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COMPLEX HYPERBOLIC SURFACES OF ABELIAN TYPE

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Dedicated to Hironori Shiga on occasion of his 60-th birthday

Abstract. We call a complex (quasiprojective) surface *of hyperbolic type*, iff – after removing finitely many points and/or curves – the universal cover is the complex two-dimensional unit ball. We characterize abelian surfaces which have a birational transform of hyperbolic type by the existence of a reduced divisor with only elliptic curve components and maximal *singularity rate* (equal to 4). We discover a Picard modular surface of Gauß numbers of bielliptic type connected with the rational cuboid problem. This paper is also necessary to understand new constructions of Picard modular forms of 3-divisible weights by special abelian theta functions.

1. Introduction. The phenomenal change from a flat to a hyperbolic metric (with negative constant curvature) in complex dimension 2 has been discovered by F. Hirzebruch in [8]. More precisely, for some elliptic curves with

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complex multiplication field K he posed the following problem: Has the abelian surface $E \times E$ a model which is Picard modular ? Starting from $E \times E$ he constructed for the field K of Eisenstein numbers covering models of general type, which are compactifications of ball quotient surfaces, see also [1], I.4.A. In [9] we proved that they are Picard modular. This means that the corresponding uniformizing ball lattices are commensurable with the (full) Picard modular group $\mathbb{U}((2,1),\mathfrak{O}_K)$. For other CM-fields K the problem remained open.

Each compact hyperbolic surface is of general type. Therefore it cannot be a model of an abelian surface. So only non-compact surfaces with complete hyperbolic structure and an abelian model are possible. The abelian model has to support at least one elliptic curve coming from compactifying non-compact ball quotient surfaces.

In section 2 we define elliptic configurations D. For abelian surfaces B we give a simple counting criterion (see 2) in Theorem 2.5), which is necessary for the components of such divisor to bound a (neat) open ball quotient model of B. The model is constructed by blowing up all intersection points of D-components. With the method of cyclic coverings we prove that the criterion 2) is also sufficient (Theorem 2.5). For the proof in section 2 we combine the Miyaoka-Yau criterion for neat ball quotient surfaces with the Cyclic Covering Theorem. We use the theory of orbital heights on orbital surfaces developed in [3]. An important role plays a quotient of two special orbital heights, which appears as *singularity rate* of elliptic configurations on abelian surfaces. From the construction it is easy to see that all the coverings support (Zariski-locally) a fibration of explicit equation type $Y^n = f$, where $f = 0$ is a (local) equation of the divisor D on B, over an elliptic base curve $E \subset B$. The fibres are n-cyclic covers of an elliptic curve (with moving branch loci). That's what we call a *cycloelliptic fibration*.

For a neat 2-ball lattice Γ the invariant (Bergmann) metric on the ball $\mathbb{B}$ goes down to a complete Kähler-Einstein metric on $\mathbb{B}/\Gamma$ with negative constant holomorphic sectional curvature. Such metrics on surfaces we call hyperbolic. For the role of ball lattices in connection with Picard-Fuchs systems of partial differential equations we refer to [21], [6], [25]. The cusp points (or their resolving cusp curves) appear as degeneration locus of the hyperbolic metric.

On this way we discover new hyperbolic surfaces by finite quotients and coverings of $E \times E$, E elliptic CM-curve with Gauß number multiplication. In a forthcoming paper we will show that all these models are quotients of Picard modular groups of the field of Gauss numbers, which can be determined precisely. Among them the K3 (Kummer) surface $(E \times E)/\langle -\mathbf{1} \rangle$ is most interesting because it is closely connected with rational cuboid problems: Find rational cuboids with (some) rational diagonals. For details and new starts we refer to [20], [4], [7]. There is a modular approach to the congruence number problem (dedicated to rational rectangular triangles with rational area) due to Tunnell [23], see also Koblitz' book [15]. I think that a Picard modular approach to the rational cuboid problems is now possible and could be fruitful.

In [12] we already used the results of this paper for the construction of Picard modular forms of weights divisible by 3 by abelian functions. Our abelian method lifts old theta constructions of Jacobi and Weierstraß from the first to the second dimension. In contrast, Matsumoto's theta constants produce Picard modular forms for a Gauß lattice of weights divisible by 4, see [18]. The first explicit and complete construction of modular forms of a Picard modular group (of Eisenstein numbers with theta constants was given by H. Shiga in [22]. There is also an abelian approach which should be written down. For hyperbolic surfaces of abelian type we found in general explicit dimension formulas for automorphic forms of the corresponding ball lattices of all weights divisible by 3, see [13].

2. Numerical ball quotient criterion for abelian surface models. Let B be an abelian surface, $D \in \mathrm{Div}^+ B$ a reduced curve on B and $Y' = B' \longrightarrow B$ the blowing up of all intersection points of the irreducible components of D. The proper transform of D on Y' is denoted by D'. We look for curves D' such that the open surface $Y := Y' \setminus \mathrm{supp}\, D'$ is a neat ball quotient surface $\mathbb{B}/\Gamma$, where

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2;\ |z|^2 = |z_1|^2 + |z_2|^2 < 1\}$$

is the two-dimensional complex unit ball and

$$\Gamma \subset \mathrm{Aut}_{\mathrm{hol}}\, \mathbb{B} = \mathbb{PU}((2,1), \mathbb{C}) =:\ \mathbb{G}$$

is a neat ball lattice. A ball lattice is a discrete subgroup of $\mathrm{Aut}_{hol}\, \mathbb{B}$ with fundamental domain of finite volume with respect to a $\mathbb{G}$-invariant hermitian metric on $\mathbb{B}$. Γ is *neat* , iff the eigenvalues of each element $\gamma \in \Gamma$ generate a torsion free subgroup of $\mathbb{C}^*$. In this case the analytic quotient morphism $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ is the universal covering of $\mathbb{B}/\Gamma$ and the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ is a (projective) algebraic surface with finitely many *cusp singularities* compactifying $\mathbb{B}/\Gamma$. The cusp singularities are of simple elliptic type, which means that they have an elliptic curve as singularity resolution. For details and proofs we refer to [3], Ch.IV.

In order to get Y' as (smoothly compactified) neat ball quotient surface, it is clear that the irreducible components of D have to be elliptic curves. Its proper image D' on Y' must be a disjoint sum of elliptic curves. It follows that the intersections of two components of D have to be transversal. Fortunately, this condition is automatically satisfied. Namely, assume that two different elliptic curves F, F' on B meet in P. Then the embeddings F, $F' \hookrightarrow B$ can be lifted via universal coverings to embeddings of lines L, $L' \hookrightarrow \mathbb{C}^2$. So the tangent lines of F, F' at P, hence F, F' themselves, cross each other in P.

Moreover, it follows that the abelian surface B splits up to isogeny into a product of two elliptic curves. Namely, the existence of only one elliptic curve on B induces such a splitting.

Alltogether we found the following (necessary) basic conditions:

(i) all irreducible components of D are elliptic curves;

(ii) these components have (at most) transversal intersections with each other;

(iii) the irreducible components of D' have negative selfintersection;

(iv) B is isogeneous to a product of two elliptic curves.

On abelian surfaces B the third property is equivalent to

(iii′) each irreducible component of D intersects properly with at least one other component.

Namely, the adjunction formula

$$-e(C) = (C \cdot (C + K_X)), \tag{1}$$

C a smooth curve on a smooth (compact) surface X, K_X a canonical divisor, $e(C) = 2 - 2g(C)$ the Euler number of C, yields

$$0 = (E^2) + (E \cdot O) = (E^2) \tag{2}$$

for elliptic curves E on any abelian surface B because the canonical class of B is trivial. It becomes negative after blowing up some points of B if and only if at least one of these points lies on E.

Definition 2.1. *A reduced effective divisor D on an abelian surface B with only elliptic components is called* elliptic configuration. *It is called an* intersecting elliptic configuration *if and only if (additionally) there are (at least two) components intersecting each other properly.*

It is clear that the properties (i),(ii),(iii) $\sim$ (iii′) are satisfied for intersecting elliptic configurations. They could be used as definition. Namely, looking at the simultaneous universal covering of the abelian surface B and the embedded elliptic curve $E \hookrightarrow B$ via tangential spaces it is clear that E does not intersect another elliptic curve E' if and only if the affine tangential lines T_E and $T_{E'}$ at points on E or E', respectively, are parallel in the affine tangential plane T_B. The intersection must be transversal, so property (ii) is satisfied automatically. Moreover, if there are two components of D intersecting each other properly, then each third component has to intersect at least one of these two first components, because its universal covering line cannot be parallel to T_E and $T_{E'}$ at the same time. So, also the properties (iii′) $\sim$ (iii) are satisfied. It follows also that intersecting elliptic configurations are connected.

Let $Y' = B' \longrightarrow \hat{Y}$ be the contraction of all components of D'. The image $\hat{D}$ of D' is considered as set (or cycle) of *cusp points* . We consider (Y', D'), Y or $(\hat{Y}, \hat{D})$ as orbital surfaces in the sense of [3]. There we defined orbital Euler and signature heights $H_e(Y)$, $H_\tau(Y)$ of open orbital surfaces, namely:

$$H_e(Y) = e(Y') = \text{Euler number of } Y',$$

$$H_\tau(Y) = \tau(Y') - \frac{1}{3}(D'^2),\ \tau(Y') = \text{signature of } Y'.$$

We set

$$\mathrm{Prop}(Y) = \mathrm{Prop}(B, D) := H_e(Y) - 3H_\tau(Y).$$

In [3], see Ch. IV, (4.8.1), (4.8.2) we proved

Proposition 2.2. *If Y is a ball quotient, then* $\mathrm{Prop}(Y) = 0$.

Definition 2.3. *An intersecting elliptic configuration D on the surface B sa-tisfying* $\mathrm{Prop}(B, D) = 0$ *is called* proportional.

Let $S = S(D)$ be the set of intersection points of all pairs of D-components and $s := \#S$ its number of elements. For abelian surfaces B we know that

$$e(B) = 0 = \frac{1}{3}((K_B^2) - 2e(B)) = \tau(B),$$

hence

$$e(Y') = H_e(Y) = s,\ \tau(Y') = -s,\ \mathrm{Prop}(Y) = 4s + (D'^2). \tag{3}$$

Going back to B we write $D = \sum_{i=1}^{N} D_i$, D_i irreducible, and set

$$S_i = S(D_i) = S_D(D_i) := S \cap D_i,\ s_i := \#S_i.$$

Then we get with (1) for the proper transforms D_i' on Y' the selfintersections $(D_i'^2) = -s_i$, hence

$$(D'^2) = \sum (D_i'^2) = -\sum s_i,\ \operatorname{Prop}(Y) = 4s - (s_1 + \cdots + s_N), \tag{4}$$

and the

Corollary 2.4. *If B is an abelian surface with intersecting elliptic configuration D such that Y is a ball quotient, then*

$$4s = s_1 + \cdots + s_N. \tag{5}$$

The basic result of this paper is the following

Theorem 2.5. *Let A be an abelian surface, $C = \sum C_j$, an intersecting elliptic configuration on A, $s = \#S(C)$, $s_j = \#S(C_j)$ defined as above, $A' \longrightarrow A$ the blowing up of A at all points of $S(C)$, C' the proper transform of C and $A'_{fin} := A' \setminus \operatorname{supp} C'$. Then it holds that*

1) $4s \geqslant \sum s_j$.
2) *A'_{fin} is a neat ball quotient surface (with smooth compactification A') if and only if C is proportional, or, equivalently*
$$4s = \sum s_j.$$
3) *If the properties of C in 2) are satisfied, then A is isogeneous to the square $E \times E$ of an elliptic curve E.*

We start the proof with

Proposition 2.6. *Let $\overline{f} : B \longrightarrow A$ be an isogeny of abelian surfaces, C an intersecting elliptic configuration on A and $D := \overline{f}^{-1}(\operatorname{supp} C)$ the preimage of the curve C identified with its reduced inverse image. Then D is an intersecting elliptic configuration on B. If C is proportional, then also D is.*

P r o o f. Let E be an elliptic curve on A. By the base change property for étal morphisms (see e.g. [19], I, Prop. 3.3) the restriction $\overline{f}^{-1}(E) \longrightarrow E$ of $\overline{f}$ is étal, too. Especially, $\overline{f}^{-1}(E)$ is smooth, hence this preimage is a disjoint finite union of smooth irreducible curves. These curves have to be elliptic because this is the only possibility of unramified covers of elliptic curves by Hurwitz genus formula.

We proved that property (i) lifts from C to D. The lift of the intersection property (iii$'$) $\sim$ (iii) to D is obvious.

Now let $\sigma : X' \longrightarrow A$ be the blowing up of $S = S(C)$ and $\rho : Y' \longrightarrow B$ the blowing up of $S(D) = \overline{f}^{-1}(S)$ with proper preimages D', C' of D or C, respectively. Contracting D' and C' we get a commutative diagram

(6)
$$\begin{tikzcd}
B \arrow[d, "\overline{f}"'] & Y' \arrow[l, "\rho"'] \arrow[r, "\hat{q}"] \arrow[d, "f'"'] & \hat{Y} \arrow[d, "\hat{f}"] & Y = Y' \setminus D' \arrow[l] \arrow[d, "f"] \\
A & X' \arrow[l, "\sigma"'] \arrow[r, "\hat{p}"] & \hat{X} & X = X' \setminus C' \arrow[l]
\end{tikzcd}$$

with vertical Galois coverings of order d, say. Counting preimage points, it is easy to see, that together with $\overline{f}$ also f' is unramified. Namely, over the exceptional rational curve $M_P = \sigma^{-1}(P)$, $P \in S$, lie precisely d exceptional rational curves L_Q, $Q \in \overline{f}^{-1}(P)$. Therefore each $R \in M_P$ has at least d preimage points, each in one L_Q. But it cannot have more, because its number is restricted by the degree d of f'. Therefore f' is unramified everywhere. This property restricts to f. This means that the orbital quotient surface Y/G, $G = \mathrm{Ker}\,\overline{f}$, coincides with X. Hence $Y \longrightarrow X$ is a finite orbital morphism. By definition of orbital heights we get the relations

$$H_e(Y) = d \cdot H_e(X)\ ,\ H_\tau(Y) = d \cdot H_\tau(X)$$

(see [3], III, Prop. 3.7.6). Therefore the proportionality relation $H_e(X) = 3H_\tau(X)$ lifts to $H_e(Y) = 3H_\tau(Y)$. $\square$

Corollary 2.7. *If an abelian surface A has a proportional elliptic configuration C, then each abelian surface B isogeneous to A has infinitely many of them. More precisely, for arbitrary $N \in \mathbb{N}$ there exist on B a proportional elliptic configuration with more than N components.*

P r o o f. For $n \in \mathbb{N}$, $n > 1$, we consider the isogeny $\mu_n : A \longrightarrow A$ multiplying each point with n. Let E be a component of C such that $O = O_A \in E$. There is a unique addition on E with zero point O. The embedding $E \hookrightarrow A$ is a homomorphism, this means the addition on A restricts to the addition on E. The multiplication morphism with n on E is denoted by n_E. Since $n_E : E \longrightarrow E$ is an isogeny of degree $n^2 = \#\ \mathrm{Ker}\, n_E$, each point $P \in E$ has precisely n^2 preimages on E but n^4 preimages on A. Therefore $\mu_n^{-1}(E)$ consists of n^2 disjoint components consisting of the translates $E + t$ of E by n-division points $t \in A$.

More generally, we need not assume that E goes through O. Then for any point $Q \in E$ we have $E = Q + E_0$ with an elliptic curve E_0 through O. Counting preimages it is easy to see now, that also $\mu_n^{-1}(E) = \mu_n^{-1}(E_0) + \mu_n^{-1}(Q)$ consists of n^2 components. Its number of components is greater than N, if $\sqrt{n} > \sqrt{N}$. With notations and implication of Proposition 2.6 we know that $\overline{f}^{-1}(\mu_n^{-1}(C))$ is a proportional elliptic divisisor on B. Obviously, its number of components is also greater than N. $\square$

Corollary 2.8. *If an abelian surface B supports a proportional elliptic configuration, then it is isogeneous to $E \times E$ for a suitable elliptic curve E.*

P r o o f. With the assumption of the corrollary we know that B is isogeneous to $E_1 \times E_2$ for two elliptic curves E_1, E_2 (see iv). There exists an isogeny $E_1 \times E_2 \longrightarrow B$. By Proposition 2.6 it suffices to show that $E_1 \times E_2$ has no proportional elliptic configuration, if E_1 and E_2 are not isogeneous. We assume this latter property. Each elliptic curve F on $E_1 \times E_2$ must be a fibre of one of the natural projections of $E_1 \times E_2$ onto E_1 or E_2, because F cannot be a covering of E_1 and E_2 at the same time. Otherwise E_1 and E_2 would be isogeneous to F, hence to each other, in contradiction to our latter assumption. Therefore each elliptic configuration $D \in \operatorname{Div} E_1 \times E_2$ is a sum of horizontal fibres $H_n \cong E_1$ and vertical fibres $V_m \cong E_2$:

$$D = \sum_{m=1}^{M} V_m + \sum_{n=1}^{N} H_n.$$

We show that D is not proportional checking the proportionality condition (5) of Corollary 2.4. We have

$$s = \#S(D) = M \cdot N,\ \#S(V_m) = N,\ \#S(H_n) = M,$$

hence

$$4s = 4M \cdot N \neq M \cdot N + N \cdot M = \sum \#S(V_m) + \sum \#S(H_n). \qquad \square$$

Remark 2.9. We have the estimation

$$2s \leq s_1 + \cdots + s_N, \text{ with } s = s(D),\ s_j = s_j(D),$$

for arbitrary intersecting elliptic configurations $D = \sum_{i=1}^{N} D_i$ on abelian surfaces B.

Namely, on the right hand side we count each intersecting point of D at least twice because of (iii′). So a sum of fibres on $E \times E$ takes the minimal value 2 of the *singularity rate*

$$\sigma(D) = \left(\sum_{i=1}^{N} s_i \right) / s$$

of D. By the way we proved statement 3) of Theorem 2.5.

3. Cyclic coverings of general type. We want to prove that abelian surfaces with proportional elliptic configurations D become neat ball quotient after blowing up $S(D)$. For this purpose we look first for finite cyclic coverings of general type satisfying the (neat) proportionality condition $H_e = 3H_\tau$. The strategy is given by the following two general results.

Ball Uniformization Theorem 3.1 (see [11], Th. 0.1 or [10], Introduction)**.** *For an orbital surface* $\mathbf{X} = (X, \mathbf{Z})$ *the following conditions are equivalent:*

(i) $\mathbf{X}$ *has a ball uniformization*

(ii) *The proportionality conditions*

(Prop 2) $H_e(\mathbf{X}) = 3H_\tau(\mathbf{X}) > 0$

(Prop 1) $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C}) < 0$ *for all orbital curves* $\mathbf{C} \subset \mathbf{Z}$

are satisfied, and there exists a finite uniformization Y *of* $\mathbf{X}$, *which is of general type.*

Cyclic Cover Theorem 3.2 (cit. in [6], proof e.g. in [17])**.** *Let* V *be a smooth algebraic variety,* $d \geqslant 2$ *a natural number,* Δ *a reduced effective divisor on* V *whose linear equivalence class* $\overline{\Delta}$ *is divisible by* d *in* $\operatorname{Pic} V$. *Then:*

(a) *There exist* d*-sheeted cyclic coverings* $V(\overline{\delta}) \longrightarrow V$ *with branch locus* Δ *and totally branched there.*

(b) *These cyclic covers* $V(\overline{\delta})$ *are in one-to-one correspondence with the "*d*-th roots" (tensor language)* $\overline{\delta}$ *of* $\overline{\Delta}$ *in* $\operatorname{Pic} V$, *that means with all* $\overline{\delta} \in \operatorname{Pic} V$ *satisfying* $d \cdot \overline{\delta} = \overline{\Delta}$.

We start with an abelian surface B and a reduced divisor $D = \sum D_k$ on B with properties (i), (ii), (iii) $\sim$ (iii′). As in the upper row of diagram (6) we blow up the intersection point set $S = S(D)$. We use the notations there and assume that the class of D is divisible by $n > 1$ in $\operatorname{Pic} B$. Then also the class of the proper image $D' = \sum D'_k$ is n-divisible in $\operatorname{Pic} Y'$. By the Cyclic Cover Theorem there exists a n-cyclic covering $\zeta' : W' \longrightarrow Y'$ (totally) branched over D'. The surface W' is smooth because D' is a disjoint sum by (ii). The normalization of B in the function field $\mathbb{C}(W')$ along ζ' is denoted by $\overline{W}$. The components of the preimage of D'_k in W' are contractible because they have together with $\zeta^*(D_k)$

negative selfintersection. The latter is equal to $n\cdot(D_k^2)$, which is negative by (iii). Alltogether we get a commutative diagram with vertical n-cyclic coverings

$$\begin{tikzcd} \overline{W} \arrow[d, "\overline{\zeta}"'] & W' \arrow[l] \arrow[r] \arrow[d, "\zeta'"'] & \hat{W} \arrow[d, "\hat{\zeta}"] & W \arrow[l] \arrow[d, "\zeta"] \\ B & Y' \arrow[l, "\rho"'] \arrow[r, "\hat{q}"] & \hat{Y} & Y \arrow[l] \end{tikzcd} \tag{7}$$

In contrast to W', the surfaces $\overline{W}$ and $\hat{W}$ are not smooth. We use orbital heights for calculating the Chern numbers of W'. For this purpose we consider the Galois quotient Y' of W' as support of the orbital surface $\mathbf{Y}' = (Y', \mathbf{Z}')$ with orbital cycle $\mathbf{Z}' = \sum \mathbf{D}'_k$, where $\mathbf{D}'_k$ is the orbital curve nD'_k (without orbital points, because the curves D'_k do not intersect each other). Each component D'_k has a unique preimage D''_k on W' with identical restriction $\zeta'_k : D''_k \leftrightarrow D'_k$ of ζ'. According to [3], chapters II, III, we have the following orbital curve heights

$$h_\tau(D''_k) = (D''^2_k),\ h_e(\mathbf{D}'_k) = e(D'_k) = e(D_k) = 0,$$

$$h_\tau(\mathbf{D}'_k) = \frac{1}{n}\cdot(D'^2_k) = \frac{1}{n}(D_k^2 - s_k) = -\frac{s_k}{n},\ s_k = \#S(D_k).$$

and the orbital relation (degree formula)

$$h_\tau(D''_k) = (\deg \zeta'_k)\cdot h_\tau(\mathbf{D}'_k) = h_\tau(\mathbf{D}'_k).$$

because $W' \longrightarrow \mathbf{Y}'$ is a finite orbital covering. It turns out that

$$(D''^2_k) = -\frac{s_k}{n}.$$

The orbital heights of W', $\mathbf{Y}'$ are

$$H_e(W') = e(W'),\ H_\tau(W') = \tau(W'),$$

$$H_e(\mathbf{Y}') = e(Y') - \sum\left(1-\frac{1}{n}\right)h_e(\mathbf{D}'_k) = e(Y') = s = \#S,$$

$$H_\tau(\mathbf{Y}') = \tau(Y') - \frac{1}{3}\sum\left(n-\frac{1}{n}\right)h_\tau(\mathbf{D}'_k) = -s + \frac{1}{3}\left(1-\frac{1}{n^2}\right)\sum s_k$$

with relations

$$H_e(W') = (\deg\zeta')\cdot H_e(\mathbf{Y}') = n\cdot H_e(\mathbf{Y}'),$$

$$H_\tau(W') = (\deg \zeta') \cdot H_\tau(\mathbf{Y}') = n \cdot H_\tau(\mathbf{Y}').$$

We assume $n > 1$. Using the Riemann-Roch formulas $(K^2_{W'}) = 2e(W') + 3\tau(W')$ for the selfintersection of canonical class, $\chi(W') = \dfrac{1}{12}(e(W') + (K'^2_W))$ for the arithmetic genus, and $s \le -s + \sum s_k$ by Remark 2.9 it follows that

$$\begin{aligned} &e(W') = n \cdot e(Y') = n \cdot s > 0, \\ &\tau(W') = -n \cdot s + \frac{1}{3}\left(n - \frac{1}{n}\right) \sum s_k, \\ &(K^2_{W'}) = -n \cdot s + \left(n - \frac{1}{n}\right) \sum s_k \ge n \cdot s - \frac{1}{n} \sum s_k \\ &\chi(W') = \frac{1}{12}\left(n - \frac{1}{n}\right) \sum s_k > 0. \end{aligned} \tag{8}$$

Most interesting is the Chern quotient

$$\frac{c_1^2}{c_2}(W') = (K^2_{W'})/e(W') = -1 + \left(1 - \frac{1}{n^2}\right) \frac{1}{s} \sum s_k. \tag{9}$$

Denoting the singularity rate by

$$\sigma = \sigma(D) := \frac{1}{s} \sum s_k$$

we can write

$$\begin{aligned} &e(W')/s = n, \\ &\tau(W')/s = -n + \frac{1}{3}\left(n - \frac{1}{n}\right) \sigma(D), \\ &(K^2_{W'})/s = -n + \left(n - \frac{1}{n}\right) \sigma(D) \ge n - \frac{2}{n}, \\ &\chi(W')/s = \frac{1}{12}\left(n - \frac{1}{n}\right) \sigma(D), \\ &\frac{c_1^2}{c_2}(W') = -1 + \left(1 - \frac{1}{n^2}\right) \sigma(D). \end{aligned} \tag{10}$$

The estimation comes from $\sigma(D) \ge 2$, see Remark 2.9. For proportional divisors

D we have $\sigma(D) = 4$ by Corollary 2.4, hence

$$\begin{aligned} 3\tau(W')/s &= n - \frac{4}{n}, \\ (K^2_{W'})/s &= 3n - \frac{4}{n}, \\ 3\chi(W')/s &= n - \frac{1}{n}, \\ \frac{c_1^2}{c_2}(W') &= 3 - \frac{4}{n^2}. \end{aligned} \tag{11}$$

Proposition 3.3. *Let B be an abelian surface with intersecting elliptic configuration D, which is n-divisible in* $\operatorname{Pic} B$, *$n > 1$. Then each n-cyclic cover W' of Y' totally branched over D' is a smooth surface of general type. The contraction $W' \longrightarrow \overline{W}$ is the minimal singularity resolution. Moreover, W' is the unique minimal model in its birational equivalence class.*

Proof. We already mentioned that W' is smooth. Now we show that there is no exceptional curve of first kind (-1 line) on W'. Assume there is one, denote it by M. Then its ζ'-image L is rational too. On the abelian surface B there is no rational curve. Therefore $L = L_Q$ is the blowing up of a point $Q \in S(D)$. The $\overline{\zeta}$-preimage P of Q is a unique point because Q is the intersection of some components of D, say $Q \in D_k$, and $\overline{\zeta}^{-1}(D_k) \longrightarrow D_k$ is bijective. The point P is the contraction of $M =: M_P$. We have an orbital Galois covering $M \longrightarrow L$ with Galois group $G := G_P = \operatorname{Gal}(W'/Y') \cong \mathbb{Z}/n\mathbb{Z}$. The number of branch points coincides with the number $t(Q) = t_D(Q) \geq 2$ of elliptic components of D through Q. We calculate orbital heights of

$$\begin{aligned} \mathbf{L} &= (L_Q,\ t(Q) \text{ smooth curve germs of weight } n \text{ crossing } L_Q) \\ h_e(\mathbf{L}) &= e(L) - t(Q)\left(1 - \frac{1}{n}\right) = 2 - t(Q)\left(1 - \frac{1}{n}\right), \\ h_\tau(\mathbf{L}) &= (L^2) = -1. \end{aligned}$$

Therefore

$$\begin{aligned} e(M) = h_e(M) &= n \cdot h_e(\mathbf{L}) = (2 - t(Q))(n-1) + 2, \\ \text{genus } g(M) &= (2 - e(M))/2 = \frac{1}{2}(t(Q) - 2)(n-1), \\ (M^2) = h_\tau(M) &= n \cdot h_\tau(\overline{L}) = -n \leq -2. \end{aligned}$$

The curve M is rational if and only if $t(Q) = 2$, but $(M^2) < -1$. Therefore M is not exceptional of first kind. We proved that W' is minimal in its birational class, hence $W' \longrightarrow \overline{W}$ is the minimal singularity resolution.

The Kodaira dimension $\varkappa(Y')$ is not negative because B is abelian. For any non-constant morphism $X \longrightarrow Y'$, X an irreducible compact complex algebraic surface, it holds that $\varkappa(X) \geq \varkappa(Y')$. Since W' covers Y' finitely, we get $\varkappa(W') \geq 0$. Surfaces with non-negative Kodaira dimension have a unique minimal model. This proves the last statement of the proposition.

From (10) we know that the selfintersection of the canonical class of W' is positive. But for minimal surfaces X of Kodaira dimension 0 and 1 one knows that (K_X^2) vanishes (see e.g. [2]). Therefore the Kodaira dimension of W' is equal to 2. This means that W' is of general type. □

Now let A be an abelian surface with proportional elliptic configuration $C = \sum C_j$. It defines birational morphisms

$$A \xleftarrow{\ \sigma\ } X' \xrightarrow{\ \hat{p}\ } \hat{X} \longleftarrow X = X' \setminus \operatorname{supp} C'$$

as described in the bottom of Diagram (6) for B instead of A. Consider the isogeny $\overline{\mu} = \mu_n :\ A \longrightarrow A$ of multiplication with $n > 1$ of degree n^4. Following the proof of Corollary 2.7 we know that each component $E = Q + E_0$ of C has preimage

$$\overline{\mu}^{-1}(E) = \overline{\mu}^{-1}(E_0) + \overline{\mu}^{-1}(Q)$$

consisting of n^2 components, which are translations of each other. The corresponding sheaves on E are isomorphic (via the translations). So all of them represent the same element in $\operatorname{Pic} A$ consisting of isomorphy classes of invertible sheaves (line bundles). Therefore $\overline{\mu}^{-1}(E)$ and also $D = D_n \ := \overline{\mu}^{-1}(C)$ is n-divisible in $\operatorname{Pic} A$ (even n^2-divisible). Moreover, $\overline{\mu}^{-1}(C)$ is an elliptic proportional divisor by Proposition 2.6. We use it for the construction of n-cyclic coverings as in Diagram (7) with (A, D) instead of (B, D). Together with Diagram (7) we get the following tower of birational morphism triples (for each fixed n).

$$\begin{tikzcd}
\overline{W} \arrow[d, "\overline{\zeta}"'] & W' \arrow[l] \arrow[r] \arrow[d, "\zeta'"'] & \hat{W} \arrow[d, "\hat{\zeta}"] & W \arrow[l] \arrow[d, "\zeta"] \\
A \arrow[d, "\overline{\mu}"'] & Y' \arrow[l, "\rho"'] \arrow[r, "\hat{q}"] \arrow[d, "\mu'"'] & \hat{Y} \arrow[d, "\hat{\mu}_n"] & Y = Y' \setminus D' \arrow[l] \arrow[d, "\mu"] \\
A & X' \arrow[l, "\sigma"'] \arrow[r, "\hat{p}"] & \hat{X} & X = X' \setminus C' \arrow[l]
\end{tikzcd} \tag{12}$$

Now we are well-prepared for the

Proof of Theorem 2.5. 1). By the above diagrams – choose one for each natural number $n > 1$ – we dispose on a series of minimal surfaces $W' = W'_n = W'(\mu'_n, \zeta)$ of general type. The well-known Miyaoka-Yau Theorem says that the Chern quotient c_1^2/c_2 is not greater than 3 for smooth compact algebraic surfaces of general type. Combined with the quotient formula in (10) we get

$$\frac{c_1^2}{c_2}(W'_n) = -1 + \left(1 - \frac{1}{n^2}\right)\sigma(D) \leq 3$$

for all n. This is only possible if $\sigma(D) \leq 4$. This relation is the same as $\sigma(C) \leq 4$ by the next proposition. The latter relation coincides with 1) of Theorem 2.5.

Proposition 3.4. *The singularity rate of intersecting elliptic configurations on abelian surfaces is an isogeny invariant.*

This means that for isogenies $\overline{f} : B \longrightarrow A$, intersecting elliptic configurations C on A, $D = \overline{f}^{-1}(C)$ considered as reduced intersecting elliptic configuration on A (see Proposition 2.6), the singularity rates $\sigma(C)$ and $\sigma(D)$ coincide.

Proof. We use the notations of Diagram (6). From (4), (3) and the definition of $\mathrm{Prop}(Y)$ before follows that the singularity rate

$$\begin{aligned}\sigma(D) &= -(D'^2)/s = (4s - \mathrm{Prop}(Y))/s \\ &= (4H_e(Y) - H_e(Y) + 3H_\tau(Y))/H_e(Y) = 3(H_e(Y) + H_\tau(Y))/H_e(Y)\end{aligned}$$

is a quotient of orbital heights. But $f : Y \longrightarrow X$ is a $\mathbb{B}$-orbital unramified finite morphism. For each orbital height H the degree formula $H(Y) = d \cdot H(X)$, with $d = \deg f$, holds. Therefore

$$\sigma(D) = -3(H_e(Y) + H_\tau(Y))/H_e(Y) = -3(H_e(X) + H_\tau(X))/H_e(X) = \sigma(C)$$

$\square$

Corollary 3.5. *The Chern-quotients of the minimal surfaces* $W' = W'_n = W'(\mu'_n, \zeta)$ *of general type constructed in Diagram* (12) *approach the extreme value* 3 *for* $n \to \infty$ *if and only if the intersecting elliptic basic divisor* C *on* A *is proportional.*

Proof. This is now an immediate consequence of the last formula of (10):

$$\frac{c_1^2}{c_2}(W'_n) = -1 + (1 - \frac{1}{n^2})\sigma(D_n) = -1 + (1 - \frac{1}{n^2})\sigma(C).$$

with limit $-1 + \sigma(C)$. □

Proof of Theorem 2.5. 2). One direction has already been proved before the statement 2), see Corollary 2.4. Now assume that C is a proportional divisor on the abelian surface A. For an arbitrary fixed natural number $n > 1$ we construct diagram (12). The cyclic covering $\zeta : W \longrightarrow Y$ is unramified because we omitted the branch locus ($Y = Y' \setminus \operatorname{supp} D$). We consider again ζ as morphism in the category of open $\mathbb{B}$-orbital surfaces because we omitted elliptic curves with negative selfintersections. Together with C also D is proportional elliptic by Proposition 2.6. So we have the relation

$$\operatorname{Prop}(Y) = H_e(Y) - 3H_\tau(Y) = 0$$

by Definition 2.3 and (4). Multiplication with $n = \deg \zeta$ yields

$$\operatorname{Prop}(W) = n \cdot H_e(Y) - 3n \cdot H_\tau(Y) = H_e(W) - 3H_\tau(W) = 0.$$

The theorem of Miyaoka-Kobayashi-Yau (MKY) for open surfaces (generalizing the compact version, see e.g. [16]) says that an open surface Z with negative elliptic curve compactification Z' of general type satisfying $\operatorname{Prop}(Z) = 0$ is a neat ball quotient. This theorem is now part of the most general Ball Uniformization Theorem 3.1 (proved also by R. Kobayashi [16] in the case of surfaces of general type). The MKY-theorem is applicable to $Z = W$, because W' is of general type, see Proposition 3.3. Therefore W is a neat ball quotient, with Baily-Borel compactification $\hat{W}$.

Both ζ and μ_n are unramified coverings. Therefore X has the same universal covering as Y and W, namely the two ball $\mathbb{B}$. It follows that Y and X themselves are neat ball quotient surfaces. The proof of Theorem 2.5 is finished. □

4. Bisectional proportional elliptic configurations. It is not easy to find proportional elliptic configurations on abelian surfaces. Theorem 2.5,

3) and Corollary 2.7 reduce the existence problem to abelian biproduct surfaces $E \times E$, E an arbitrary elliptic curve. The endomorphism algebra is

$$\mathrm{End}^\circ E \times E = \mathrm{Mat}_2(\mathrm{End}^\circ E) = \mathrm{Mat}_2(\mathbb{Q}) \text{ or } \mathrm{Mat}_2(K),$$

K an imaginary quadratic number field. We concentrate our attention on the latter (decomposed CM-) case, which happens iff E has complex multiplication. Then we dispose on the matrix ring $\mathrm{Mat}_2(\mathfrak{O})$ acting on $E \times E$, $\mathrm{End}\, E \cong \mathfrak{O}$, $\mathfrak{O}$ an order of K, which is enough to produce a few special, but arithmetically important, examples.

As in linear algebra the action of $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}_2(\mathfrak{O})$ can be described by

$$E \times E \ni \begin{pmatrix} P \\ Q \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} := \begin{pmatrix} \alpha P+\beta Q \\ \gamma P+\delta Q \end{pmatrix} = \begin{pmatrix} \alpha(P)+\beta(Q) \\ \gamma(P)+\delta(Q) \end{pmatrix}$$

$G : E\times E \longrightarrow E\times E$ is an isogeny iff $\det G = \alpha\delta - \beta\gamma \neq 0$. It is an automorphism iff $G \in \mathbb{G}L_2(\mathfrak{O})$. The multiplicative semigroup of isogenies is denoted by $\mathrm{Isog}\, E\times E$. We identify

$$\mathrm{End}\, E \times E = \mathrm{Mat}_2(\mathfrak{O})\ , \ \mathrm{Aut}_O\, E \times E =: \mathrm{End}^*\, E \times E = \mathbb{G}l_2(\mathfrak{O}) \ (\textit{unit group}).$$

The isogenies G applied to fibres produce elliptic curves on $E \times E$, e.g.

$$E_1(G) := G(E \times O) = \left\{ \begin{pmatrix} \alpha P \\ \gamma P \end{pmatrix} ;\ P \in E \right\},$$
$$E_2(G) := G(O \times E) = \left\{ \begin{pmatrix} \beta Q \\ \delta Q \end{pmatrix} ;\ Q \in E \right\}$$

Transposing columns we get the same class of elliptic curves on $E \times E$ through O:

$$(\mathrm{Isog}\, E \times E)(E \times O) = (\mathrm{Isog}\, E \times E)(O \times E).$$

Identifying E with $E \times O$ the isogeny G induces an isogeny

$$g :\ E \leftrightarrow E \times O \longrightarrow G(E \times O),\ P \mapsto (P, O) \mapsto \begin{pmatrix} \alpha(P) \\ \gamma(P) \end{pmatrix}$$

with kernel

$$\mathrm{Ker}\, g = g^{-1}(O \times O) = E_{\alpha-\mathrm{tor}} \cap E_{\gamma-\mathrm{tor}} = \mathrm{Ker}\, \alpha \cap \mathrm{Ker}\, \gamma. \tag{13}$$

For each ideal $\mathfrak{J}$ of $\mathfrak{O}$ we set $E_{\mathfrak{J}-\mathrm{tor}} := \{T \in E;\ \mathfrak{J}T = O\}$.

Lemma 4.1. *For any $G \in \mathrm{Mat}_2(\mathfrak{O})$ as above, the restriction g to $E \times O$ is an isomorphism onto $G(E \times O)$ iff*
(a) $\mathrm{Ker}\, \alpha \cap \mathrm{Ker}\, \gamma = O$.
This condition is satisfied if

(b) $\mathfrak{I} := \mathfrak{O}\alpha + \mathfrak{O}\gamma = \mathfrak{O}$.
In the principal case $\mathfrak{O} = \mathfrak{O}_K$, both properties (a) and (b) are equivalent.

Proof. The first statement follows from (13). It is clear that

$$\operatorname{Ker}\alpha \cap \operatorname{Ker}\gamma = E_{\mathfrak{I}-\mathrm{tor}}, \tag{14}$$

hence (a) is a consequence of (b).

In any case we have $E = E(\mathbb{C}) = \mathbb{C}/\mathfrak{a}$, $\mathfrak{a}$ an ideal of $\mathfrak{O}$ with

$$[\mathfrak{a} : \mathfrak{a}]_K := \{c \in K;\ c\mathfrak{a} \subset \mathfrak{a}\}.$$

The (natural) torsion points of E are represented by K, more precisely, $E_{\mathrm{tor}} = K/\mathfrak{a}$. In the principal case $\mathfrak{O}$ is a Dedekind domain. Then we know for ideals $\mathfrak{I} \subsetneqq \mathfrak{O}$ that

$$[\mathfrak{a} : \mathfrak{I}]_K = \mathfrak{a} \cdot \mathfrak{I}^{-1} \supsetneqq \mathfrak{a}, \tag{15}$$

hence there is an element $c \in K \setminus \mathfrak{a}$ such that $c\mathfrak{I} \subseteq \mathfrak{a}$. The class $c \bmod \mathfrak{a}$ is a non-trivial $\mathfrak{I}$-torsion point of E. By (14) condition (a) is not satisfied. We proved the implication (a) $\Rightarrow$ (b) in the principal case. $\square$

Let p_1, p_2 be the projections of $E \times E$ onto the first or second factor, respectively. By abuse of language, the curve $C \subset E \times E$ is called a *horizontal (vertical)* section iff p_1 (p_2) induces an isomorphism $C \longleftrightarrow E$. It is called a *bisection*, iff C is simultaneously a horizontal and vertical section. The image curve $g(E) = G(E \times O)$ is a horizontal section iff the implication $\alpha(P) = \alpha(Q) \Rightarrow \gamma(P) = \gamma(Q)$ holds for all pairs $P, Q \in E$. Now the first three statements of the following corollary are immediately clear.

Corollary 4.2. *With the notations of the lemma it holds that:*
The image curve $G(E \times O)$ is a horizontal section iff $\operatorname{Ker}\alpha \subseteq \operatorname{Ker}\gamma$. It is a vertical section iff $\operatorname{Ker}\gamma \subseteq \operatorname{Ker}\alpha$. The curve $G(E \times O)$ is a bisection iff $E_{\mathfrak{I}-\mathrm{tor}} = \operatorname{Ker}\alpha = \operatorname{Ker}\gamma$. The morphism g is an isomorphism onto a bisection if and only if α and γ are units in $\mathfrak{O}$.

Proof. We have only to check the last statement. The if-direction is trivial. Together with (a) and (13) it is easy to see now that the isomorphy and bisectional assumptions are equivalent with

$$O = \operatorname{Ker}\alpha \cap \operatorname{Ker}\gamma = \operatorname{Ker}\alpha = \operatorname{Ker}\gamma.$$

Therefore the E-endomorphisms α and γ are invertible because they are also surjective. $\square$

We want to count intersection points of End$(E \times E)$-induced elliptic curves. It is immediately clear that for $G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $G' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ we have surjective homomorphisms

$$\begin{aligned} &\mathrm{Ker}_{E\times E}\begin{pmatrix} \alpha & -\alpha' \\ \gamma & -\gamma' \end{pmatrix} \longrightarrow E_1(G) \cap E_1(G') \\ &\mathrm{Ker}_{E\times E}\begin{pmatrix} \alpha & -\beta' \\ \gamma & -\delta' \end{pmatrix} \longrightarrow E_1(G) \cap E_2(G') \end{aligned} \tag{16}$$

with kernels $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\alpha' \cap \mathrm{Ker}\,\gamma \cap \mathrm{Ker}\,\gamma'$ and $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta' \cap \mathrm{Ker}\,\gamma \cap \mathrm{Ker}\,\delta'$, respectively. For instance, the surjection in the first row sends $\begin{pmatrix} P \\ Q \end{pmatrix}$ to $\begin{pmatrix} \alpha(P) \\ \gamma(P) \end{pmatrix} = \begin{pmatrix} \alpha'(Q) \\ \gamma'(Q) \end{pmatrix}$.

Lemma 4.3. *Assume that these kernels in* (16) *are finite. The number of intersection points are*

$$\begin{aligned} \#(E_1(G) \cap E_1(G')) &= N(\det \begin{pmatrix} \alpha & \gamma \\ \alpha' & \gamma' \end{pmatrix}) \\ \#(E_1(G) \cap E_2(G')) &= N(\det \begin{pmatrix} \alpha & \gamma \\ \beta' & \delta' \end{pmatrix}), \end{aligned}$$

where $N = N_{K/\mathbb{Q}}$ denotes the absolute norm.

Proof. Along the uniformizing exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}^2 \longrightarrow E \times E \longrightarrow 0$$

we lift, for instance, the curves $E_1(G)$, $E_2(G')$ to the universally covering lines

$$\mathbb{C}^2 \supset L_1(G):\ \gamma Z_1 - \alpha Z_2 = 0 \text{ or } L_2(G'):\ \delta' Z_1 - \beta' Z_2 = 0. \tag{17}$$

The number of intersection points of $E_1(G)$, $E_2(G')$ coincides with the norm of the determinant of the coefficient matrix of the system of two linear equations in (17). For this result we refer to [1], I.5.G (8), or originally, to [9], Lemma II.5. This proves the second equality of the lemma. The proof of the first is the same. □

Example 4.4 (Hirzebruch [8], see also [1], I.4.A)**.** Let $K = \mathbb{Q}(\rho)$, $\rho = e^{2\pi i/3}$ primitive third unit root, the field of Eisenstein numbers, $E = \mathbb{C}/\mathfrak{O}_K$ and $G = \begin{pmatrix} 1 & -\rho \\ 1 & 1 \end{pmatrix}$. Then $D = E \times O + O \times E + E_1(G) + E_2(G)$ is a proportional elliptic configuration on $E \times E$. After blowing up the zero point of $E \times E$ one gets a D-compactified neat ball quotient surface.

Proof. The elliptic curves $E_1(G)$, $E_2(G)$ are bisections by Corollary 4.2. Therefore they intersect each horizontal and vertical fibre in one point only. Since

$\det G$ is a unit, the curves $E_1(G)$, $E_2(G)$ have also only $O = O_{E\times E}$ as intersection point by Lemma 4.3. So D is an intersecting elliptic configuration with

$$s = \#S(D) = 1,$$
$$s(E\times O) = \#S_D(E\times O) = s(O\times E) = s(E_1(G)) = s(E_2(G)) = 1.$$

The proportionality condition ($4\cdot 1 = 1+1+1+1$) is satisfied. Now Theorem 2.5, 2) yields the conclusion. □

Fail Example 4/5 ([1], I.4.G,H)**.** For the ring $\mathfrak{O} = \mathbb{Z} + \mathbb{Z}i$ of Gaussian integers and the elliptic curve $E = \mathbb{C}/\mathfrak{O}$ the authors of [1] present on $E\times E$ the intersecting elliptic configuration

$$D = E_1(F) + E_2(F) + E_1(G) + E_2(G) + E_1(H) + E_2(H)$$

with $F = \left(\begin{smallmatrix} 0 & 1-i \\ 1 & 1 \end{smallmatrix}\right)$, $G = \left(\begin{smallmatrix} 1 & 1 \\ 1 & i \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1+i \end{smallmatrix}\right)$, $E_1(F) = O\times E$, $E_1(H) = E\times O$,

$$s(D) = 4,$$
$$s(E_1(F)) = s(E_2(F)) = s(E_1(G)) = s(E_2(G)) = s(E_1(H)) = s(E_2(H)) = 2.$$

The proportionality condition of Theorem 2.5 2) is not satisfied:

$$4\cdot 4 > 2+2+2+2+2+2.$$

So the example fails to be a Picard modular (after blowing up intersection points). The authors of [1] used this example for the construction of a smooth compact surface with $c_1^2 = 3c_2$ by means of a special Kummer covering of small degree. Knowing proportionality relation 2) of Theorem 2.5 we are able to construct a proportional elliptic configuration on this surface.

Main Example 4.6. Take the same abelian surface $E\times E$ as in the previous (fail) example. The matrices $G = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} i & -i \\ 1 & 1 \end{smallmatrix}\right)$ define four bisectional (see Corollary 4.2) elliptic curves

$$E_1 := E_1(G)\ ,\ E_2 := E_2(G)\ ,\ E_3 := E_1(H)\ ,\ E_4 := E_2(H)$$

on $E\times E$. With the formulas of Lemma 4.3 it is easy to calculate the *numerical intersection matrix* N (number of intersection points as entries) for these curves:

$$N = \begin{pmatrix} \infty & 4 & 2 & 2 \\ 4 & \infty & 2 & 2 \\ 2 & 2 & \infty & 4 \\ 2 & 2 & 4 & \infty \end{pmatrix}.$$

For a matrix $A \in \mathrm{Mat}_2(\mathfrak{O})$, $\det A \neq 0$, we set

$$(E \times E)_{A-\mathrm{tor}} := \mathrm{Ker}_{E\times E}\, A.$$

Since the adjoint matrix $A' \in \mathrm{Mat}_2(\mathfrak{O})$ of A satisfies $AA' = A'A = (\det A)\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, we have the inclusions

$$\begin{array}{rcl} (E \times E)_{A-\mathrm{tor}} \subseteq (E \times E)_{\det A-\mathrm{tor}} & = & E_{\det A-\mathrm{tor}} \times E_{\det A-\mathrm{tor}} \\ & & \quad\quad\quad\;\; \mid\!\cap \\ E_{N(\det A)-\mathrm{tor}} \times E_{N(\det A)-\mathrm{tor}} & = & (E \times E)_{N(\det A)-\mathrm{tor}}, \end{array} \tag{18}$$

$$E_{\det A-\mathrm{tor}} \subseteq E_{N(\det A)-\mathrm{tor}} \cong (\mathbb{Z}/N(\det A)\mathbb{Z})^2.$$

The latter relations transfer to our elliptic curves E_j, $j = 1, 2, 3, 4$. Restricting diagonal endomorphisms of $E \times E$ to E_j we get

$$E_{j,\lambda-\mathrm{tor}} = E_j \cap (E \times E)_{\lambda-\mathrm{tor}} \quad \text{for all} \quad \lambda \in \mathfrak{O}. \tag{19}$$

For $A = G$ or H we have $|\det A| = 2$, $N(\det A) = 4$. Therefore the four intersection points of E_1, E_2 or of E_3, E_4 coincide with the four 2-torsion points of these curves, respectively. For example, according to (16) we have

$$E_1(G) \cap E_2(G) \cong (E \times E)_{G-\mathrm{tor}} \subseteq E_{2-\mathrm{tor}} \times E_{2-\mathrm{tor}} = (E \times E)_{2-\mathrm{tor}}.$$

(The minus sign in the second column of G' in (16) can be omitted if only 2-torsion points appear in the kernel). Therefore, by (19),

$$E_1 \cap E_2 \subseteq (E \times E)_{2-\mathrm{tor}} \cap E_j = E_{j,2-\mathrm{tor}},\ j = 1, 2.$$

The inclusion is the identity because the number of elements is 4 on both sides. To be more explicit we set $T_{mn} := (T_m, T_n) \in E \cdot E$ with the vector

$$(T_0, T_1, T_2, T_3) = (0, \frac{1}{2}, \frac{1+i}{2}, \frac{i}{2}) \bmod \mathfrak{O}$$

of 2-torsion points of E and get

$$\begin{aligned} (E \times E)_{2-\mathrm{tor}} &= \{T_{mn};\ 0 \leqslant m, n \leqslant 3\}, \\ (E \times E)_{(1+i)-\mathrm{tor}} &= \{O, T_{02}, T_{20}, T_{22}\} \cong (\mathbb{Z}/2\mathbb{Z})^2; \\ E_{1,2-\mathrm{tor}} &= \{O, T_{11}, T_{22}, T_{33}\} = \langle T_{11}\rangle \times \langle T_{33}\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2. \end{aligned}$$

because E_1 is the diagonal curve on $E \times E$. We proved that

$$\begin{aligned} E_1 \cap E_2 &= \langle T_{11}\rangle \times \langle T_{33}\rangle, \\ E_{1,(1+i)-\mathrm{tor}} &= E_{2,(1+i)-\mathrm{tor}} = \{O, T_{22}\} = \langle T_{22}\rangle \cong \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

For further intersections one needs only to look at the inverses A^{-1} of matrices A constructed by pairs of two different columns taken from G and H. Namely, the columns $\mathfrak{c}$ of A^{-1} satisfy $A\mathfrak{c} \in \mathfrak{O} \times \mathfrak{O}$, therefore $\mathfrak{c} \bmod \mathfrak{O} \in E = \mathbb{C}/\mathfrak{O}$ belongs to $(E \times E)_{A-\mathrm{tor}}$. This allows already to fill the numerical intersection matrix N to get the following point intersection scheme P for E_1, E_2, E_3, E_4:

$$P = \begin{pmatrix} E_1 & \langle T_{11}\rangle\times\langle T_{33}\rangle & \langle T_{22}\rangle & \langle T_{22}\rangle \\ \langle T_{11}\rangle\times\langle T_{33}\rangle & E_2 & \langle T_{22}\rangle & \langle T_{22}\rangle \\ \langle T_{22}\rangle & \langle T_{22}\rangle & E_3 & \langle T_{13}\rangle\times\langle T_{31}\rangle \\ \langle T_{22}\rangle & \langle T_{22}\rangle & \langle T_{13}\rangle\times\langle T_{31}\rangle & E_4 \end{pmatrix}$$

The elliptic configuration $C := E_1 + E_2 + E_3 + E_4$ is not proportional:

$$S(C) = \{O, T_{11}, T_{22}, T_{33}, T_{13}, T_{31}\}\ ,\ \#S(E_k) = 4,\ k = 1, 2, 3, 4;$$

$$4 \cdot \#S(C) = 4 \cdot 6 > 4 + 4 + 4 + 4.$$

But we can enrich it by adding some horizontal and vertical fibres. We take

$$H_1 := E \times T_1,\ H_3 := E \times T_3,\ V_1 := T_1 \times E,\ V_3 := T_3 \times E$$

and consider the elliptic configuration

$$D := E_1 + E_2 + E_3 + E_4 + H_1 + H_3 + V_1 + V_3 = C + F \tag{20}$$

Since the elliptic curves E_k are bisections, they have only one intersection point with each fibre. The intersection indices are equal to 1. Identifying divisors with supports we have

$$S(F) = \{T_{11}, T_{33}, T_{13}, T_{31}\} = C \cap F \subset S(C),$$

hence

$$\begin{gathered} S = S(D) = S(C),\ S(E_k) = S_D(E_k) = S \subseteq (E_k),\ k = 1, 2, 3, 4, \\ S(H_m) = S_D(H_m) = S_F(H_m) = \{T_{1m}, T_{3m}\},\ m = 1, 3, \\ S(V_m) = S_D(V_m) = S_F(V_m) = \{T_{m1}, T_{m3}\}, m = 1, 3. \end{gathered}$$

Counting the intersection points of the components we get the proportionality relation

$$4 \cdot \#S = 4 \cdot 6 = 4 + 4 + 4 + 4 + 2 + 2 + 2 + 2 \tag{21}$$

we looked for.

With Theorem 2.5 we get

Proposition 4.7. *Blowing up $E \times E$ at $S(D)$, $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$, D the intersec-ting elliptic configuration (20), we get a compactified neat ball quotient surface $(E \times E)'$. The compactification divisor is the proper transform D' of D on $(E \times E)'$.*

5. Explicit cycloelliptic fibrations. We want to understand more explicitly our surface models W' as curve fibrations over elliptic curves. Since ball quotients are extreme from the metric (or other numerical) view point one should expect that specializations of the curves over finite fields have also extreme properties, which are interesting in Coding Theory. We present one of the simplest explicit example starting from an elliptic curve over its own function field. It is then easy to generalize the method to other cases.

Let $\mathfrak{k} = \mathbb{C}(x,y)$ be the function field of the elliptic curve $E: Y^2 = X^3 - X$ and $\widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}_{\mathbb{C}(x)}$ the normalization of the projective plane elliptic curve $\mathfrak{C}:\ T^2 = (U-x)(U+x)(U-1)(U+1)$ over the rational function field $\mathbb{C}(x)$. By base change from $\mathbb{C}(x)$ to $\mathfrak{k}$ we get the following Galois tower of curves over $\mathfrak{k}$:

$$\begin{array}{l} \widetilde{\mathfrak{C}}_{\mathfrak{k}} : V^2 = U^3 - U,\ T^2 = (U-x)(U+x)(U-1)(U+1) \\ \quad | \\ \mathfrak{C}_{\mathfrak{k}} : V^2 = U^3 - U \\ \quad | \\ \mathbb{P}^1_{\mathfrak{k}} \end{array}$$

with $(2:1)$-Galois quotient morphisms $(u,v,t) \mapsto (u,v) \mapsto u$. The top curve $\widetilde{\mathfrak{C}}_k$ is understood as normalization of the projective model of the space curve described by the two affine equations above. One has only to desingularise the point at infinity lying over $\infty_{\mathfrak{C}} = (0:1:0)$. The ramification locus of $\widetilde{\mathfrak{C}}_{\mathfrak{k}}$ over $\mathfrak{C}_{\mathfrak{k}}$ consists of six points:

$$Ram(\widetilde{\mathfrak{C}}_{\mathfrak{k}}/\mathfrak{C}_{\mathfrak{k}}) = \{(x,\pm y,0),\ (-x,\pm iy,0),\ (\pm 1,0,0)\};$$

the branch locus on $\mathbb{P}^1_{\mathfrak{k}}$ is

(22) $\{e_1,e_2,e_3,e_4;h_1,h_3\}=\{(x,y),(-x,iy),(x,-y),(-x,-iy);(1,0),(-1,0)\}.$

By Hurwitz' formula we get the genus

$$g(\widetilde{\mathfrak{C}}_{\mathfrak{k}}) = 1 + (g(\mathfrak{C}_{\mathfrak{k}})-1) + 6/2 = 4.$$

The elliptic curve $\mathfrak{C}_{\mathfrak{k}}/\mathfrak{k}$ is nothing else but the general fibre of the (vertical) projection $E \times E \longrightarrow E$ onto the first component. Looking back to the main example, especially to (20), we see that the branch locus of $\widetilde{\mathfrak{C}}_{\mathfrak{k}}/\mathfrak{C}_{\mathfrak{k}}$ is the intersection (pull back) of the bisectional elliptic curves E_1, E_2, E_3, E_4 and the horizontal fibres H_1, H_3 with the general fibre $\mathfrak{E}_\mathfrak{k}$. Namely, the set of the four bisections is the $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \rangle$-orbit of the diagonal curve in $E \times E$. Their equations on $E \times E$ are $(u, v) = ((-1)^k x, i^k y)$, $k = 0, 1$. On the other hand, the points $(1, 0)$ and $(-1, 0)$ are obviously the odd 2-torsion points on E or $\mathfrak{E}_\mathfrak{k}$.

On the global surface $E \times E$ with general fibre $\mathfrak{E}_\mathfrak{k}$ we add to the above six sections the vertical fibres V_1 and V_3 to get the divisor D as in (20). The framed surface $(E \times E, D)$ restricts to $(\mathfrak{E}_\mathfrak{k}, e_1 + e_2 + e_3 + e_4 + h_1 + h_3)$ with the same divisor as described in (22). The components of D are $E \times E$- isomorphic with each other. Therefore D is 8-divisible in $\mathrm{Pic}\, E \times E$, especially 2-divisible. By the Cyclic Cover Theorem 3.2 we dispose of a global 2-cyclic covering diagram (7) with $B = E \times E$. The framed 2-cyclic surface coverings $(W, D)/(E \times E, D)$ and also $(W', D')/((E \times E)', D')$ "restrict" to $\widetilde{\mathfrak{C}}_\mathfrak{k}/\mathfrak{E}_\mathfrak{k}$ over the general point $\mathrm{Spec}\, \mathfrak{k}$; but W/E and also W'/E "restrict" to $\widetilde{\mathfrak{C}}_\mathfrak{k}/\mathfrak{k}$. We see that W'/E is a genus 4 fibration over the horizontal elliptic basic curve $E : Y^2 = X^3 - X$. The fibres are the 2-cyclic coverings $C_{x,y}$ of the vertical elliptic curve $E : V^2 = U^3 - U$ with moving branch locus described in (22).

Proposition 5.1. *The surface W' supporting the cycloelliptic genus-4 family $\{C_{x,y}\}$ has a complete hyperbolic metric degenerating along D'. It is a minimal smooth surface of general type with Chern numbers*

$$\tau(W') = 0, \; e(W') = 12, \; (K^2_{W'}) = 24, \; \chi(W') = 3.$$

P r o o f. Since D is a proportional divisor on $E \times E$ by (21) we know from the Theorem 2.5 that $Y = (E \times E)' \setminus D'$ is a neat ball quotient. Repeating arguments, the unramified covering W of Y, see (7), has the same universal covering ball $\mathbb{B}$ as Y. For the calculation of Chern numbers we use (11) with $s = \#S(D) = 6$ and $\sigma = 4$ (proportionality). For the properties of minimality and general type we use the following

Corollary 5.2. *of Proposition* 3.3 *and Theorem* 2.5.
Let B be an abelian surface with proportional elliptic configuration D, which is n-divisible in $\mathrm{Pic}\, B$, *$n > 1$. Then each n-cyclic cover W' of Y' totally branched over D' is a smoothly compactified neat ball quotient surface of general type. The contraction $W' \longrightarrow \overline{W}$ is the minimal singularity resolution. Moreover, W' is the unique minimal model in its birational equivalence class.* □

Following this way and the proof of Theorem 2.5 one can construct further explicit n-cycloelliptic curve families over elliptic curves with Gauß or Eisenstein complex multiplication supporting a complete hyperbolic metric. The equations for the fibre curves are as explicit as the algebraic description of n-torsion points on the elliptic basic curve E. It is an open question to find such fibred hyperbolic models over other elliptic curves.

6. Going down to rational and Kummer surfaces. Let D' be the proper transform of an intersecting elliptic B-divisor D along the blowing up $\beta : Y' := B' \longrightarrow B$ of $S(D)$, B an abelian surface. We look for finite Galois quotients $X' = Y'/G$ of $Y' = B'$, which are ball quotients with compactification curve D'/G. This means that $X := (Y' \setminus D')/G = Y/G = \mathbb{B}/\Gamma$ for a suitable ball lattice $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$. Obviously, G must be a finite subgroup of

$$\mathrm{Aut}_{\mathrm{hol}}(B,D) := \{g \in \mathrm{Aut}_{\mathrm{hol}}(B); g(D) = D\}.$$

Proposition 6.1. *The surface $X = Y/G$ is a ball quotient $\mathbb{B}/\Gamma$ if D is proportional.*

Proof. From Theorem 2.5 we know that Y is an open neat ball quotient $\mathbb{B}/\Gamma'$. The action of G on Y lifts along the universal covering $\mathbb{B} \longrightarrow Y$. This yields an exact sequence of group homomorphisms

$$1 \longrightarrow \Gamma' := \pi_1(Y) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \tag{23}$$

with inclusion $\Gamma' \subseteq \Gamma$ without loss of generality. Therefore $X = Y/G = \mathbb{B}/\Gamma$ is a ball quotient. $\square$

We apply this proposition to our Main Example 4.6 on $E \times E$, $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$ with proportional elliptic divisor D described in (20). The bicyclic group

$$G := \langle \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle \times \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \rangle \cong (\mathbb{Z}/4\mathbb{Z})^2 \subset \mathrm{Aut}\, E \times E$$

acts transitively on the columns of

$$\left(\begin{smallmatrix} 1 & -1 & i & -i \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right) \mathrm{mod}^{\times} \mathfrak{O}^*$$

defining $C = E_1 + E_2 + E_3 + E_4$ via column pairs. Therefore G acts also on $S(C) = S(D)$, thereby transitively on its even part $\{O, T_{22}\}$ and on its odd part $S(F) = \{T_{11}, T_{13}, T_{31}, T_{33}\}$. Moreover, the generators of G

$$I := \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right) \; J := \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$$

send vertical and horizontal fibres to fibres of the same type. Therefore G acts on the four fibres through horizontal and vertical pairs of the odd points, hence transitively on $\{V_1, V_3\}$ and on $\{H_1, H_3\}$. Alltogether we have an action of G on

$D = C + F$. Along β the action pulls back to $(E \times E)'$, D' and to the inverse image

$$\tilde{D} = D' + L_{00} + L_{22} + L_{11} + L_{33} + L_{13} + L_{31}, \; L_{ij} := \beta^{-1}(T_{ij}) \cong \mathbb{P}^1 \tag{24}$$

of D with G-orbits

$$\begin{aligned} G(E_1) &= \{E_1, E_2, E_3, E_4\}, \\ G(H_1) &= \{H_1, H_3\}, G(V_1) = \{V_1, V_3\} \\ G(L_{00}) &= \{L_{00}\}, G(L_{22}) = \{L_{22}\}, \\ G(L_{11}) &= \{L_{11}, L_{33}, L_{13}, L_{31}\}. \end{aligned} \tag{25}$$

Corollary 6.2. *For each subgroup U of $G = \langle I, J\rangle$ the surface $(E\times E)'/U$ is a compactified ball quotient surface with cusp curve D'/U.*

Beside of interesting rational surface models among quotients of $E \times E$ by subgroups of G there is an important case closely connected with *Rational Cuboid Problems*, see [20], [7], [4]. We take $U = \langle -\mathbf{1}\rangle = \langle (IJ)^2\rangle$ to get a K3-quotient.

Corollary 6.3. *The Kummer surface $\overline{S} := (E \times E)/\langle -\mathbf{1}\rangle$ has the compactified ball quotient model $S' = (E \times E)'/\langle -\mathbf{1}\rangle$ with cusp divisor*

$$\overline{B}'_\infty = \overline{D}' = \overline{E}'_1 + \overline{E}'_2 + \overline{E}'_3 + \overline{E}'_4 + \overline{H}'_1 + \overline{H}'_3 + \overline{V}'_1 + \overline{V}'_3$$

being a disjoint sum of smooth rational curves

$$\overline{E}'_1, \overline{E}'_2, \overline{E}'_3, \overline{E}'_4, \overline{H}'_1, \overline{H}'_3, \overline{V}'_1, \overline{V}'_3,$$

which are the images of the D'-components along $(E \times E)' \longrightarrow S'$. The cusp singularities of the corresponding Baily-Borel model $\hat{S} = \widehat{\mathbb{B}/\Gamma_S}$ are rational of type $(2, 2, 2, 2)$. The open orbital ball quotient on

$$S = \mathbb{B}/\Gamma_S = (E \times E)' \setminus D')/\langle -\mathbf{1}\rangle$$

is

$$\mathbf{S} = \mathbb{B}/\mathbf{\Gamma_S} = (S, \mathbf{B}^*) = (S, \mathbf{B}_1^* + \mathbf{B}_0)$$

with open disconnected orbital 1-cycle

$$\mathbf{B}_1^* = \overline{\mathbf{L}}_{00}^* + \overline{\mathbf{L}}_{11}^* + \overline{\mathbf{L}}_{22}^* + \overline{\mathbf{L}}_{33}^* + \overline{\mathbf{L}}_{13}^* + \overline{\mathbf{L}}_{31}^*$$

with smooth rational components all of weight 2*, selfintersection* -2*, and with* 0*-cycle*

$$\mathbf{B}_0 = \overline{T}_{02} + \overline{T}_{20}$$

consisting of two isolated cyclic surface singularities of type $\langle 2,1\rangle$.

Notations. The upper index * means that we omit cusp points lying on the curve, and bar markes image curves along the $\langle -\mathbf{1}\rangle$-quotient maps. Rational cusp type $(2,2,2,2)$ means that the (rational) cusp curve is crossed by four curves of branch weight 2 and no other orbital curves, see [3], III.

Proof. It is easy to verify that $\overline{S}$ is a Kummer surface, whose minimal smooth model is K3. We refer to [24] or to [20], [7], [4] for this simple fact. The action of $-\mathbf{1}$ on $E\times E$ has precisely sixteen isolated fixed points, namely $(E\times E)_{2-\text{tor}} = \{T_{mn};\ 0\leqslant m,n\leqslant 3\}$. The image points $\overline{T}_{mn}$ are the singularities of $\overline{S}$, all of type $\langle 2,1\rangle$. In order to get S' we have to blow up six of them. Their preimages form a divisor

$$B_1' := \overline{L}_{00}' + \overline{L}_{11}' + \overline{L}_{22}' + \overline{L}_{33}' + \overline{L}_{13}' + \overline{L}_{31}',$$

which is a disjoint sum of -2-lines. The reduced branch cycle of the covering $(E\times E)' \longrightarrow S'$ is $B' = B_1' + B_0$, where B_0 is the sum of 10 points $\overline{T}_{kl}$ with double index set complementary to the index set used for the B_1'-components. Since the action of $-\mathbf{1}$ on each elliptic curve H_k, V_k, E_{k+1}, $0\leqslant k\leqslant 3$, is not trivial, their images $\overline{H}_k, \overline{V}_k, \overline{E}_{k+1}$ on $\overline{S}$, hence also the proper transforms $\overline{H}_k', \overline{V}_k', \overline{E}_{k+1}'$, are rational (and smooth). From Proposition 4.7 and Corollary 6.2 we know that $Y = (E\times E)'\setminus D'$ is a neat open ball quotient $\mathbb{B}/\Gamma_Y$. It follows immediately that $S = S'\setminus \overline{D}'$ is a ball quotient $\mathbb{B}/\Gamma_S$ with exact sequence

$$1 \longrightarrow \Gamma_Y \longrightarrow \Gamma_S \longrightarrow \langle -\mathbf{1}\rangle \cong \mathbb{Z}/2\mathbb{Z} \longrightarrow 1, \tag{26}$$

see Proposition 6.1 with S instead of X. Only the 2-torsion points T_{02} and T_{20} survive after removing D' from $(E\times E)'$. Obviously, the ramification indices the B_1'-components are all equal to 2. Since $\mathbb{B}\longrightarrow Y$ is unramified, the ramified coverings $\mathbb{B}\longrightarrow S$ and $Y\longrightarrow S$ have the same orbital cycle. So we get the orbital cycle $\mathbf{B}^*$ as defined in the corollary. $\square$

7. The Kummer surface of rational cuboid problem and other quotients are Picard modular. In a forthcoming paper we will show that the cycloelliptic covers and the U-quotients, $U\subset\langle I,J\rangle$, of the main example, especially the above orbital Kummer surface, are Picard modular. More precisely, the corresponding ball lattices are well-determined congruence subgroups of $\Gamma := \mathbb{SU}((2,1),\mathbb{Z}+\mathbb{Z}i)$. Let $\pi := 1+i$ be the Gauss prime dividing 2. Consider the inclusion chain

$$\Gamma(4) \longrightarrow \Gamma(2\pi) \longrightarrow \Gamma(2) \longrightarrow \Gamma(\pi) \longrightarrow \Gamma$$

of principal congruence subgroups of Γ. The index

$$[\Gamma : \Gamma(4)] = \frac{1}{4} \cdot 4^8 \cdot (1 - 2^{-2}) \cdot (1 - 0 \cdot 2^{-3}) = 3 \cdot 2^{12}$$

can be read off from the general (2,1)-unitary index formula for natuaral principal congruence subgroups in [3], Proposition 5A.2.14. We refine the chain by the following diagram of inclusions:

$$\begin{tikzcd}
 & \Gamma(4) \arrow[r, "="] \arrow[d] & \Gamma(4) \arrow[d, "Z_2^5"] & & \\
 & {} \arrow[d] & \Gamma(2\pi) \arrow[d, "Z_2"] & & \\
Z_2^8 \; \{ & {} \arrow[d] & \Gamma_{E\times E} \arrow[r, "="] \arrow[d, "\langle -1 \rangle"] & \Gamma_{E\times E} \arrow[d] & \\
(27) & {} \arrow[d] & \Gamma_{K3} \arrow[dl, "\langle \overline{J}^2 \rangle"] \arrow[dr, "\langle \overline{I}, \overline{J} \rangle"'] & {} \arrow[d] & \{ \; \langle I, J \rangle \cong Z_4^2 \\
 & \Gamma(2) \arrow[r, "Z_2"'] \arrow[d] \arrow[dr, "Z_2^3"] & {} \arrow[r, "Z_2"'] & \Gamma_2 \arrow[dl, "Z_2"'] & \\
2\mathbb{O} \; \{ & {} \arrow[d] & \Gamma(\pi) \arrow[d, "S_3"] & & \\
 & \Gamma \arrow[r, "="] & \Gamma & &
\end{tikzcd}$$

At the arrows we wrote the corresponding factor groups. For instance, $\Gamma/\Gamma(2)$ is the binary octahedron group $2\mathbb{O}$ of order 48 defined as preimage of the octahedron group $\mathbb{O}$ along the classical group epimorphism $\mathbb{SU}(2) \longrightarrow S\mathbb{O}(3)$ with kernel $\langle -1\rangle$. This has been proved in [10], Proposition 8.3. In the same paper, see (35) in section 8 there, we proved that $\Gamma(2)/\Gamma(4)$ is a power of Z_2, where Z_2 is the cyclic group of order 2. Comparing indices we get $\Gamma(2)/\Gamma(4) \cong Z_2^8$.

The corresponding diagram of Galois coverings of ball quotient surfaces is the following one:

(28)

$$\begin{tikzcd}
 & X'_{\Gamma(4)} \arrow[r, "="] \arrow[dddd] & X'_{\Gamma(4)} \arrow[d, "Z_2^5"] & & \text{general type} \\
 & & X'_{\Gamma(2\pi)} \arrow[d, "Z_2"] & & \text{general type} \\
Z_2^8\,\{ & & (E\times E)' \arrow[r, "="] \arrow[d, "\langle -1\rangle"] & (E\times E)' \arrow[dd] & \text{abelian} \\
 & & X'_{K3} \arrow[dl, "\langle \overline{J}^2\rangle"] \arrow[dr, "\langle \overline{I},\overline{J}\rangle"'] & & \}\ \langle I,J\rangle \cong Z_4^2 \\
F' \arrow[r, "="] & X'_{\Gamma(2)} \arrow[r, "Z_2"'] \arrow[dr, "Z_2^3"'] \arrow[dd] & {} \arrow[r, "Z_2"'] & (\mathbb{P}^1\times\mathbb{P}^1)' \arrow[dl, "Z_2"] & \text{rational} \\
2\mathbb{O}\,\{ & & (\mathbb{P}^2_{\mathrm{Apoll}})' \arrow[d, "S_3"] & & \text{rational} \\
 & X'_\Gamma \arrow[r, "="] & X'_\Gamma & & \text{rational}
\end{tikzcd}$$

Except for the $K3$-surface X'_{K3} we announced the rough Kodaira classification type of the surfaces of each line in the last column. For the non-general types we announce also the fine classifications:

$E \times E$ is the abelian surface of our main example with $E \cong \mathbb{C}/\mathbb{Z}+\mathbb{Z}i$, and $(E \times E)'$ the blowing up of $E \times E$ at the six intersection points of eight elliptic components of the divisor (20).

$\mathbb{P}^2_{\text{Apoll}}$ denotes the projective plane with the *Apollonius cycle* consisting of a plane quadric together with three tangent lines, and $(\mathbb{P}^2_{\text{Apoll}})'$ is the blowing up of $\mathbb{P}^2$ at the three tangent points, which are precisely the cusp points of the Baily-Borel compactification of $\mathbb{B}/\Gamma(\pi)$, see [10] or [11].

On $\mathbb{P}^1 \times \mathbb{P}^1$ one finds the three cusp points on the diagonal curve. They have to be blown up to get the model $(\mathbb{P}^1 \times \mathbb{P}^1)'$ in the diagram.

We remark that $X'_{\Gamma(2)}$ comes near to the Picard modular Theta surface constructed by van Geemen in [5], which could not precisely classified until now. Our $X'_{\Gamma(2)}$ is understood now as a special degeneration of $E7$-del-Pezzo surfaces. In simpler words, we found the following construction. Take four points on $\mathbb{P}^2$ in general position. The configuration of six lines through the pairs of the four points is known as *complete quadrilateral.* The quadrilateral, considered as plane curve, has seven singular points: four intersection points of three lines and three intersection points of precisely two lines of the configuration. The blowing up of these seven points yield the smoothly compactified ball quotient surface $X'_{\Gamma(2)}$ of Diagram (28). The proper transforms of the six lines have selfintersection -2 on $X'_{\Gamma(2)}$. So they can be contracted to singular points. The arising surface $\hat{X}_{\Gamma(2)}$ is the Baily-Borel compactification of $\mathbb{B}/\Gamma(2)$ with these six cusp points.

The link with Picard modular groups comes with the Apollonius model. This main point of proof is well prepared in [10] or [11]. It needs also some effort to determine the factor groups in Diagram (27) precisely and the orbital cycles with their weights. Then one compares with the quotients of $(E \times E)'$ and discovers coincidences. This will be done in a forthcoming paper dedicated to Picard modular forms.

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