VERSITA

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 14, NUMBER 1 (2011) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

SURVEY PAPER

MAXIMUM PRINCIPLE AND ITS APPLICATION FOR THE TIME-FRACTIONAL DIFFUSION EQUATIONS

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Abstract

Dedicated to Professor Rudolf Gorenflo
on the occasion of his 80th anniversary

In the paper, maximum principle for the generalized time-fractional diffusion equations including the multi-term diffusion equation and the diffusion equation of distributed order is formulated and discussed. In these equations, the time-fractional derivative is defined in the Caputo sense. In contrast to the Riemann-Liouville fractional derivative, the Caputo fractional derivative is shown to possess a suitable generalization of the extremum principle well-known for ordinary derivative. As an application, the maximum principle is used to get some a priori estimates for solutions of initial-boundary-value problems for the generalized time-fractional diffusion equations and then to prove uniqueness of their solutions.

MSC 2010: 26A33, 33E12, 35B45, 35B50, 35K99, 45K05

Key Words and Phrases: time-fractional diffusion equation; time-fractional multi-term diffusion equation; time-fractional diffusion equation of distributed order; extremum principle; Caputo fractional derivative; generalized Riemann-Liouville fractional derivative; initial-boundary-value problems; maximum principle; uniqueness results

pp. 110–124 , DOI: 10.2478/s13540-011-0008-6

Springer

1. Introduction

Mathematical modelling of many physical, technical, biological, chemical, economical etc. processes with ordinary and partial differential equations experiences in some sense a rebirth during the last few decades: practically all important and useful models were "fractionalized", i.e. replaced with models containing the Fractional Calculus operators (see e.g. [1]-[15], [20]-[35] and references there). The recent book [35] is completely devoted to different applications of the fractional differential equations in physics, chemistry, technique, astrophysics, etc. and contains several dozens of interesting case studies. Unfortunately, in many cases a "fractionalization" of the models appears to be just a formal procedure. In applications, two formal approaches are mainly used for formulating models with the fractional derivatives. In the framework of the first approach, integer order derivatives are replaced by the fractional derivatives in differential equations describing a process. Within the second approach, variation principles such as Hamilton's principle are used as a starting point for deriving equations of a process. In this case, fractionalization of the classical case is achieved by replacing some (or all) integer order derivatives by the fractional derivatives in the Lagrange function. Of course, by applying the models obtained by means of these formal fractionalization procedures one has to check if they can describe the underlying processes better or at least not worse compared to the conventional models. As a rule, this can be done by analysing the experimental data and comparing them with the results of the numerical simulations obtained with the help of the fractional models.

Still, some fractional models can be deduced either directly from the first principles (e.g. the Abel integral equation for the tautochrone problem or for determination of the thermal flux through the boundary of a blast-furnace wall) or from the micro models like the so called continuous time random walk models. Especially this last approach was applied very successfully within the last years to deduce the fractional models for so called anomalous diffusion processes, where conventional diffusion equation does not work. In some sense, fractional models of the anomalous diffusion (see e.g. [1], [4], [6], [8], [20], [26], [27] and references there) along with fractional models in the linear viscoelasticity (see e.g. [5], [22], [23], [25], [32], [33] and references there) build a showcase of the applications of Fractional Calculus.

In this paper, the generalized time-fractional diffusion equation along with some of its important generalizations (multi-term equation and equation of distributed order) is considered from the mathematical viewpoint. This equation corresponds to the continuous time random walk model where the characteristic waiting time elapsing between two successive jumps diverge, but the jump length variance remains finite and is proportional to t^α. For the detailed description of these models see e.g. [6], [8], [26], [27] and references there.

In real life anomalous diffusion processes, the exponent α of the mean square displacement proportional to t^α often does not remain constant and changes, say, in the interval from 0 to 1, from 1 to 2 or even from 0 to 2. To adequately describe these phenomena with the fractional models, several approaches were suggested in the literature. One of them introduces the fractional derivatives of the variable order, i.e., the derivatives with the order that can change with the time or/and depending on the spatial coordinates (for the definitions and applications see e.g. [3], [15], [29]). Another, more simple, method is to model the variable exponent with a linear combination of the power functions in the form $\sum_{k=1}^m \lambda_k t^{\alpha_k},\ 0 \le \lambda_k,\ 0 \le \alpha_k < 2$. Following this line, so called multi-term time-fractional diffusion equation appears to be a suitable model. A detailed discussion of this equation along with many references to the related papers can be found in [16].

On the other hand, recently the sub-diffusion processes with the mean square displacement with a logarithmic growth have been introduced (see e.g. [2], [28], [34] and references there). One of the approaches for modelling of such processes is to employ time-fractional diffusion equations of distributed order (see e.g. [18] for the mathematical analysis of this equation and further references). A derivative of the distributed order is introduced as a mean value of the fractional derivatives with the orders from an interval (say, [0, 1]) weighted with a non-negative weight function $\omega(\alpha)$. One important particular case of the time-fractional diffusion equation of distributed order is the multi-term time-fractional diffusion equation. In this case the weight function is taken in form of a finite linear combination of the Dirac δ-functions with the positive weight coefficients.

For the numerous references to the literature dealing with different methods and techniques for the analysis of the partial fractional differential equations, especially those considered in this paper, we refer the interested reader to the papers [16]-[19].

The rest of this paper is organized as follows. In the second section, the notions, definitions and problems formulations we deal with in the further discussions are introduced. The third section is devoted to the discussion of the extremum principle for the generalized Riemann-Liouville fractional derivatives. In particular, we show that a suitable extremum principle is valid only for the Caputo fractional derivative. In the fourth section, the maximum principles for the generalized time-fractional diffusion equation and its generalizations - the multi-term time-fractional diffusion equation and the time-fractional diffusion equation of distributed order - are presented. Finally, in the last section the uniqueness of the solution of the initial-boundary-value problem for the corresponding equations is shown. This solution - if it exists - continuously depends on the data given in the problem.

2. Definitions and problem formulation

In this paper, first the generalized time-fractional diffusion equation is considered. For an unknown function $u = u(x,t)$ it has the form

$$(D_t^\alpha u)(t) = L_x(u) + F(x,t), \tag{1}$$

$$0 < \alpha \leq 1, (x,t) \in \Omega_T := G \times (0,T), \quad G \subset R^n,$$

where the operator L_x acts with respect to the spatial variables x according to the formula

$$L_x(u) := \mathrm{div}(p(x)\,\mathrm{grad}\,u) - q(x)u, \tag{2}$$

$$p \in C^1(\bar{G}), q \in C(\bar{G}),\ 0 < p(x), 0 \leq q(x), x \in \bar{G}, \tag{3}$$

the fractional derivative

$$(D_t^\alpha f)(t) := (I^{1-\alpha} f')(t),\ 0 < \alpha \leq 1, \tag{4}$$

with respect to the time variable t is defined in the Caputo sense, whereas I^α is the Riemann-Liouville fractional integral

$$(I^\alpha f)(t) := \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad 0 < \alpha < 1, \\ f(t), \quad \alpha = 0, \end{array} \right. \tag{5}$$

and the domain G with the boundary S is open and bounded in R^n

The operator L_x is a linear second order elliptic differential operator that can be represented in the form

$$L_x(u) = \sum_{k=1}^n \left(p(x)\frac{\partial^2 u}{\partial x_k^2} + \frac{\partial p}{\partial x_k}\frac{\partial u}{\partial x_k} \right) - q(x)u,$$

or, more shortly, in the form

$$L_x(u) = p(x)\triangle u \,+\, (\mathrm{grad}\,p,\, \mathrm{grad}\,u) - q(x)u, \tag{6}$$

$\triangle$ being the Laplace operator.

If $\alpha = 1$, the equation (1) is a standard linear second-order parabolic PDE. The theory of this equation is well-known, so that the focus in the further discussions will be on the case $0 < \alpha < 1$.

In some practical situations the underlying processes cannot be described by the equation (1), but by its generalization - the multi-term time-fractional diffusion equation that is given by

$$(P_{\alpha,\alpha_1,\ldots,\alpha_m}(D_t)u)(t) = L_x(u) + F(x,t), \tag{7}$$

$$(x,t) \in \Omega_T := G \times (0,T),\ G \subset R^n,$$

where the operator L_x is defined by (2) and

$$P_{\alpha,\alpha_1,\ldots,\alpha_m}(D_t) := D_t^{\alpha} + \sum_{i=1}^{m} \lambda_i \, D_t^{\alpha_i}, \tag{8}$$

$$0 < \alpha_m < \cdots < \alpha_1 < \alpha \le 1,\ 0 \le \lambda_i,\ i = 1, \cdots, m,\ m \in N_0,$$

D_t^{α} being the Caputo fractional derivative (4) of order α.

The next generalization of the equation (1) (and of the equation (7)) that will be discussed in this paper is the time-fractional diffusion equation of the distributed order:

$$(\mathbb{D}_t^{w(\alpha)} u)(t) = L_x(u) + F(x,t), \tag{9}$$

$$(x,t) \in \Omega_T := G \times (0,T),\ G \subset R^n.$$

The fractional derivative $\mathbb{D}_t^{w(\alpha)}$ of distributed order is defined by

$$(\mathbb{D}_t^{w(\alpha)} f)(t) = \int_0^1 (D_t^{\alpha} f)(t) \, w(\alpha) \, d\alpha \tag{10}$$

with the Caputo fractional derivative D_t^{α} defined by (4) and with a continuous non-negative weight function $w : [0,1] \to R$ that is not identically equal to zero on the interval $[0,1]$, such that the conditions

$$0 \le w(\alpha),\ w \not\equiv 0,\ \alpha \in [0,1],\ \int_0^1 w(\alpha) \, d\,\alpha \ = \ W \ > 0 \tag{11}$$

hold true.

Of course, each of the equations (1), (7), (9) has in general case an infinite number of solutions. In the real world situations that are modelled with these equations, certain conditions that describe an initial state of the underlying process and the observations of its visible parts ensure the deterministic character of the corresponding processes. In the paper, the initial-boundary-value problem

$$u\big|_{t=0} = u_0(x),\ x \in \bar{G}, \tag{12}$$

$$u\big|_{S} = v(x,t),\ (x,t) \in S \times [0,T] \tag{13}$$

for the equations (1), (7), (9) is considered.

Definition 1. A solution of the problem (1), (12), (13) (of the problem (7), (12), (13) or of the problem (9), (12), (13), respectively) is called a function $u = u(x,t)$ defined in the domain $\bar{\Omega}_T := \bar{G} \times [0,T]$ that belongs to the space $C_{xt}(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ and satisfies both the equation (1) (the equation (7) or the equation (9), respectively) and the initial and boundary conditions (12)-(13). By $W_t^1((0,T))$ the space of the functions $f \in C^1((0,T])$ such that $f' \in L((0,T))$ is denoted.

If the initial-boundary-value problem (12), (13) for the equation (1) (the equation (7), or the equation (9)) possesses a solution in the sense of Definition 1, then the functions F, u_0 and v given in the problem have to belong to the spaces $C(\Omega_T)$, $C(\bar{G})$ and $C(S \times [0,T])$, respectively. In the further discussions, we always suppose these inclusions to be valid.

3. Extremum principle

In this paper, we mainly deal with the uniqueness of the solution of the initial-boundary-value problem (12), (13) for the equations (1), (7), and (9). The uniqueness will be proved by means of an appropriate maximum principle for these equations. In its turn, the proof of the maximum principle is based on an extremum principle for the Caputo fractional derivative (4). Moreover, we show in this section that the Caputo derivative is the only particular case of the generalized Riemann-Liouville fractional derivative for which an extremum principle is valid.

The generalized Riemann-Liouville fractional derivative

$$(D_t^{\alpha,\beta} f)(t) := (I^{\beta(1-\alpha)} \frac{d}{dt}(I^{(1-\beta)(1-\alpha)} f))(t), \quad t > 0 \tag{14}$$

with the Riemann-Liouville integral I^β defined by (5) was introduced in [10]. Here the order α obeys $0 < \alpha \le 1$ and the type β obeys $0 \le \beta \le 1$. The type β allows to interpolate continuously from the Riemann-Liouville case $D^{\alpha,0} \equiv D_{RL}^\alpha$ to the Caputo case $D^{\alpha,1} \equiv D_t^\alpha$ given by (4), the Riemann-Liouville derivative of order α ($0 < \alpha \le 1$) being defined by

$$(D_{RL}^\alpha f)(t) := \frac{d}{dt}(I^{1-\alpha} f)(t), \ t > 0. \tag{15}$$

For an overview of the properties and applications of the generalized Riemann-Liouville fractional derivative see e.g. [11]. In particular, it was proved in [11] that there does not exist a probabilistic interpretation for the solutions of the fractional diffusion equation of the type (1) with the constant coefficients p and q and with the fractional derivatives of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ instead of the Caputo derivative whenever

$0 \leq \beta < 1$. This statement is in the direct relation to the results presented in this section regarding the absence of an extremum principle for the fractional derivatives of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ whenever $0 \leq \beta < 1$. Because the extremum principle is an essential part of the proof of the maximum principle for the fractional differential equations (1), (7) and (9), our hypothesis is that the maximum principle for the generalized diffusion equations (1), (7) and (9) with the fractional derivatives of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ instead of the Caputo derivative does not hold whenever $0 \leq \beta < 1$. This hypothesis will be considered in details elsewhere.

Whereas the Riemann-Liouville definition is often employed in the mathematical treatises, the Caputo derivative is preferably used in modelling of the applied problems. In this section one more argument for the relevance of the Caputo derivative for modelling of the real life processes is given. This choice of the definition is justified by the fact that from the whole range of the fractional derivatives in the Hilfer definition (14) of the generalized Riemann-Liouville derivative the Caputo derivative is the only one that possesses a suitable generalization of the extremum principle well-known for the ordinary derivative. For other derivatives including the Riemann-Liouville derivative, the extremum principle and possibly the maximum principle do not hold true.

THEOREM 1. *Let a function $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $\tau = t_0,\ t_0 \in (0,T]$. Then the generalized Riemann-Liouville fractional derivative (14) of the function f is non-negative at the point t_0 for any $\alpha,\ 0 < \alpha < 1$, i.e.*

$$(D_t^{\alpha,\beta} f)(t_0) \geq 0,\ 0 < \alpha < 1 \tag{16}$$

if and only if $\beta = 1$, i.e. only in the case of the Caputo derivative (4).

For the proof of the sufficient condition, i.e. that the inequality (16) is valid for the Caputo derivative ($\beta = 1$) we refer the reader to [19].

Now let us prove the necessary condition by the method of contradiction.

Let us consider a family of functions in the form

$$f(t) := -at^2 + bt + c,\ 0 < a,\ 0 < b \leq 2a,\ c \in R \tag{17}$$

on the closed interval $[0,1]$. The conditions on the parameters of the function f ensure the existence of the maximum point $t = b/(2a)$ that belongs to the interval $[0,1]$. The function f is evidently a $C^1([0,1])$-function and thus fulfils all conditions of Theorem 1.

We now evaluate the generalized Riemann-Liouville fractional derivative with $0 < \alpha < 1$ and $0 \le \beta < 1$ of the function f at the maximum point $t = b/(2a)$. In this case, the formula

$$(D_t^{\alpha,\beta} t^\gamma)(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha} \tag{18}$$

is valid under the condition $\gamma + (1-\beta)(1-\alpha) > 0$, that can be proved by the direct calculations (see e.g. [11]). Then we get the formulae

$$(D_t^{\alpha,\beta} f)(t) = -\frac{2at^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{bt^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}$$

for $0 < \alpha < 1$ and $0 \le \beta < 1$ that leads to the formula

$$(D_t^{\alpha,\beta} f)(t)\Big|_{t=\frac{b}{2a}} = \left(\frac{b}{2a}\right)^{-\alpha} \left(\frac{b^2(1-\alpha)}{2a\Gamma(3-\alpha)} + \frac{c}{\Gamma(1-\alpha)}\right).$$

Because the coefficient c can be chosen to be both positive and negative and with an arbitrary absolute value, the fractional derivative $D^{\alpha,\beta} f$ can be evidently made both positive and negative and cannot be hold nonnegative in a maximum point of the function f.

A careful reader must have noticed that the right-hand sides of the last three formulae for the fractional derivative $D^{\alpha,\beta}$ do not depend on the parameter β. In fact this is only true for the values $0 \le \beta < 1$; in the case $\beta = 1$, i.e. for the Caputo fractional derivative, the formula (18) is not valid anymore because of the fact that the Caputo derivative of a constant function is equal to zero:

$$(D_t^{\alpha,1} t^0)(t) \equiv 0, \;\; \forall t \ge 0. \tag{19}$$

It is this formula that makes a difference between the Caputo derivative and all other derivatives with the type β ($0 \le \beta < 1$) including the Riemann-Liouville derivative. In this sense, the Caputo derivative is a "degenerating case" in the whole range of the generalized Riemann-Liouville fractional derivatives with $0 \le \beta \le 1$ just like the conventional derivatives are nothing else then a "singular" but a very useful case in the family of the Caputo fractional derivatives. In the next sections we deal with the equations (1), (7), and (9) containing the Caputo derivatives.

4. Maximum principle and its applications

The maximum principle plays a very essential role in the theory of the partial differential equations of the parabolic and elliptic type (see e.g. [31]). From the physical viewpoint, this principle for an integer order system means that during a diffusion (or a heat conduction) process in a finite body without sources, the maximum concentration (or the maximum temperature) is reached either in a certain point at the start of the process, i.e. at $t = 0$, or in a boundary point for $t > 0$. Similarly, from the physical viewpoint, one can expect the same behavior from the generalized fractional diffusion equations (1), (7), and (9) that are known to be employed by modelling of the so called anomalous diffusion processes. In this section this fact, i.e. the maximum principle for the equations (1), (7), and (9), is presented and applied to get some a priory estimates of the solutions and then to consider their uniqueness.

THEOREM 2. *Let a function $u \in C_{xt}(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ be a solution of the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), or the diffusion equation (9) of the distributed order in the domain Ω_T and $F(x,t) \le 0,\ (x,t) \in \Omega_T$.*

Then either $u(x,t) \le 0,\ \forall (x,t) \in \bar{\Omega}_T$ or the function u attains its positive maximum on the part $S_G^T := (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,

$$u(x,t) \le \max\{0, \max_{(x,t)\in S_G^T} u(x,t)\},\ \forall (x,t) \in \bar{\Omega}_T. \tag{20}$$

For the detailed proofs of this theorem we refer the reader to [17], [19] in the case of the generalized time-fractional diffusion equation (1), to [16] in the case of the generalized multi-term time-fractional diffusion equation (7), and to [18] in the case of the generalized time-fractional diffusion equation (9) of distributed order.

An appropriate minimum principle is valid, too.

THEOREM 3. *Let a function $u \in C_{xt}(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ be a solution of the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), or the diffusion equation (9) of the distributed order in the domain Ω_T and $F(x,t) \ge 0,\ (x,t) \in \Omega_T$. Then either $u(x,t) \ge 0,\ (x,t) \in \bar{\Omega}_T$ or the function u attains its negative minimum on the part $S_G^T = (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,*

$$u(x,t) \ge \min\{0, \min_{(x,t)\in S_G^T} u(x,t)\},\quad \forall (x,t) \in \bar{\Omega}_T. \tag{21}$$

Like in the classical case of the parabolic and elliptic PDEs, one of the important applications of the maximum principle is the possibility to investigate some a priori properties of the solution of the initial-boundary-value problem (12)-(13) for the corresponding equations and to prove the solution uniqueness. In this section we show that the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), and the diffusion equation (9) of the distributed order possess at most one solution each and this solution - if it exists - continuously depends on the data given in the problem.

First, some a priori estimates for the solution norm are established.

THEOREM 4. *Let u be a solution of the initial-boundary-value problem (12)-(13) for the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), or the diffusion equation (9) of the distributed order and F belong to the space $C(\bar{\Omega}_T)$ with the norm $M := \|F\|_{C(\bar{\Omega}_T)}$. Them the following estimate of the solution norm holds true:*

$$\|u\|_{C(\bar{\Omega}_T)} \le \max\{M_0,\ M_1\} + C_\alpha\, M, \tag{22}$$

where

$$M_0 := \|u_0\|_{C(\bar{G})},\ M_1 := \|v\|_{C(S\times[0,T])}, \tag{23}$$

and the constant C_α is given by

$$C_\alpha := \begin{cases} \dfrac{T^\alpha}{\Gamma(1+\alpha)} & \text{for the equation (1)}, \\[2ex] \dfrac{T^\alpha}{(1+\sum_{i=1}^m \lambda_i)\Gamma(1+\alpha)} & \text{for the equation (7)}, \\[2ex] \dfrac{T^\alpha}{W\,\Gamma(1+\alpha)} & \text{for the equation (9)}, \end{cases} \tag{24}$$

W being determined by the weight function w of the distributed order derivative (8) as follows

$$W := \int_0^1 w(\alpha)\, d\alpha\ > 0.$$

P r o o f. To illustrate the proof of the theorem, we present here the reasoning for its simplest case: the equation (1) (for the details, see [17], [19]); other cases can be found in [16] (the equation (7)) and [18] (the equation (9)). To start with the proof, we first introduce an auxiliary function w:

$$w(x,t) := u(x,t) - \frac{M}{\Gamma(1+\alpha)}\, t^\alpha,\ (x,t) \in \bar{\Omega}_T.$$

Evidently, the function w is a solution of the problem (1), (12)-(13) with the functions

$$F_1(x,t) := F(x,t) - M - q(x)\frac{M}{\Gamma(1+\alpha)}\,t^{\alpha},$$
$$v_1(x,t) := v(x,t) - \frac{M}{\Gamma(1+\alpha)}\,t^{\alpha}$$

instead of F and v, respectively. To get the expression for the function F_1, the formula (18) is used. The function F_1 evidently satisfies the condition $F_1(x,t) \le 0,\ (x,t) \in \bar{\Omega}_T$. Then the maximum principle applied to the solution w leads to the estimate

$$w(x,t) \le \max\{M_0, M_1\},\ (x,t) \in \bar{\Omega}_T, \tag{25}$$

where the constants $M_0,\ M_1$ are defined as in (23). For the function u, we get

$$u(x,t) = w(x,t) + \frac{M}{\Gamma(1+\alpha)}\,t^{\alpha} \le \tag{26}$$
$$\max\{M_0, M_1\} + \frac{T^{\alpha}}{\Gamma(1+\alpha)}\,M,\ (x,t) \in \bar{\Omega}_T.$$

The minimum principle from Theorem 3 applied to the auxiliary function

$$w(x,t) := u(x,t) + \frac{M}{\Gamma(1+\alpha)}\,t^{\alpha},\ (x,t) \in \bar{\Omega}_T$$

leads to the estimate ($(x,t) \in \bar{\Omega}_T$)

$$u(x,t) \ge -\max\{M_0, M_1\} - \frac{T^{\alpha}}{\Gamma(1+\alpha)}\,M,$$

that together with the estimate (26) finishes the proof of the theorem for the case of the equation (1). □

The a priori estimates established in the previous theorem can be used to show the uniqueness of the solution of the initial-boundary-value problem for the corresponding generalized time-fractional diffusion equations. This result is given in the next theorem.

Theorem 5. *The initial-boundary-value problem (12)-(13) for the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), or the diffusion equation (9) of the distributed order possesses at most one solution. This solution continuously depends on the data given in the problem in the sense that if*

$$\|F - \tilde{F}\|_{C(\bar{\Omega}_T)} \le \epsilon,$$
$$\|u_0 - \tilde{u}_0\|_{C(\bar{G})} \le \epsilon_0,\ \|v - \tilde{v}\|_{C(S\times[0,T])} \le \epsilon_1,$$

then the estimate

$$\|u - \tilde{u}\|_{C(\bar{\Omega}_T)} \leq \max\{\epsilon_0, \epsilon_1\} + C_\alpha\, \epsilon \tag{27}$$

for the corresponding solutions u and $\tilde{u}$ and with the constant C_α given by (24) holds true.

P r o o f. The proof of this theorem is an easy consequence from the a priori estimates for the solution of the corresponding equations established in Theorem 4. Consider e.g. the uniqueness of solution of the initial-boundary-value problem (12)-(13) for the generalized time-fractional multi-term diffusion equation (7). The homogeneous problem (7), (12)-(13) with zero initial and boundary conditions, i.e. the problem with the data $F \equiv 0$, $u_0 \equiv 0$, and $v \equiv 0$, evidently possesses the trivial solution $u(x,t) \equiv 0$, $(x,t) \in \bar{\Omega}_T$. This solution is unique due to the a priori estimate (22) that says that the solution norm has to be zero in the case of zero initial and boundary conditions. Because the problem under consideration is a linear one, the uniqueness of solution of the problem (7), (12)-(13) in the general case follows from the uniqueness of the homogeneous problem with zero initial and boundary conditions.

Finally, the inequality (27) is obtained from the estimate (22) for the function $u - \tilde{u}$ that is a solution of the initial-boundary-value problem (12)-(13) for the equation (7) with the functions $F - \tilde{F}$, $u_0 - \tilde{u}_0$, and $v - \tilde{v}$ instead of the functions F, u_0, and v, respectively. □

5. Conclusions

The maximum principle enables us to obtain information about solutions of differential equations and the a priori estimates for them without any explicit knowledge of the solutions themselves, and thus is a valuable tool in scientific research. In the paper, the maximum principle for the generalized time-fractional diffusion equation (1), the multi-term diffusion equation (7), and the diffusion equation (9) of the distributed order was discussed and applied for proving the uniqueness of the initial-boundary-value problem (12)-(13) for these equations. Of course, following the lines of the application of the maximum principle for the parabolic and elliptic PDEs (see e.g. the recent book [31]), a lot of other properties of the solutions to the time-fractional partial differential equations can be established. In particular, the maximum principle can be applied for some classes of the non-linear equations of the fractional order, too.

Another important and interesting problem that is still waiting for its solution would be to try to extend the maximum principle to the space- and time-space-fractional partial differential equations. These equations are employed nowadays very actively in modelling of several relevant complex phenomena like the anomalous diffusion in inhomogeneous and porous mediums, Levy processes and Levy flights and the so called fractional kinetics (see e.g. [1], [6], [36] and references there). Like in the time-fractional equations, several different definitions of the space-fractional derivatives are used in these models. A clear understanding what definitions enable the maximum principles that are expected to be fulfilled from the physical viewpoint would help in the attempts towards modelling of the real phenomena with the space- and time-space-fractional partial differential equations very essentially.

All these questions and problems are still open and will be considered elsewhere.

Acknowledgements: Partially supported under Project D ID 02 / 25 / 2009: "Integral Transform Methods, Special Functions and Applications", National Science Fund - Ministry of Education, Youth, and Science, Bulgaria.

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Received: September 30, 2010