Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **30** (2004), 111–134

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

$\mathbb{Z}_2$-GRADED POLYNOMIAL IDENTITIES FOR SUPERALGEBRAS OF BLOCK-TRIANGULAR MATRICES*

Onofrio M. Di Vincenzo

Communicated by V. Drensky

Abstract. We present some results about the $\mathbb{Z}_2$-graded polynomial identities of block-triangular matrix superalgebras $R = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$. In particular, we describe conditions for the T_2-ideal of a such superalgebra to be factorable as the product $T_2(A)T_2(B)$. Moreover, we give formulas for computing the sequence of the graded cocharacters of R in some interesting case.

1. Introduction. In the theory of polynomial identities for associative algebras over a field of characteristic zero a basic role is played by the superalgebras and their $\mathbb{Z}_2$-graded identities (see [19]). For instance, as proved by Kemer, any proper T-ideal of the free algebra, $\mathbb{F}\langle X\rangle$, is the ideal of the polynomial identities satisfied by the Grassmann envelope, $G(A)$, of a suitable finite dimensional superalgebra A. If A is any PI-algebra then we will denote by $T(A)$

2000 *Mathematics Subject Classification:* Primary 16R50, Secondary 16W55.

Key words: Graded cocharacter sequence, triangular matrices, superalgebra, polynomial identity.

*Partially supported by MURST COFIN 2003 and Università di Bari.

the T-ideal of all ordinary polynomial identities of A. Since in characteristic zero any T-ideal is generated by multilinear polynomials, then it is enough to study the vector space, $V_n(A)$, of the multilinear polynomials of $\mathbb{F}\langle X\rangle$ of degree n modulo $T(A)$, for any $n \in \mathbb{N}$. The dimension, $c_n(A)$, of this space is called the n-th codimension of A. It is well know that the sequence $c_n(A)$ is exponentially bounded ([22]). Recently, it has been proved that the limit $\lim_n \sqrt[n]{c_n(A)}$ does exist for any non trivial PI-algebra (see [12] and [13]) and it is a non-negative integer, called the *PI-exponent* of A. This invariant can be used in order to classify the varieties of PI-algebras, as suggested by the mentioned papers. In [15] the authors prove that the minimal varieties with respect to a fixed exponent are determined by the T-ideals of the Grassmann envelope of the so-called "minimal superalgebras". For an algebraically closed field, such superalgebras can be realized as $\mathbb{Z}_2$-graded subalgebras of block-triangular matrix algebras equipped with a suitable $\mathbb{Z}_2$-grading. Precisely, the blocks along the main diagonal are simple superalgebras of finite dimension. It is important to notice that, as proved by Kemer, any non trivial verbally prime variety of associative algebras is generated by the Grassmann envelope of one of these simple superalgebras. Hence it is an interesting problem to investigate the $\mathbb{Z}_2$-graded polynomial identities of the mentioned block-triangular superalgebras. In this paper we present some recent results concerning this matter.

2. $\mathbb{Z}_2$-graded cocharacters. Let $\mathbb{F}$ be a field of characteristic zero and let A be an associative $\mathbb{F}$-algebra. We say that A is a *superalgebra*, or a *$\mathbb{Z}_2$-graded algebra*, if $A = \bigoplus_{i\in\mathbb{Z}_2} A_i$, where $A_i \subseteq A$ are subspaces and $A_iA_j \subseteq A_{i+j}$ holds for any $i, j \in \mathbb{Z}_2$. The subspace A_i is called the *homogeneous component of A of degree i*. We say that the elements $a \in A_i$ are *homogeneous of degree i* and we denote their degrees as: $|a| = i$. Moreover, we say that $a \in A$ is an *even* element if $|a| = 0 \in \mathbb{Z}_2$; similarly a is an *odd* element if $|a| = 1$. By definition, a subspace $W \subseteq A$ is a *$\mathbb{Z}_2$-graded subspace* if $W = \bigoplus_{i\in\mathbb{Z}_2} W_i$, where $W_i = W \cap A_i$ for all $i \in \mathbb{Z}_2$. Finally, if $A = A_0 \oplus A_1$ and $B = B_1 \oplus B_2$ are superalgebras then a homomorphism of algebras $\varphi : A \to B$ is called a *$\mathbb{Z}_2$-graded homomorphism* if it holds $\varphi(A_i) \subseteq B_i$, for all $i \in \mathbb{Z}_2$.

One defines a free object in the class of superalgebras by considering the free $\mathbb{F}$-algebra over the disjoint union of two countable sets of variables, Y and Z, whose elements are regarded as *even* and *odd* respectively. We shall denote by $\mathbb{F}\langle Y, Z\rangle$ the free $\mathbb{F}$-algebra generated by $Y \cup Z$. The even component of $\mathbb{F}\langle Y, Z\rangle$ is the space spanned by those monomials in which an even number of elements from Z occurs. The remaining monomials span the odd component of $\mathbb{F}\langle Y, Z\rangle$.

A polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ in $\mathbb{F}\langle Y, Z\rangle$ is called a $\mathbb{Z}_2$-graded polynomial identity for a superalgebra A if it is in the kernel of all possible $\mathbb{Z}_2$-graded homomorphisms $\varphi : \mathbb{F}\langle Y, Z\rangle \to A$. In other words, f is a graded polynomial identity for A if it vanishes under all the possible substitutions of the variables by elements of A with the same parity only: the y_i replaced by $a_i \in A_0$ and the z_i by $b_i \in A_1$. One often calls these substitutions *graded substitutions*.

The set $T_2(A)$ of all graded polynomial identities of A is an ideal of the free superalgebra invariant under all graded endomorphisms. It is called the T_2-ideal of (the graded polynomial identities of) A. The factor algebra $\mathbb{F}\langle Y, Z\rangle/T_2(A)$ inherits the superalgebra structure of the free superalgebra, and is a free object for the class of the superalgebras B such that $T_2(A) \subseteq T_2(B)$. This factor algebra is called the *relatively free superalgebra* associated to A. In order to study this relatively free superalgebra, we may use the powerful tools of representation theory of the symmetric groups.

More precisely, let us define the $\mathbb{Z}_2$-*graded multilinear* polynomials in $\mathbb{F}\langle Y, Z\rangle$ as follows.

Definition 2.1. *For $n \in \mathbb{N}$, the vector space*

$$V_n^{\mathbb{Z}_2} := \operatorname{span}\langle x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(n)} \mid \sigma \in S_n,\ x_i \in \{y_i, z_i\}\rangle$$

is called the space of $\mathbb{Z}_2$-graded multilinear polynomials.

Since the characteristic of the ground field $\mathbb{F}$ is zero, a standard process of multilinearization shows that $T_2(A)$ is generated, as a T_2-ideal, by the subspaces $V_n^{\mathbb{Z}_2} \cap T_2(A)$. Actually, it is more convenient to study the factor space

$$V_n^{\mathbb{Z}_2}(A) := \frac{V_n^{\mathbb{Z}_2}}{V_n^{\mathbb{Z}_2} \cap T_2(A)}.$$

As we said above, an effective tool to this aim is provided by the representation theory of the symmetric groups.

Indeed, one can notice that $V_n^{\mathbb{Z}_2}$ is an S_n-module with respect to the natural left action, and $V_n^{\mathbb{Z}_2} \cap T_2(A)$ is an S_n-submodule of it. We shall denote by $\chi_n^{\mathbb{Z}_2}(A)$ the character of such representation and we call it the *n-th $\mathbb{Z}_2$-graded cocharacter of the superalgebra* A or equivalently *of the ideal* $T_2(A)$. Similarly, we shall denote by $c_n^{\mathbb{Z}_2}(A)$ the dimension of the factor space $V_n^{\mathbb{Z}_2}(A)$ and we call it the *n-th $\mathbb{Z}_2$-graded codimension of* A. One can define a "*superexponent*" by setting

$$\exp^{\mathbb{Z}_2}(A) := \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(A)},$$

if this limit does exist. Very recently, it has been proved the existence of this superexponent for any finite dimensional superalgebra, or more generally for any finitely generated superalgebra which satisfies an ordinary polynomial identity [3].

We remark that in the study of the $\mathbb{Z}_2$-graded polynomial identities of the superalgebra A we can consider "smaller" spaces of multilinear polynomials. To be more precise, for fixed h, k, let

$$V_{h,k} := \text{span}\langle m \text{ monomials of } V_{h+k}^{\mathbb{Z}_2} \mid \text{just } y_1, \ldots, y_h, z_{h+1}, \ldots, z_{h+k} \text{ occur in } m\rangle.$$

Setting $n := h + k$, and $\mathcal{H}_{h,k} := Sym(\{1, \ldots, h\}) \times Sym(\{h + 1, \ldots, n\}) \leqslant S_n$, the space $V_{h,k}$ is an $\mathcal{H}_{h,k}$-module, and the subspace $V_{h,k} \cap T_2(A)$ is a submodule. Therefore one can consider the factor $\mathcal{H}_{h,k}$-module

$$V_{h,k}(A) := \frac{V_{h,k}}{V_{h,k} \cap T_2(A)}.$$

We shall denote by $\chi_{h,k}(A)$ its $\mathcal{H}_{h,k}$-character, and by $c_{h,k}(A)$ its dimension.

We briefly recall that if H is a subgroup of a group G and M is an H-module, we can turn M into a G-module by considering the induced G-module structure. In other words, one sets $M^G := \mathbb{F}G \otimes_{\mathbb{F}H} M$. This is the so-called G-module induced by M. The relation between the S_n-structure of $V_n^{\mathbb{Z}_2}(A)$ and the $\mathcal{H}_{h,k}$-structure of $V_{h,k}(A)$ then is displayed by the following result (see [2], [5]):

Theorem 2.2. *Let A be a superalgebra. Then for all $n \in \mathbb{N}$*

$$V_n^{\mathbb{Z}_2}(A) \cong \sum_{k=0}^{n} (V_{n-k,k}(A))^{S_n}$$

as S_n-modules. In particular,

$$c_n^{\mathbb{Z}_2}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k,k}(A).$$

In this way the study of the S_n-structure of $T_2(A)$ is reduced to the study of the modules $V_{n-k,k}(A)$.

Since the characteristic of the field $\mathbb{F}$ is zero, then any representation of the groups $\mathcal{H}_{h,k} = S_h \times S_k$ $(h + k := n)$ is completely reducible. The irreducible

$\mathcal{H}_{h,k}$-characters are in a bijective correspondence with the pairs of partitions (λ, μ) where $\lambda \vdash h$ and $\mu \vdash k$. More precisely, if χ_ν denotes the irreducible $S_{|\nu|}$-character associated to the partition ν, then the irreducible $\mathcal{H}_{h,k}$-character associated to (λ, μ) is $\chi_{\lambda,\mu} = \chi_\lambda \otimes \chi_\mu$.

In order to simplify the notation, we shall often identify an irreducible character χ_ν of the symmetric group with the corresponding partition $\nu = (\nu_1, \ldots, \nu_r)$. So for instance, we shall write

$$\chi_{h,k}(A) = \sum_{\substack{\lambda \vdash h \\ \mu \vdash k}} m_{\lambda,\mu}\, \lambda \otimes \mu$$

for certain multiplicities $m_{\lambda,\mu} = m_{\lambda,\mu}(A)$.

Let $E = E_0 \oplus E_1$ be the Grassmann (or exterior) algebra of a vector space of countable dimension equipped with its natural $\mathbb{Z}_2$-grading. For any superalgebra A, the *Grassmann envelope* of A is defined as the following superalgebra:

$$G(A) = (A_0 \otimes E_0) \oplus (A_1 \otimes E_1)$$

The relationship between the graded identities of the superalgebras $A, G(A)$ is described in [18] by means of an involution $I \mapsto I^*$ defined on the lattice of the T_2-ideals of the free superalgebra $\mathbb{F}\langle Y, Z\rangle$. Using the language of the representation theory, one has the following relationship between the sequences of graded cocharacters of A and $G(A)$:

$$\chi_{h,k}(A) = \sum_{\mu,\nu} m_{\mu,\nu} \mu \otimes \nu \;\Leftrightarrow\; \chi_{h,k}(G(A)) = \sum_{\mu,\nu} m_{\mu,\nu} \mu \otimes \nu' \tag{1}$$

where $\nu' \vdash k$ is the conjugate partition of ν. We recall that the involution $*$ satisfies also the property $(IJ)^* = I^* J^*$.

The relation between the sequences of $\mathbb{Z}_2$-graded cocharacters of the T_2-ideals I, J and IJ is described in the following result. It has been obtained in [8] as generalization of the previous result in [4] about ordinary T-ideals.

More precisely, if χ', χ'' are sequences of characters $\chi'_{k,l}$ and $\chi''_{k,l}$ $(k, l \geq 0)$ of the product group $S_k \times S_l$, we define $(\chi' \circ \chi'')_{k,l}$ to be the following sequence of characters:

$$(\chi' \circ \chi'')_{k,l} = \sum_{i=0}^{k} \sum_{j=0}^{l} \chi'_{i,j} \,\hat{\otimes}\, \chi''_{k-i,l-j}$$

where $\hat{\otimes}$ is the outer tensor product of the characters of the symmetric group. Explicitly for the irreducible characters $\chi_{\mu,\nu}, \chi_{\rho,\tau}$, where μ, ν, ρ, τ are partitions of m, n, r, t respectively, one has:

$$\chi_{\mu,\nu}\hat{\otimes}\chi_{\rho,\tau} = (\chi_\mu\hat{\otimes}\chi_\rho) \otimes (\chi_\nu\hat{\otimes}\chi_\tau) = (\chi_\mu \otimes \chi_\rho)^{S_{m+r}} \otimes (\chi_\nu \otimes \chi_\tau)^{S_{n+t}}.$$

We are ready to state the following result:

Theorem 2.3. *Let I, J be T_2-ideals of the superalgebras A and B respectively. Denote by R any superalgebra whose T_2-ideal factorizes as the product IJ. Then, the $\mathbb{Z}_2$-graded cocharacters $\chi_{k,l}(R)$ of this superalgebra verifies:*

$$\begin{aligned} \chi_{k,l}(R) = {} & \chi_{k,l}(A) + \chi_{k,l}(B) + \chi_{(1),\varnothing}\hat{\otimes}(\chi(A) \circ \chi(B))_{k-1,l} \\ & + \chi_{\varnothing,(1)}\hat{\otimes}(\chi(A) \circ \chi(B))_{k,l-1} - (\chi(A) \circ \chi(B))_{k,l} \end{aligned} \tag{2}$$

These results, together with the classification of the simple superalgebras of finite dimension, allow us to reduce the study in this paper just to the matrix algebras with entries in the field $\mathbb{F}$.

3. Block-triangular superalgebras.

Let A, B be $\mathbb{Z}_2$-graded algebras and W be a $\mathbb{Z}_2$-graded A-B-bimodule, that is $W = W_0 \oplus W_1$ where W_i are subspaces of W and $A_i W_j B_h \subseteq W_{i+j+h}$ for any $i, j, h \in \mathbb{Z}_2$ We denote by R the block-triangular matrix algebra defined as the following:

$$R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$$

The algebra R can be graded by $\mathbb{Z}_2$ in a natural way by putting for any $i \in \mathbb{Z}_2$:

$$R_i = \begin{bmatrix} A_i & W_i \\ 0 & B_i \end{bmatrix}$$

With respect to such $\mathbb{Z}_2$-grading, we have clearly that $T_2(A)T_2(B) \subseteq T_2(R)$. We shall describe in a greater detail the relations between $T_2(R)$ and the $\mathbb{Z}_2$-graded identities of A and B in some relevant case. We begin with an easy example:

Example 3.1. *A, B are PI-algebras over $\mathbb{F}$ and*

$$A_1 = B_1 = W_1 = 0$$

In this case, R is a superalgebra with the trivial $\mathbb{Z}_2$-grading, that is $R = R_0$ and $R_1 = 0$. Therefore, the odd indeterminates z are always in $T_2(R)$

and the polynomial $f(y_1, \ldots, y_q)$ is a $\mathbb{Z}_2$-graded polynomial identity for R if and only if $f(x_1, \ldots, x_q)$ lies in $T(R)$. As a consequence we obtain $\mathbb{F}\langle Y, Z\rangle/T_2(R) \approx \mathbb{F}\langle X\rangle/T(R)$. Moreover, $V_{h,k}(R) = 0$ if $k > 0$ and $V_{n,0}(R) \approx V_n(R) = V_n/V_n \cap T(R)$ as S_n-modules. Therefore, by Theorem 2.2 one has

$$\chi_n^{\mathbb{Z}_2}(R) = \chi_{n,0}(R) = \chi_n(R), \quad c_n^{\mathbb{Z}_2}(R) = c_n(R)$$

Moreover, if $A = M_m, B = M_n$ and $W = M_{m\times n}$ the vector space of $m \times n$ rectangular matrices then $R = \mathrm{UT}_{m,n}$. In this case it is well know that $T(\mathrm{UT}_{m,n}) = T(M_m(\mathbb{F}))T(M_n(\mathbb{F}))$ (see [14]). This decomposition is a particular case of deep result of Giambruno and Zaicev. More precisely, in [15] they solve in the positive a conjecture due to Drensky [10, 11] about the factorability of the T-ideals of minimal varieties as a product of verbally prime T-ideals. In [16] Formanek gave a formula for the Hilbert series of the product of a couple of T-ideals as a function of the Hilbert series of the factors. The proof of this result given in [17] works for arbitrary homogeneous ideals of the free algebra. Using the result of Formanek, Berele and Regev [4] proved a formula that relates the sequence of ordinary cocharacters of a product of T-ideals to the sequences of cocharacters of these ideals. In our case we have:

$$\chi_n^{\mathbb{Z}_2}(R) = \chi_n(A) + \chi_n(B) + \left(\chi_{(1)} \otimes \chi_{n-1}(A)\right)^{S_n} +$$
$$\left(\chi_{(1)} \otimes \chi_{n-1}(B)\right)^{S_n} - \sum_{p=0}^{n} \left(\chi_p(A) \otimes \chi_{n-p}(B)\right)^{S_n}.$$

The second instance is:

Example 3.2. *A, B are PI-algebras over $\mathbb{F}$ and*

$$A_1 = B_1 = W_0 = 0$$

In this case the superalgebra $R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$ is equipped with the canonical $\mathbb{Z}_2$-grading:

$$R_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}$$

If we assume that W is a *free* $A - B$ bimodule, then the main result of [6] allows us to describe a generating set for the $\mathbb{Z}_2$-graded polynomial identities for the superalgebra R in terms of the ordinary polynomial identities of A and B.

More precisely, Theorem 1 of [6] can be written as the following:

Theorem 3.3. *Let the T-ideals $T(A \oplus B), T(A)$ and $T(B)$ have bases*

$$\{f_l(x) \mid l \in L\}, \quad \{g_{l'}(x) \mid l' \in L'\} \quad \text{and} \quad \{h_{l''}(x) \mid l'' \in L''\}$$

respectively. Then the T_2-ideal of the $\mathbb{Z}_2$-graded polynomial identities of the superalgebra $R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$ has a basis

$$\{z_1 z_2, f_l(y), g_{l'}(y) z_1, z_1 h_{l''}(y) \mid l \in L, l' \in L', l'' \in L''\}.$$

A result with a similar flavor has been obtained in [9]. In fact, the authors describe the graded cocharacter sequence of the superalgebra R in terms of the ordinary cocharacter sequences associated to the polynomial identities of A and B. More precisely, with the same notation of the previous theorem one has (see Theorem 3.1 of [9]):

Theorem 3.4. *The $\mathbb{Z}_2$-graded cocharacter sequence for the superalgebra R is the following*

$$\begin{aligned} &\chi_{n,0}(R) = \chi_n(A \oplus B) \\ &\chi_{n,1}(R) = \sum_{p=0}^{n} (\chi_p(A) \otimes \chi_{n-p}(B))^{S_n} \otimes \chi_{(1)} \\ &\chi_{n,k}(R) = 0 \quad \text{for} \quad k \geqslant 2 \\ &\qquad (n \in \mathbb{N}) \end{aligned}$$

Now, as in [9] it is easy to show the following result about the graded codimension of R.

Corollary 3.5. *The graded codimension sequence of R is related to the ordinary codimension sequences of A, B and $A \oplus B$ by the following formula:*

$$c_n^{\mathbb{Z}_2}(R) = c_n(A \oplus B) + n \sum_{h+k=n-1} \binom{n-1}{h} c_h(A) c_k(B). \tag{3}$$

Proof. By Theorem 2.2 one has

$$c_n^{\mathbb{Z}_2}(R) = \sum_{i=0}^{n} \binom{n}{i} c_{n-i,i}(R).$$

By [6], Theorem 1, it follows that $c_{n-i,i}(R) = 0$ if $i \geqslant 2$, hence

$$c_n^{\mathbb{Z}_2}(R) = c_{n,0}(R) + nc_{n-1,1}(R).$$

The explicit formula follows then as a consequence of Theorem 3.4 $\square$

Using this, and the results of Giambruno and Zaicev about the PI-exponent [12], [13], we obtain

Corollary 3.6. *The $\mathbb{Z}_2$-graded PI-exponent of R is*

$$\exp^{\mathbb{Z}_2}(R) := \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(R)} = \exp(A) + \exp(B).$$

Now we recall the general setting, that is A, B are superalgebras, W is a $\mathbb{Z}_2$-graded $A - B$-bimodule and $R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$. As we said above, if $R_i = \begin{bmatrix} A_i & W_i \\ 0 & B_i \end{bmatrix}$ then $T_2(A)T_2(B) \subseteq T_2(R)$. The final result of this section consists in describing a suitable condition for the structures A, B, W such that one has $T_2(A)T_2(B) = T_2(R)$. For this purpose, the main tool is the Lewin's Theorem [20].

Let I and J be any two-sided ideals of $\mathbb{F}\langle X\rangle$. Consider the factor algebras $\mathbb{F}\langle X\rangle/I, \mathbb{F}\langle X\rangle/J$ and let M be a $\mathbb{F}\langle X\rangle/I$-$\mathbb{F}\langle X\rangle/J$-bimodule. We define:

$$\tilde{R} = \begin{bmatrix} \mathbb{F}\langle X\rangle/I & M \\ 0 & \mathbb{F}\langle X\rangle/J \end{bmatrix}$$

Assume $\{w_i\}$ is a countable set of elements of M. Then an algebra homomorphism $\varphi : x_i \in \mathbb{F}\langle X\rangle \mapsto a_i \in \tilde{R}$ is defined, where:

$$a_i = \begin{pmatrix} x_i + I & w_i \\ 0 & x_i + J \end{pmatrix}$$

For the kernel $\ker(\varphi)$of the homomorphism φ we get immediately:

$$IJ \subseteq \ker(\varphi) \subseteq I \cap J.$$

We have:

Theorem 3.7 (Lewin, [20]). *If $\{w_i\}$ is a countable free set of elements of the bimodule M, then for the homomorphism φ defined by $\{w_i\}$, we have:*

$$\ker(\varphi) = IJ.$$

Consider now the free superalgebra $\mathbb{F}\langle Y, Z\rangle$, let I, J be T_2-ideals and let M be a $\mathbb{Z}_2$-graded $\mathbb{F}\langle Y, Z\rangle/I$-$\mathbb{F}\langle Y, Z\rangle/J$-bimodule. Of course, the algebra:

$$\tilde{R} = \begin{bmatrix} \mathbb{F}\langle Y, Z\rangle/I & M \\ 0 & \mathbb{F}\langle Y, Z\rangle/J \end{bmatrix} \tag{4}$$

is also $\mathbb{Z}_2$-graded and one has $IJ \subseteq T_2(\tilde{R})$. Moreover, let $u_i, v_i \in M$ be homogeneous elements of even and odd degree respectively, for all $i \geq 1$. Let $\varphi : \mathbb{F}\langle Y, Z\rangle \to \tilde{R}$ be the homomorphism defined by the set $\{u_i, v_i\}$, then φ is a $\mathbb{Z}_2$-graded homomorphism and hence $T_2(\tilde{R}) \subseteq \ker(\varphi)$. If $\{u_i, v_i\}$ is a free subset of the $\mathbb{F}\langle Y, Z\rangle/I$-$\mathbb{F}\langle Y, Z\rangle/J$-bimodule, then by the Lewin's Theorem we have that $\ker(\varphi) = IJ$. Hence we can conclude:

Corollary 3.8. *If the $\mathbb{Z}_2$-graded bimodule M contains a countable free set $\{u_i, v_i\}$ of homogeneous elements such that $|y_i| = |u_i|$ and $|z_i| = |v_i|$ for any $i \geq 1$, then:*

$$T_2(\tilde{R}) = IJ.$$

4. A free construction for matrix superalgebras. Let us consider $\mathbb{Z}_2$-gradings on matrix algebras. Let $M_m = M_m(\mathbb{F})$ be the algebra of matrices of order m with entries in $\mathbb{F}$ and fix a map $|\ | : \{1, 2, \ldots, m\} \to \mathbb{Z}_2$. If $e_{ij} \in M_m$ is any unit matrix, then such map can be extended to these elements in the following way:

$$|e_{ij}| = |j| - |i|.$$

Since $|e_{ij}e_{jk}| = |e_{ik}| = |k| - |i| = |e_{ij}| + |e_{jk}|$, in this way a $\mathbb{Z}_2$-grading is defined on M_m. Clearly, such grading is the elementary grading defined by the vector $(|1|, \ldots, |m|) \in \mathbb{Z}_2^m$ (see [1]). We write $(M_m, |\ |)$ for the matrix superalgebra M_m endowed with the $\mathbb{Z}_2$-grading defined by the map $|\ | : \{1, 2, \ldots, m\} \to \mathbb{Z}_2$.

Moreover, the superalgebra $(M_m, |\ |)$ is simply denoted as $M_{k,l}(\mathbb{F})$ if $|i| = 0$ for $1 \leq i \leq k$ and $|i| = 1$ for $k+1 \leq i \leq k+l = m$.

By the classification of the finite dimensional simple superalgebras over an algebraically closed field (see [27], [19]), it holds that there are exactly two class of such superalgebras up to isomorphisms: $M_{k,l}(\mathbb{F})$ with $k \geq l \geq 0$ $(k \neq 0)$ and $M_m \oplus tM_m$ with $m > 0, t^2 = 1$. Moreover, we can regard the latter superalgebra as a $\mathbb{Z}_2$-graded subalgebra of $M_{m,m}(\mathbb{F})$. More precisely, we consider the $\mathbb{Z}_2$-graded monomorphism, $\varphi : M_m \oplus tM_m \to M_{m,m}(\mathbb{F})$, defined as follows:

$$a_0 + ta_1 \mapsto \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \tag{5}$$

Now, if $(M_m, |\ |_m)$ and $(M_n, |\ |_n)$ are matrix superalgebras, then we define the map $|\ | : \{1, 2, \ldots, m+n\} \to \mathbb{Z}_2$ by putting $|i| = |i|_m$ for $i \leq m$ and $|i| = |i-m|_n$ for $i > m$. We consider then the matrix algebra M_{m+n} endowed with the $\mathbb{Z}_2$-grading defined by the map $|\ |$. Now consider the $\mathbb{F}$-vector space $W = M_{m \times n}$ of the $m \times n$ rectangular matrices, and let A, B be $\mathbb{Z}_2$-graded subalgebras respectively of M_m, M_n. Clearly the space W is an A-B-bimodule. In this way the superalgebra $R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$ is a $\mathbb{Z}_2$-graded subalgebra of $(M_{m+n}, |\ |)$.

For a given superalgebra R of this type, we will exhibit explicitly a superalgebra $\overline{R}$ isomorphic to the superalgebra $\tilde{R}$ (see equation 4) and such that $T_2(R) = T_2(\overline{R})$. We say that $R, \overline{R}$ are *$\mathbb{Z}_2$-graded PI equivalent*. The notion of "generic superalgebra" is very useful for this purpose. More precisely, we say that a superalgebra Ω is a *generic superalgebra associated to a superalgebra S* if it holds:

$$\Omega \approx \mathbb{F}\langle Y, Z\rangle / T_2(S).$$

In particular, this implies that $T_2(\Omega) = T_2(S)$.

If S has finite dimension, then one has a canonic way to define a $\mathbb{Z}_2$-graded generic algebra. In fact, assume that the superalgebra S has a $\mathbb{F}$-linear basis $E = \{e_1, \ldots, e_n\}$ whose elements are all homogeneous. Denote:

$$P(S) = \mathbb{F}[\, u_i^{(h)}, v_i^{(h)} \,|\, 1 \leq i \leq n, h \geq 1\,]$$

the polynomial ring in the countable set of commuting variables $u_i^{(h)}, v_i^{(h)}$. We call $P(S)$ the *polynomial ring associated to the finite dimensional superalgebra S*. Note that the following tensor product over the field $\mathbb{F}$:

$$S \otimes P(S) = \bigoplus_{i \in \mathbb{Z}_2} S_i \otimes P(S)$$

is a superalgebra such that $T_2(S \otimes P(S)) = T_2(S)$. We consider in $S \otimes P(S)$ the $\mathbb{Z}_2$-graded subalgebra S' generated, for all $h \geq 1$, by the following homogeneous elements:

$$a_h = \sum_{|e_i|=0} u_i^{(h)} e_i \text{ and } b_h = \sum_{|e_i|=1} v_i^{(h)} e_i$$

where the index i ranges over $1 \leq i \leq n$. We can easily prove:

$$S' \approx \mathbb{F}\langle Y, Z\rangle / T_2(S).$$

Note that if $S = M_m$ then we choose canonically the set of the unit matrices e_{ij} as $\mathbb{F}$-linear basis (for the non-graded case, see for instance [23]).

Consider now the block triangular superalgebra R. Of course, we can produce a $\mathbb{Z}_2$-homogeneous linear basis of R by considering the disjoint union of the bases for A and B with the canonical basis $\{e_{ij}\}$ $(1 \leq i \leq m, m+1 \leq j \leq m+n)$ of W. Let $P = P(R)$ be the polynomial ring associated to R, then $R \otimes P$ contains the generic free superalgebras R', A' and B' associated in the canonic way to R, A and B respectively.

Let us consider the $\mathbb{Z}_2$-graded subalgebra of $R \otimes P$ defined as:

$$\overline{R} = \begin{bmatrix} A' & W' \\ 0 & B' \end{bmatrix} \tag{6}$$

where W' is the A'-B'-bimodule contained in $R \otimes P$ generated, for all $h \geq 1$, by the following homogeneous elements:

$$\overline{u}_h = \sum_{|e_{ij}|=0} u_{ij}^{(h)} e_{ij} \text{ and } \overline{v}_h = \sum_{|e_{ij}|=1} v_{ij}^{(h)} e_{ij} \tag{7}$$

with $1 \leq i \leq m$, $m+1 \leq j \leq m+n$. Then we have:

Proposition 4.1.

$$T_2(R) = T_2(R') = T_2(\overline{R})$$

Proof. It is sufficient to note that $T_2(R') = T_2(R) = T_2(R \otimes P)$ and moreover $R' \subseteq \overline{R} \subseteq R \otimes P$. □

Since $A' \approx \mathbb{F}\langle Y, Z\rangle / T_2(A)$ and $B' \approx \mathbb{F}\langle Y, Z\rangle / T_2(B)$, by Corollary 3.8 in order to prove the factorization of $T_2(R) = T_2(\overline{R})$ it sufficient to show that the homogeneous elements $\overline{u}_h, \overline{v}_h$ defined in (7) form a free set of the bimodule W'.

For this purpose we need to introduce the notion of "$\mathbb{Z}_2$-regularity" of a matrix subalgebra.

Let us consider a matrix superalgebra $(M_m, |\ |)$. For any fixed element $g \in \mathbb{Z}_2$ and any commutative $\mathbb{F}$-algebra C we define the following $\mathbb{F}$-linear map $\pi_g : M_m(C) \to M_m(C)$:

$$\sum_{i,j} a_{ij} e_{ij} \mapsto \sum_{|i|=g,j} a_{ij} e_{ij}$$

where $1 \leq i, j \leq m$. Clearly $\pi_0 + \pi_1 = \mathrm{id}$, the identity map. If A is a $\mathbb{Z}_2$-graded subalgebra of $(M_m, |\ |)$, denote as usual $P = P(A)$ the polynomial ring associated to A. Since for the generic superalgebra A' the following chain of immersions holds:

$$A' \subseteq A \otimes P \subseteq M_m \otimes P = M_m(P)$$

we can define $\hat{\pi}_g : A' \to M_m(P)$ as the restriction of π_g to A'. In the same way, we can define also the $\mathbb{F}$-linear map $\pi_g^* : M_m(P) \to M_m(P)$

$$\sum_{i,j} a_{ij} e_{ij} \mapsto \sum_{i,|j|=g} a_{ij} e_{ij}$$

and its restriction $\hat{\pi}_g^* : A' \to M_m(P)$. As in [8], we have:

Proposition 4.2. *The maps $\hat{\pi}_g$ are all injective if and only if the maps $\hat{\pi}_g^*$ are so, for all $g \in \mathbb{Z}_2$.*

Proof. Let $\varphi : \mathbb{F}\langle Y, Z\rangle \to A'$ denote the canonic $\mathbb{Z}_2$-graded epimorphism such that $\ker(\varphi) = T_2(A)$. Let a' be a matrix of A' and $f \in \mathbb{F}\langle Y, Z\rangle$ be a polynomial such that $\varphi(f) = a'$. Clearly, the condition $\hat{\pi}_g(a') = 0$ is equivalent to $\pi_g(\nu(f)) = 0$, for any $\mathbb{Z}_2$-graded substitution $\nu : \mathbb{F}\langle Y, Z\rangle \to A$. Therefore, if the element a' is homogeneous of degree $h \in \mathbb{Z}_2$ and $\hat{\pi}_g(a') = 0$ then $\hat{\pi}_{g+h}^*(a') = 0$ too. □

Moreover, one has:

Definition 4.3. *A $\mathbb{Z}_2$-graded subalgebra $A \subseteq M_m$ is said to be* $\mathbb{Z}_2$-regular *if the maps $\hat{\pi}_g$ (or equivalently the maps $\hat{\pi}_g^*$) are injective, for any $g \in \mathbb{Z}_2$.*

With the notation of equation (6) and (7), we have:

Proposition 4.4. *Let A, B be $\mathbb{Z}_2$-graded subalgebras respectively of M_m, M_n. If one of such subalgebras is $\mathbb{Z}_2$-regular then the homogeneous elements $\overline{u}_h, \overline{v}_h$ of the graded A'-B'-bimodule W' form a countable free set such that $|\overline{u}_h| = |y_h|$ and $|\overline{v}_h| = |z_h|$ for all $h \geq 1$.*

This proposition is a particular case of one among the results in [8]. We include here its proof for the sake of completeness.

P r o o f. We assume that B is a $\mathbb{Z}_2$-regular subalgebra of M_n. Since the non-zero entries of the matrices $\overline{u}_h, \overline{v}_h$ are distinct variables for all the indices h, clearly it is sufficient to prove that each element $\overline{u}_h, \overline{v}_h$ is torsion-free. Then, let $\sum_s a_s w\, b_s = 0$ with $a_s \in A', b_s \in B'$ and w in the set $\{\overline{u}_h, \overline{v}_h\}$. Suppose that the matrices b_s are linearly independent and by contradiction that $a_s \neq 0$ for any index s. From the row-by-column product, it follows that for any pair of indices (i, q) we have:

$$\sum_s \sum_{j,p} (a_s)_{ij} w_{jp} (b_s)_{pq} = 0.$$

Note that $w_{jp} \neq 0$ if and only if $|p| - |j| = |w|$. Moreover, the entries $w_{jp} \neq 0$ are variables that are distinct from those of the polynomials $(a_s)_{ij}$ and $(b_s)_{pq}$. It follows:

$$\sum_s (a_s)_{ij} (b_s)_{pq} = 0$$

for any quadruple of indices (i, j, p, q) such that $|p| - |j| = |w|$. Since $a_1 \neq 0$, there are indices i_1, j_1 such that $(a_1)_{i_1,j_1} \neq 0$. By putting $g = |j_1| + |w|$ we have then:

$$\sum_s (a_s)_{i_1 j_1} (b_s)_{pq} = 0$$

for any indices p, q with $|p| = g$. By multiplying now this equation for the unit matrix e_{pq} and by summing over the indices p, q, we finally obtain:

$$\sum_s (a_s)_{i_1 j_1} \hat{\pi}_g (b_s) = 0.$$

Note that the matrices $\hat{\pi}_g(b_s)$ are linearly independent since $\hat{\pi}_g$ is a monomorphism. Since $(a_1)_{i_1 j_1} \neq 0$, we get then a contradiction. We argue in a similar way if A is a $\mathbb{Z}_2$-regular subalgebra of M_m. $\square$

A similar proof works for the following proposition

Proposition 4.5. *Let A, B be $\mathbb{Z}_2$-graded subalgebras of $(M_m, |\ |_m)$ and $(M_n, |\ |_n)$ respectively. If both the maps $|\ |_m$ and $|\ |_n$ are constant then for the homogeneous elements $\overline{u}_h, \overline{v}_h$ of the graded A'-B'-bimodule W' it holds:*

(1) *If $|1|_m + |1|_n = 0 \in \mathbb{Z}_2$ then $W'_1 = 0$ and $\{\overline{u}_h \mid h \geq 1\}$ is a countable free set of even elements*

(2) *If* $|1|_m + |1|_n = 1 \in \mathbb{Z}_2$ *then* $W_0' = 0$ *and* $\{\overline{v}_h \mid h \geq 1\}$ *is a countable free set of odd elements.*

5. Applications. First of all we state one of the main result of this paper. More precisely, from the Corollary 3.8 and the Propositions 4.1 and 4.4 it follows:

Theorem 5.1. *Let* R *be the following* $\mathbb{Z}_2$*-graded block-triangular matrix algebra:*

$$R = \left[\begin{array}{cc} A & U \\ 0 & B \end{array} \right]$$

where $A \subseteq M_m, B \subseteq M_n$ *are graded subalgebras and* $U = M_{m\times n}$*. If one of such subalgebras is* $\mathbb{Z}_2$*-regular, then the* T_2*-ideal* $T_2(R)$ *factorizes as:*

$$T_2(R) = T_2(A)T_2(B).$$

Let us recall the following results about the $\mathbb{Z}_2$-regularity of matrix superalgebras, which are special cases of more general results for gradings by an arbitrary group (see [8]).

Proposition 5.2. *Let* $A = (M_m, |\ |)$ *be a complete matrix superalgebra. Then* A *is* $\mathbb{Z}_2$*-regular if and only if the map* $|\ |$ *is surjective and its fibers are equipotent.*

Proposition 5.3. *Let* A *be a* $\mathbb{Z}_2$*-graded subalgebra of* $(M_m, |\ |)$ *and set the* $\mathbb{Z}_2$*-grading on* M_{2m} *by the vector* $(|1|, \ldots, |m|, 1 + |1|, \ldots, 1 + |m|)$*. Then the map* $\varphi : M_m \to M_{2m}$ *sending* $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ *is a graded monomorphism and* $\varphi(A)$ *is a* $\mathbb{Z}_2$*-regular subalgebra of* M_{2m}*.*

With regard to the finite dimensional simple superalgebra $M_m(\mathbb{F}) \oplus tM_m(\mathbb{F})$, we recall that it is isomorphic to the $\mathbb{Z}_2$-graded subalgebra

$$D_{m,m}(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in M_m \right\}$$

of $M_{m,m}(\mathbb{F})$ (see equation 5). It is easy to see that the $D_{m,m}(\mathbb{F})$ is $\mathbb{Z}_2$-regular.

Finally we have:

Theorem 5.4. *Let R be a matrix superalgebra of type:*

$$R = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$$

where $A = (M_m, |\ |_m), B = (M_n, |\ |_n)$ are complete matrix superalgebras and $W = M_{m\times n}$. The T_2-ideal of R factorizes as $T_2(R) = T_2(A)T_2(B)$ if and only if one of the superalgebras A or B is $\mathbb{Z}_2$-regular.

The sufficient condition follows by the Theorem 5.1. For the necessary condition, assume that the superalgebras A and B are both non-regular. Then it is possible to define a polynomial $f \in \mathbb{F}\langle Y, Z\rangle$ such that $f \in T_2(R)$ but $f \notin T_2(A)T_2(B)$.

The proof of this fact in [8] is based on the following argument. Recall that the $\mathbb{Z}_2$-grading of R is defined by the vector $(|1|_m, \dots, |m|_m, |1|_n, \dots, |n|_n)$. Note that we can obtain a new $\mathbb{Z}_2$-grading for the algebra R by the vector

$$(|1|_m, \dots, |m|_m, 1 + |1|_n, \dots, 1 + |n|_n) \in \mathbb{Z}_2^{m+n}.$$

We denote by R^* this latter superalgebra. Note that R and R^* differ only for the degree of the unit matrices in W. In particular, one has that $T_2(A)T_2(B) \subseteq T_2(R^*)$. In order to prove that the polynomial $f \notin T_2(A)T_2(B)$ it is enough to show that $f \notin T_2(R^*)$. Within this setting the construction of the polynomial f is easier than the one for the general case, as given in [8]. For convenience of the reader we explicitly construct f in this simpler case. Since the superalgebra $A = (M_m, |\ |_m)$ is not $\mathbb{Z}_2$- regular the fibers of the map $|\ |_m$ are not equipotent. Denote by p_A the greatest cardinality of these fibers, let q_A be the cardinality of the other fiber and choose $a \in \{1, \dots, m\}$ among the elements of the fiber with cardinality p_A. Similarly define p_B, q_B and b. We distinguish two cases according to $|a|_m \neq |b|_n$ either $|a|_m = |b|_n$. In the former case the required polynomial is the standard polynomial of degree $r = 2(p_A + p_B) - 1$ in variables from Y, that is

$$f = s_r(y_1, \dots, y_r).$$

In fact, R_0 is canonically isomorphic to the algebra $\mathrm{UT}_{p_A,q_B} \oplus \mathrm{UT}_{q_A,p_B}$ while R_0^* is isomorphic to $\mathrm{UT}_{p_A,p_B} \oplus \mathrm{UT}_{q_A,q_B}$ and the standard polynomial $s_t(x_1, \dots, x_t)$ is a polynomial identity for $\mathrm{UT}_{h,k}$ if and only if $t \geq 2(h + k)$. In the latter case f is the following multilinear polynomial:

$$f = s_{2q_A}(Y_A) d_r(z_1, \dots, z_r; y_1, \dots, y_{r+1}) s_{2q_B}(Y_B)$$

where $r = 2\max\{p_A q_A, p_B q_B\} + 1$, $d_r(z_1, \dots, z_r; y_1, \dots, y_{r+1}) = d_r(z; y)$ denotes the r-th Capelli polynomial, that is

$$d_r(z;y) = \sum_{\sigma \in S_r} (-1)^\sigma y_1 z_{\sigma(1)} y_2 z_{\sigma(2)} \cdots y_r z_{\sigma(r)} y_{r+1}$$

and Y_A, Y_B are disjoint subsets of $Y - \{y_1, \dots, y_{r+1}\}$.

Since $\max\{dimA_1, dimB_1\} = r - 1$ and $AB = BA = 0$ then any non vanishing graded substitution of $d_r(z;y)$ by unit matrices of R has values in the odd component W_1 of the bimodule W, that is in the subspace generated by the matrices e_{ij} with $1 \le i \le m < j \le m+n$ and $|i| \neq |j|$. Since $BW = WA = 0$ in order to show that f is a $\mathbb{Z}_2$-graded PI for R it is enough to evaluate the standard polynomial $s_{2q_A}(Y_A)$ by unit matrices of A and the standard polynomial $s_{2q_B}(Y_B)$ by unit matrices of B. Now, let $g \in Z_2$ such that $g = |a|_m = |b|_n$ and $\pi_g, \pi_{g+1} : M_{m+n} \to M_{m+n}$ the $\mathbb{F}$-linear maps defined above. Then we have:

$$A_0 = \pi_g(A_0) \oplus \pi_{g+1}(A_0) \approx M_{p_A} \oplus M_{q_A}$$

and

$$B_0 = \pi_g(B_0) \oplus \pi_{g+1}(B_0) \approx M_{p_B} \oplus M_{q_B}$$

Now the result follows by the Amitsur-Levitzki's Theorem and the equation

$$\pi_g(A_0) W_1 \pi_g(B_0) = 0,$$

because $s_{2q_A}(A_0) \subseteq \pi_g(A_0)$ and $s_{2q_B}(B_0) \subseteq \pi_g(B_0)$. Considering the superalgebra R^*, one has $\pi_{g+1}(B_0) \approx M_{p_B}$ hence $s_{2q_B}(B_0) \subseteq \pi_{g+1}(B_0)$. It is possible to find a non vanishing graded substitution of f on R^* by mean of a straightforward computation and Theorem 1.4.34 of [23] about Capelli polynomials.

We close this section with some example where we apply the previous results.

Example 5.5. *Let the $\mathbb{Z}_2$-grading on M_m and M_n be given by the maps:*

$$|i|_m = 0 \in \mathbb{Z}_2, \text{ for all } 1 \le i \le m \text{ and } |i|_n = 1 \in \mathbb{Z}_2, \text{ for all } 1 \le i \le n.$$

In this case R is a $\mathbb{Z}_2$-graded subalgebra of $M_{m,n}(\mathbb{F})$, more precisely we have

$$R_0 := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad R_1 := \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}.$$

By equation 6 a similar decomposition holds for the superalgebra $\overline{R} = \begin{bmatrix} A' & W' \\ 0 & B' \end{bmatrix}$. Moreover, by Proposition 4.5 for all $\mathbb{Z}_2$-graded subalgebras $A \subseteq M_m$ and $B \subseteq M_n$ W' is a free $A'-B'$ bimodule. Precisely, it is freely generated by the odd elements

$$\overline{v}_h = \sum_{i=1}^{m} \sum_{j=m+1}^{m+n} v_{ij}^{(h)} e_{ij} \quad (h \in \mathbb{N}).$$

Therefore $T_2(\overline{R})$ and its $\mathbb{Z}_2$-graded cocharacter sequence are described by the results concerning the Example 3.2. Moreover, we obtain the same conclusions about the graded identities of R, because $T(A) = T(A'), T(B) = T(B')$ and $T(R) = T(\overline{R})$ by Proposition 4.1.

As an instance of this procedure we consider in the superalgebra $M_{1,1}(\mathbb{F})$ the $\mathbb{Z}_2$-graded subalgebra UT_2. We have:

Proposition 5.6. *Let $R = \mathrm{UT}_2$ be the superalgebra of the 2×2 upper triangular matrices with the non trivial grading defined by the vector $(0,1) \in \mathbb{Z}_2^2$. Then a basis of its $\mathbb{Z}_2$-graded polynomial identities is:*

$$z_1 z_2, \quad [y_1, y_2].$$

Moreover, its $\mathbb{Z}_2$-graded cocharacter sequence is determined by:

- $\chi_{n,0}(\mathrm{UT}_2) = (n)$
- $\chi_{n,1}(\mathrm{UT}_2) = \sum_{a+b=n} m_{(a,b)}\Big((a,b) \otimes (1)\Big)$ *where* $m_{(a,b)} = a - b + 1$
- $\chi_{n,k}(\mathrm{UT}_2) = 0$ *for* $k \geqslant 2$

P r o o f. Let us use the notation of Example 5.5. In this case $m = n = 1$ and $A = B = \mathbb{F}$. Hence $T(A) = T(B) = T(A \oplus B)$ and this T-ideal is generated by the polynomial $[x_1, x_2]$. Hence, the polynomials $z_1 z_2$ and $[y_1, y_2]$ generate $T_2(R)$ by Theorem 3.3. The result about the cocharacter sequence follows by Theorem 3.4 and the Young rule. In fact we have:

- $\chi_{n,0}(R) = \chi_n(\mathbb{F} \oplus \mathbb{F}) = \chi_n(\mathbb{F}) = (n)$
- $\chi_{n,1}(R) = \sum_{p=0}^{n} (\chi_p(\mathbb{F}) \hat{\otimes} \chi_{n-p}(\mathbb{F})) \otimes \chi_{(1)} = \sum_{p=0}^{n} ((p) \otimes (n-p))^{S_n} \otimes (1) = \sum_{a+b=n} m_{(a,b)}\Big((a,b) \otimes (1)\Big)$ where $m_{(a,b)} = a - b + 1$

- $\chi_{n,k}(R) = 0$ for $k \geqslant 2$ $\square$

Let us remark that this result is equivalent to the decomposition of $V_n^{\mathbb{Z}_2}(UT_2)$ given in [25] by the use of the representation theory of the hyperoctahedral group.

In the same way, we can consider in the superalgebra $M_{m,n}(\mathbb{F})$ the $\mathbb{Z}_2$-graded subalgebra UT_{m+n} of the upper triangular matrices. In this case $A = \mathrm{UT}_m$, $B = \mathrm{UT}_n$ and the corresponding $\mathbb{Z}_2$-grading of UT_{m+n} is the elementary one induced by the vector $\mathbf{g} := (\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{n}) \in \mathbb{Z}_2^{m+n}$.
Assume $m \geq n$, then $T(A \oplus B) = T(A) \cap T(B) = T(A)$ and it is well know (see [21]) that the ordinary polynomial identities of UT_m are consequences of the polynomial $[x_1, x_2] \ldots [x_{2m-1}, x_{2m}]$. Hence in this case a basis of the graded polynomial identities of $\mathrm{UT}_{m+n} \subseteq M_{m,n}(\mathbb{F})$ is (see Corollary of [6])

$$z_1 z_2, \quad [y_1, y_2] \ldots [y_{2m-1}, y_{2m}], \quad z_1[y_1, y_2] \ldots [y_{2n-1}, y_{2n}].$$

Let us recall that in [26] has been proved that if G is a finite abelian group and the field $\mathbb{F}$ is algebraically closed of characteristic zero, then any G-grading on UT_m is isomorphic to an elementary one. Moreover, in [7], the authors describe generators for the ideals of the graded identities for any given elementary grading on UT_m.

We end the paper computing explicitly the cocharacters of a superalgebra which has a factorable T_2-ideal. Let us consider the following block-triangular matrix algebra:

$$R = \begin{bmatrix} A & U \\ 0 & B \end{bmatrix}$$

where $A = D_{1,1}(\mathbb{F})$, $B = M_{1,0}(\mathbb{F}) = \mathbb{F}$ and $U = M_{2\times 1}$. We have:

Proposition 5.7. *Let $m_{\mu,\nu}$ denote the multiplicities of $\mu \otimes \nu$ in the decomposition of the $\mathbb{Z}_2$-graded cocharacter $\chi_{k,l}(R)$. For $l > 1$, the non zero values of $m_{\mu,\nu}$ are listed in the following table:*

μ/ν	(l)	$(l-1,1)$
(a)	$a+1$	$a+1$
(a,b)	$2(a-b+1)$	$a-b+1$
$(a,b,1)$	$a-b+1$	

where $a, b, \neq 0$. For $l = 1$, the table of the $m_{\mu,\nu}$ is the following:

μ/ν	(1)
(a)	$a+1$
(a,b)	$2(a-b+1)$
$(a,b,1)$	$a-b+1$

Finally, for $l=0$ we have:

μ/ν	$\varnothing$
(a)	1
(a,b)	$2(a-b+1)$
$(a,b,1)$	$a-b+1$

Proof. Note that A is a $\mathbb{Z}_2$-regular superalgebra and hence $T_2(R) = T_2(A)T_2(B)$ by Theorem 5.1. Then we can apply the formula (2) to compute the graded cocharacter $\chi_{k,l}(R)$. For the superalgebra A we have clearly:

$$\chi_{k,l}(A) = (k) \otimes (l).$$

Similarly one has:

$$\chi_{k,0}(B) = (k) \quad \text{and} \quad \chi_{k,l}(B) = 0 \text{ for any } l \geq 1.$$

The computation of the multiplicities is based essentially on the following equation:

$$(\chi(A) \circ \chi(B))_{k,l} = \sum_{i=0}^{k}\sum_{j=0}^{l} \chi_{i,j}(A)\hat{\otimes}\chi_{k-i,l-j}(B) = \sum_{i=0}^{k} \chi_{i,l}(A)\hat{\otimes}\chi_{k-i,0}(B)$$

$$= \sum_{i=0}^{k}\Big((i)\otimes(l)\Big)\hat{\otimes}\Big((k-i)\otimes\varnothing\Big) = \sum_{i=0}^{k}\Big((i)\otimes(k-i)\Big)^{S_k}\otimes(l)$$

$$= \sum_{a+b=k} m_{(a,b)}(a,b)\otimes(l)$$

where $m_{(a,b)} = a-b+1$. $\square$

As a last example we study the $\mathbb{Z}_2$-graded structure of one among the PI-algebras of minimal exponent:

$$S = \begin{bmatrix} E & E \\ 0 & E_0 \end{bmatrix}$$

Indeed, S can be endowed with the natural $\mathbb{Z}_2$-grading

$$S_0 = \begin{bmatrix} E_0 & E_0 \\ 0 & E_0 \end{bmatrix} \qquad S_1 = \begin{bmatrix} E_1 & E_1 \\ 0 & 0 \end{bmatrix}.$$

The structure of the T-ideal of the ordinary polynomial identities of S has been described in [24]. Here we obtain:

Corollary 5.8. *The generators of $T_2(S)$ are the polynomials:*

$$[y_1, y_2][y_3, y_4], \quad (z_1 \circ z_2)[y_3, y_4], \quad [y_1, z_2][y_3, y_4],$$

$$[y_1, y_2]z_3, \qquad (z_1 \circ z_2)z_3, \qquad [y_1, z_2]z_3$$

where $u \circ v := uv + vu$.

The non zero values of the multiplicities $m_{\mu,\nu}$ of $\mu \otimes \nu$ in the decomposition of cocharacter sequence $\chi_{k,l}(S)$ of the superalgebra S are summarized in the following tables:

- *If $l > 1$*

μ/ν	(1^l)	$(2, 1^{l-2})$
(a)	$a+1$	$a+1$
(a,b)	$2(a-b+1)$	$a-b+1$
$(a,b,1)$	$a-b+1$	

- *If $l = 1$*

μ/ν	(1)
(a)	$a+1$
(a,b)	$2(a-b+1)$
$(a,b,1)$	$a-b+1$

- *If $l = 0$*

μ/ν	$\varnothing$
(a)	1
(a,b)	$2(a-b+1)$
$(a,b,1)$	$a-b+1$

P r o o f. Notice that S is isomorphic to the Grassmann envelope of the superalgebra R of the previous proposition. Hence we have (see [18]):

$$T_2(S) = T_2(R)^* = T_2(D_{1,1}(\mathbb{F}))^* T_2(\mathbb{F})^* = T_2(E)T_2(E_0).$$

The result concerning the cocharacter sequence follows by equation (1). □

Acknowledgements. I would like to thank Vincenzo Nardozza for many useful discussions during the preparation of this paper.

REFERENCES

[1] Y. Bahturin, S. K. Sehgal, M. Zaicev. Group Gradings on Associative algebras. *J. Algebra* **241**, *2* (2001), 677–698.

[2] A. Berele. Cocharacters of $Z/2Z$-graded algebras. *Israel J. Math.* **61** (1988), 225–234.

[3] F. Benanti, A. Giambruno, M. Pipitone. Polynomial identities on superalgebras and exponential growth. *J. Algebra* **269**, *2* (2003), 422–438.

[4] A. Berele, A. Regev. Codimensions of products and of intersections of verbally prime T-ideals. *Israel J. Math.* **103** (1998), 17–28.

[5] O. M. Di Vincenzo. Cocharacters of G-graded algebras. *Comm. Algebra* **24**(10) (1996), 3293–3310.

[6] O. M. Di Vincenzo, V. Drensky. The basis of the Graded Polynomial Identities for Superalgebras of Triangular Matrices. *Comm. Algebra* **24**, *2* (1996), 727–735.

[7] O. M. Di Vincenzo, P. Koshlukov, A. Valenti. Gradings on the algebra of upper triangular matrices and their graded identities. *J. Algebra* **275** (2004), 550–566.

[8] O. M. Di Vincenzo, R. La Scala. Block-triangular matrix algebras and factorable ideals of graded polynomial identities. *J. Algebra* (2004), (to appear).

[9] O. M. Di Vincenzo, V.Nardozza. $\mathbb{Z}_2$-graded cocharacters for superalgebras of triangular matrices. *J. Pure Appl. Algebra* (2004), (to appear).

[10] V. Drensky. Extremal varieties of algebras I. *Serdica Bulg. Math. Publ.* **13** (1987), 320–332 (in Russian).

[11] V. Drensky. Extremal varieties of algebras II. *Serdica Bulg. Math. Publ.* **14** (1988), 20–27 (in Russian).

[12] A. Giambruno, M. Zaicev. On codimension growth of finitely generated associative algebras. *Adv. Math.* **140** (1998), 145–155.

[13] A. Giambruno, M. Zaicev. Exponential codimension growth of PI-algebras: an exact estimate. *Adv. Math.* **142** (1999), 221–243.

[14] A. Giambruno, M. Zaicev. Minimal varieties of exponential growth. *Adv. Math.* **174** (2003), 310–323.

[15] A. Giambruno, M. V. Zaicev. Codimension growth and minimal superalgebras. *Trans. Amer. Math. Soc.* **355** (2003), 5091–5117.

[16] E. Formanek. Noncommutative invariant theory. *Contemp. Math.* **43**, (1985), 87–119.

[17] P. Halpin Some Poincaré series related to identities of 2×2 matrices. *Pacific J. Math.* **107** (1983), 107–115.

[18] A. R. Kemer. Varieties and $\mathbb{Z}_2$-graded algebras. *Math. USSR, Izv.* **25**, *2* (1985), 359–374.

[19] A. R. Kemer. Ideals of identities of Associative Algebras. Translations of Mathematical Monographs, vol **87**, Providence, RI: American Mathematical Society (AMS), 81 p.

[20] J. Lewin. A matrix representation for associative algebras. I. *Trans. Amer. Math. Soc.* **188** (1974), 293–308.

[21] Yu. N. Maltsev. A basis for the identities of the algebra of upper triangular matrices. *Algebra i Logika* **10** (1971), 393–400 (in Russian); English translation: *Algebra and Logic* **10** (1971), 242–247.

[22] A. Regev. Existence of identities in $A \otimes B$. *Israel J. Math.* **11** (1972), 131–152.

[23] L. H. Rowen. Polynomial identities in ring theory. *Pure Appl. Math.* **84**, Academic Press, New York, 1980.

[24] A. N. Stoyanova-Venkova. Some lattices of varieties of associative algebras defined by identities of fifth degree. *C. R. Acad. Bulgare Sci.* **35**, *7* (1982), 867–868 (in Russian).

[25] A. Valenti. The graded identities of upper triangular matrices of size two. *J. Pure Appl. Algebra* **172** (2002), 325–335.

[26] A. Valenti, M. Zaicev. Abelian gradings on upper-triangular matrices. *Arch. Math.* **80**, *1* (2003), 12–27.

[27] C. T. C. Wall. Graded Brauer groups. *J. Reine Angew. Math.* **213** (1963/1964), 187–199.

Dipartimento di Matematica
Università degli Studi di Bari
via Orabona 4, 70125 Bari, Italia
e-mail: `divincenzo@dm.uniba.it`

Received March 15, 2004