly disguised formulation. Notice that Theorem **4.1** does not use the semigroup structure. The Cauchy problem (33) is important from the applications point of view too. Indeed, if $B = 0$, $\gamma = 2H - 1$ and $A = \Delta$, then (33) is the FPK equation associated with the fractional Brownian motion with the Hurst parameter $H \in (0,1)$. Two other applications are discussed below (see Theorems **4.2** and **4.3**). In more general settings, B represents a drift term. See (20) for the case when B and A are differential operators.

Theorem **4.2** below is an application of Theorem **4.1** to the symmetric version of linear fractional stable motions (LFSM) (see (30) for the definition of LFSM in the one-dimensional case). Let $L_{\alpha,H}$, $0 < \alpha < 2$, $0 < H < 1$, be a LFSM. Then its density solves the following equation

$$\frac{\partial u(t,x)}{\partial t} = \alpha H t^{\alpha H - 1} \mathbb{D}_0^{\alpha} u(t,x),\ t > 0,\ x \in \mathbb{R}^d, \tag{38}$$

where $\mathbb{D}_0^{\alpha} = -(-\Delta)^{\alpha/2}$. This operator can also be represented as a hypersingular integral

$$\mathbb{D}_0^{\alpha} h(x) = b(\alpha) \int_{\mathbb{R}^d_y} \frac{\Delta_y^2 h(x)}{|y|^{d+\alpha}} dy, \tag{39}$$

where Δ_y^2 is the second order centered finite difference in the y direction and $b(\alpha)$ is the normalizing constant

$$b(\alpha) = \frac{\alpha \Gamma(\frac{\alpha}{2}) \Gamma(\frac{d+\alpha}{2}) \sin \frac{\alpha\pi}{2}}{2^{2-\alpha} \pi^{1+d/2}}. \tag{40}$$

Denoting $\gamma = \alpha H - 1$ and $A = \mathbb{D}_0^{\alpha}$, one can rewrite equation (38) in the form (33) with $B = 0$ and the initial condition

$$u(0,x) = \varphi(x),\ x \in \mathbb{R}^d. \tag{41}$$

The operator $A = \mathbb{D}_0^{\alpha}$ is a pseudodifferential operator with the symbol $\sigma_A(\xi) = -|\xi|^{\alpha}$, so the condition on the spectrum of operator A required for Theorem **4.1** is fulfilled. Therefore, for this particular case Theorem **4.1** implies the following theorem.

Theorem **4.2**. *Let $u(t,x)$ be a solution to the Cauchy problem (38), (41). Let $f_t^{\mu}(\tau)$ be the density function of the process W_t^{μ}. Then*

$$v(t,x) = \int_0^{\infty} f_t^{\mu}(\tau) u(\tau,x) d\tau$$

satisfies the following initial value problem for a fractional distributed order differential equation

$$\mathbf{D}_{\mu} v(t,x) = \alpha H t^{\alpha H - 1} G^{\mu}_{\gamma,t} \mathbb{D}_0^{\alpha} v(t,x),\ t > 0,\ x \in \mathbb{R}^d,$$

$$v(0,x) = \varphi(x),\ \ x \in \mathbb{R}^d,$$

where $G^{\mu}_{\gamma,t}$, $\gamma = \alpha H - 1$, is defined in (35).

Theorem **4.1** can also be applied to fractional FPK equations in the infinite dimensional case. Below we consider only the simplest case. Let H be an infinite dimensional separable Hilbert space and Q be a positive definite trace operator on H. In this section we suppose $\mathcal{W}_t$ is the infinite dimensional Wiener process associated with the operator Q. Then the corresponding FPK equation has the form (see (7))

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} Tr[QD^2 u(t,x)],\ t > 0,\ x \in H, \tag{42}$$

where Tr stands for the trace, and D^2 is the second order Fréchet derivative. Note that equation (42) is the FPK equation associated with the simplest Itô SDE $dX_t = d\mathcal{W}_t$. Denote $A(\cdot) = \frac{1}{2} Tr[QD^2 \cdot]$ with $Dom(A) = UC_b^2(H)$, the space of functions $u : H \to R$ such that $D^2 u$ is uniformly continuous and bounded. Then applying Theorem **4.1** with $B = 0$ and $\gamma = 0$, one obtains the following theorem.

Theorem **4.3**. *Let $u(t,x)$ be a strong solution to equation (42) with the initial condition $u(0,x) = \varphi(x)$, $\varphi \in UC_b(H)$. Let $f_t^{\mu}(\tau)$ be the density function of the process W_t^{μ}. Then $v(t,x) = \int_0^{\infty} f_t^{\mu}(\tau) u(\tau,x) d\tau$ is a strong solution to the following initial value problem for the infinite dimensional time-fractional distributed order differential equation*

$$D_{\mu} v(t,x) = \frac{1}{2} Tr[QD^2 v(t,x)],\ t > 0,\ x \in H, \tag{43}$$

$$v(0,x) = \varphi(x),\ \ x \in H. \tag{44}$$

Remark **4.2**. The stochastic process associated with the Cauchy problem for the infinite dimensional fractional FPK type equation (43)-(44) is, obviously, $X_t = \mathcal{W}_{W_t^{\mu}}$ with $\mathcal{W}_t$ and W_t^{μ} independent. The model case considered in Theorem **4.3** can be generalized for general uniformly elliptic operators in equation (43).

5. Driving processes and other related issues

Driving processes of the SDEs associated with fractional FPK equations appear to be independent time-changes of basic processes like Brownian motion, Lévy processes, fractional Brownian motions, etc. Donsker's theorem states that, in the cádlág space $D([0,\infty),\mathbb{R}^d)$ with the Skorohod topology (5; 21), d-dimensional Brownian motion is the weak limit of the scaled sums $\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor} X_j$ where $\{X_j\}$ is a sequence of independent and identically distributed (i.i.d.) mean zero, variance one random vectors $\{X_j\}$. Alternatively, the same kind of result holds in $C([0,\infty),\mathbb{R}^d)$ with the uniform topology, if the path of the nth term is made continuous by linearly interpolating the normalized partial sums. These facts are important from the approximation point of view since an approximation of the basic driving process B_t yields, under some conditions, an approximation of other processes X_t driven by B_t.

Natural approximants of the time-changed processes B_{W_t}, L_{W_t}, etc., where W_t is the inverse to a stable subordinator, are continuous time random walks (CTRWs). A CTRW is a random walk subordinated to a renewal process. More precisely, take two independent sequences, $\{Y_i \in R^d : i \geq 1\}$ which are i.i.d. random vectors and $\{\tau_i : i \geq 1\}$ which are i.i.d. positive real-valued random variables. Then $S_n = Y_1 + \cdots + Y_n$ is the position after n jumps and $T_n = \tau_1 + \cdots + \tau_n$ is the time of the nth jump. Assume that $S_0 = 0$ and $T_0 = 0$. The stochastic process

$$X_t = S_{N_t} = \sum_{i=1}^{N_t} Y_i,$$

where $N_t \equiv \max\{n \geq 0 : T_n \leq t\}$, is called a *continuous time random walk.*

CTRWs, invented by Montrol and Weiss (36) in 1965, have rich applications in many applied sciences and the literature on CTRWs is still increasing at a rapid rate. See papers (33; 34) and references therein for a discussion of the history of development of the CTRW theory and its connections to fractional differential equations and other relevant fields. There are various approaches to the study of weak CTRW limits, depending on the topology and methods used for the proof of convergence. The methods used include master equations, constructive random walk approximations, and use of abstract continuous mapping theorems. Random walk approximations of stochastic processes associated to space-, time-, or space-time-fractional FPK type equations are constructed in (1; 9; 13; 14; 15; 16; 17; 44; 46).

Papers (30; 31; 32) establish CTRW limit theorems in the M_1-topology (which is weaker than the Skorohod topology) on the space $D([0,\infty),\mathbb{R}^d)$.

In this section we are interested in random walk approximations of stochastic processes associated with the fractional FPK type equations of the form

$$\mathbf{D}_*^{\beta}u(t,x) = \int_0^2 \mathbb{D}_0^{\alpha}u(t,x)d\rho,\ t>0,\ x\in\mathbb{R}^d, \tag{45}$$

where ρ is a finite mixing measure with $supp\,\rho \subset (0,2]$ and D_0^{α} is given by formula (39). Note that the role of the measure ρ is different from the role of the mixing measure μ used in previous sections. The measure ρ specifies a mixture of symmetric α-stable distributions in the stochastic process associated with equation (45), rather than a mixture in a time change.

First consider the case $\beta = 1$, that is the FPK type equation is given in the form

$$\frac{\partial u(t,x)}{\partial t} = \int_0^2 \mathbb{D}_0^{\alpha}u(t,x)d\rho(\alpha). \tag{46}$$

Let $\mathbb{Z}^d$ be the d-dimensional integer lattice and h be a positive real number (mesh size). Introduce for $m=(m_1,...,m_d)\neq(0,...,0)=0\in\mathbb{Z}^d$

$$Q_m(h) = \int_0^2 \frac{b(\alpha)}{(h|m|)^{\alpha}}d\rho \tag{47}$$

where $b(\alpha)$ is given in (40) and let

$$Q_0(h) = \sum_{m\neq 0}\frac{Q_m(h)}{|m|^d}. \tag{48}$$

THEOREM **5.1**. *Fix $t>0$ and let $h>0$, $\tau=t/n$. Let $Y_j\in\mathbb{Z}^d$, $j\geq 1$, be i.i.d. random vectors with the transition probabilities*

$$p_k = \begin{cases} 1-\tau Q_0(h), & \text{if } k=0;\\ \tau\frac{Q_k(h)}{|k|^d}, & \text{if } k\neq 0,\end{cases} \tag{49}$$

where $Q_k(h)$ are defined in (47), (48). Assume that

$$\sigma(\tau,h) = \tau Q_0(h) \leq 1. \tag{50}$$

Then the sequence of random vectors $S_n = hY_1+...+hY_n$ converges in law as $n\to\infty$ to Y_t whose probability density function is the solution to equation (46) with the initial condition $u(0,x)=\delta_0(x)$.

This theorem proved in (46), describes a random walk approximation in the interval $(0,t)$ of a Markovian process Y_t which is the ρ-mixture of symmetric α-stable motions. Note that Y_t itself is not stable if ρ mixes at least two stables with different indices. In this approximation the time step τ and the mesh size h are not independent, they are related through (49). So, $n \to \infty$ implies $h \to 0$, which in turn, due to (50), implies $\tau \to 0$.

If Y_t is a driving process, then the density of a time-changed process $X_t = Y_{W_t}$, where W_t is the inverse to a β-stable subordinator, solves the fractional FPK type equation (45) with the initial condition $u(0, x) = \delta_0(x)$. Since X_t is non-Markovian, an approximating random walk also can not be independent. Therefore, transition probabilities split into two different sets of probabilities:

(1) *non-Markovian transition probabilities,* which express a long non-Markovian memory of the past; and
(2) *Markovian transition probabilities,* which express transition from positions at the previous time instant.

In the particular case when the operator on the right hand side of (45) is the Laplace operator, two different random walk approximations were constructed in papers (15) and (25). In paper (1) a random walk approximant is constructed using non-Markovian transition probabilities suggested in (25). The theorem below provides a random walk approximation of Y_{W_t} with transition probabilities in which the non-Markovian part uses the technique suggested in (15).

Suppose that non-Markovian transition probabilities are given by (see (15))

$$
\begin{aligned}
c_\ell &= (-1)^{\ell+1}\binom{\beta}{\ell} = \left|\binom{\beta}{\ell}\right|, \ \ell = 1, \dots, n, \\
b_n &= \sum_{\ell=0}^{n} (-1)^\ell \binom{\beta}{\ell},
\end{aligned}
\tag{51}
$$

and Markovian transition probabilities $\{p_k\}_{k\in\mathbb{Z}^n}$ are given by

$$
p_k = \begin{cases} c_1 - \tau^\beta Q_0(h), & \text{if } k = 0; \\ \tau^\beta \frac{Q_k(h)}{|k|^d}, & \text{if } k \neq 0. \end{cases}
\tag{52}
$$

Then the probability q_j^{n+1} of sojourn of a particle at $x_j = j$ at time t_{n+1} is

$$
q_j^{n+1} = b_n q_j^0 + \sum_{\ell=1}^{n-1} c_{n-\ell+1} q_j^\ell + \Big(c_1 - \tau^\beta Q_0(h)\Big) q_j^n + \sum_{k\neq 0} p_k q_{j-k}^n.
\tag{53}
$$

THEOREM **5.2**. *Fix $t > 0$ and let $h > 0$, $\tau = t/n$. Let $Y_j \in \mathbb{Z}^d$, $j \geq 1$ be identically distributed random vectors with the non-Markovian* *and Markovian transition probabilities defined in* (51) *and in* (52), *respectively. Assume that*

$$\tau \leq \Big(\frac{\beta}{Q_0(h)}\Big)^{\frac{1}{\beta}}. \tag{54}$$

Then the sequence of random vectors $S_n = hY_1 + ... + hY_n$, *converges as* $n \to \infty$ *in law to* $X_t = Y_{W_t}$ *whose probability density function is the solution to equation (45) with the initial condition* $u(0,x) = \delta_0(x)$.

(1) Theorem **5.2** extends to the case when the left hand side of equation (45) is a time distributed fractional order differential operator with a mixing measure μ.

(2) Condition (54) generalizes the well-known Lax's stability condition arising in the finite-difference method for solution of an initial value problem for the heat equation. Selection of the non-Markovian probabilities as in (1; 25) gives a slightly different stability condition

$$\tau \leq \Big(\frac{2 - 2^{1-\beta}}{\Gamma(2-\beta)Q_0(h)}\Big)^{\frac{1}{\beta}}.$$

This condition as well as (54) coincide with Lax's stability condition if $\beta = 1$ and the operator on the right hand side of (45) is the Laplace operator.

Fractional FPK type equations with variable order functions and their associated stochastic processes have been studied less thoroughly. One interesting phenomenon is that the process may generate internal memory effects quantified as an inhomogeneous term in the equation; for details see (45). Here we demonstrate how such an inhomogeneous term arises in a single change of diffusion regime. Suppose a FPK type equation is given in the form

$$\mathcal{D}_*^{\beta(t)} u(t) = \mathcal{A}u(t),\ t > 0, \tag{55}$$

with the initial condition

$$u(0) = \varphi, \tag{56}$$

where $\mathcal{D}_*^{\beta(t)}$ is defined as an integral operator

$$\mathcal{D}_*^{\beta(t)} f(t) = \int_0^t \mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau)\frac{df(\tau)}{d\tau}d\tau. \tag{57}$$

with the kernel

$$\mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) = \frac{1}{\Gamma(1-\beta(\mu t + \nu\tau))(t-\tau)^{\beta(\mu t+\nu\tau)}}, \quad 0 < \tau < t\,. \tag{58}$$

The parameters ν and μ belong to the following *causality parallelogram*: $\Pi = \left\{(\mu,\nu) \in \mathbb{R}^2 : 0 \leq \mu \leq 1, -1 \leq \nu \leq +1, 0 \leq \mu+\nu \leq 1\right\}$.

Assume the function $\beta(t)$ takes only two values β_1 if $0 < t < T$ and β_2 if $t > T$. In other words, the diffusion regime changes at time $t = T$ from a sub-diffusive regime β_1 to a sub-diffusive regime β_2. Since the first regime is sub-diffusive, a non-Markovian memory occurs which results in the actual change appearing at time $T_* \geq T$. Here T_* depends on the parameters μ and ν; see (45) where the value of T_* is found. For simplicity, suppose $\nu = 0$ and $\mu = 1$. In this case $T_* = T$, and we assume the following continuity condition at the change of regime time $t = T$:

$$u(T) = u(T - 0). \tag{59}$$

For $0 < t < T$, equation (55) is a fractional equation of order β_1 so a solution to the Cauchy problem (55)-(56) can be found by standard methods (see, e.g. (12; 10)). If $t > T$, then one has

$$\mathcal{D}_*^{\beta(t)} u(t) = \int_0^T \mathcal{K}_{1,0}^{\beta_1}(t,\tau) \frac{du(\tau)}{d\tau} d\tau + \int_T^t \mathcal{K}_{1,0}^{\beta_2}(t,\tau) \frac{du(\tau)}{d\tau} d\tau.$$

Hence, using (19), equation (55) takes the form

$${}_T\mathbf{D}_{*,t}^{\beta_2} u(t) = \mathcal{A}u(t) + h(t),\ t > T, \tag{60}$$

with the initial condition (59). Equation (60) is no longer homogeneous, due to the nonhomogeneous term $h(t) = -\int_0^T \mathcal{K}_{1,0}^{\beta_1}(t,\tau) \frac{du(\tau)}{d\tau} d\tau$.

The fractional Duhamel principle reduces initial value problems for inhomogeneous fractional order differential equations to initial value problems with corresponding homogeneous equations. The theorem below is the fractional Duhamel principle in the simplest case with a single time-fractional derivative. For the general case the reader is referred to (47).

THEOREM **5.3**. *Let $0 < \alpha < 1$ and $V(t,\tau)$, $0 \leq \tau \leq t$, be a solution of the Cauchy problem for the homogeneous equation*

$${}_\tau\mathbf{D}_*^{\alpha} V(t,\tau) = \mathcal{A}V(t,\tau), \quad t > \tau,$$

$$V(\tau,\tau) = \mathbf{D}_{RL}^{1-\alpha} f(\tau),$$

where $\mathbf{D}_{RL}^{1-\alpha}$ is the Riemann-Liuoville fractional derivative of order $1-\alpha$ defined in (20) and $f(t)$ is a continuous function. Then the Duhamel integral

$$v(t) = \int_0^t V(t,\tau) d\tau$$

solves the inhomogeneous Cauchy problem

$$\mathbf{D}_*^{\alpha} v(t) = \mathcal{A}v(t) + f(t),\ t > 0,$$

with the initial condition $v(0) = 0$.

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Received: August 8, 2010