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LINEARLY NORMAL CURVES IN $\mathbf{P}^n$

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Abstract. We construct linearly normal curves covering a big range from $\mathbf{P}^n$, $n \geq 6$ (Theorems 1.7, 1.9). The problem of existence of such algebraic curves in $\mathbf{P}^3$ has been solved in [4], and extended to $\mathbf{P}^4$ and $\mathbf{P}^5$ in [10]. In both these papers is used the idea appearing in [4] and consisting in adding hyperplane sections to the curves constructed in [6] (for $\mathbf{P}^3$) and [15, 11] (for $\mathbf{P}^4$ and $\mathbf{P}^5$) on some special surfaces. In the present paper we apply the same idea to the curves lying on some rational surfaces from $\mathbf{P}^n$, constructed in [12, 3, 2] (see [13, 14] also).

1. Introduction. We shall work over an algebraically closed field k of any characteristic. We shall use the standard notations from [8].

Definition 1.1. *A smooth, irreducible (projective) curve $C \subset \mathbf{P}^n$ $(n \geq 3)$ is called* **linearly normal** *if it is non-degenerate (i.e. not contained in any hyperplane) and it is not a projection of a curve from a bigger projective space.*

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Remark 1.2. Let $C \subset \mathbf{P}^n$ be a smooth, irreducible curve and let's denote by $\mathcal{I}_C$ its ideal sheaf. Then: C is linearly normal $\Leftrightarrow h^j(\mathcal{I}_C(1)) = 0$, $j = 0, 1 \Leftrightarrow C$ is embedded in $\mathbf{P}^n$ using a complete linear system.

Remark 1.3. From the Riemann-Roch Theorem we deduce immediately that, if $C \subset \mathbf{P}^n$ is a linearly normal curve of degree d and genus g, then $g \geq d - n$.

For $n \in \mathbf{Z}$, $n \geq 3$, let's denote by $HC(n)$ and $LN(n)$, respectively, the following Problems:

$HC(n)$: For which pairs (d, g) of positive integers there is a smooth, irreducible, non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g?

$LN(n)$: For which pairs (d, g) of positive integers there is a (smooth, irreducible) linearly normal curve $C \subset \mathbf{P}^n$ of degree d and genus g?

We call $HC(n)$ the Halphen-Castelnuovo Problem.

Definition 1.4. *A pair of positive integers (d, g) is called a* **gap for HC(n)** *if there is no smooth, irreducible, non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g. We analogously define a* **gap for LN(n)**.

We recall from [7] the Harris-Eisenbud numbers $\pi_p = \pi_p(d, n)$ (for $n \geq 3$ and $0 \leq p \leq n - 2$), namely

$$\pi_p(d, n) := \frac{m_p(m_p - 1)}{2}(n + p - 1) + m_p(\varepsilon_p + p) + \mu_p, \tag{1.1}$$

where

$$m_p = m_p(d, n) := [(d - 1)/(n + p - 1)]_* \tag{1.2}$$

(we denote by $[x]_*$ the integer part of the real number x)

$$\varepsilon_p = \varepsilon_p(d, n) := d - 1 - m_p(n + p - 1) \tag{1.3}$$

$$\mu_p = \mu_p(d, n) := \max(0, [(p - n + 2 + \varepsilon_p)/2]_*). \tag{1.4}$$

We remark that $\mu_0 = 0$ and $\pi_p = d^2/(2(n + p - 1)) + O(d)$.

Theorem 1.5 (Castelnuovo, [1, 7]). *If $C \subset \mathbf{P}^n$ is a smooth, irreducible, non-degenerate curve of degree d and genus g, then $d \geq n$ and $0 \leq g \leq \pi_0(d, n)$.*

In [4] (for $n = 3$) and [10] (for $n = 4, 5$) is proved the following

Theorem 1.6. *If $n \in \{3, 4, 5\}$, $d, g \in \mathbf{Z}$, $d \geq n$ and $g \geq d - n$ (see Theorem 1.5 and Remark 1.3), then the pair (d, g) is a gap for $LN(n)$ if and only if it is a gap for $HC(n)$.*

Now it is natural to ask whether a similar equivalence (under the restriction $g \geq d-n$ from Remark 1.3) takes place in projective spaces $\mathbf{P}^n$ of dimension $n \geq 6$. We'll show in Section 3 (Examples 3.1, 3.2) that the answer is negative for $n \in \{6,7\}$. More precisely, we'll produce examples of pairs (d,g) such that $g = \pi_2(d,n)+1$, $g \geq d-n$, $n \in \{6,7\}$ which *are gaps* for $LN(n)$, although they *are not gaps* for $HC(n)$. With these examples in mind, we restrict ourselves to prove the following:

Theorem 1.7. *If* $n \in \{6,7\}$, $d,g \in \mathbf{Z}$, $d \geq n$ *and* $d-n \leq g \leq \pi_2(d,n)$, *then* (d,g) *is not a gap for* $LN(n)$.

Remark 1.8. From [3] and [2] we already know that the pairs (d,g) as in Theorem 1.7 are not gaps for $HC(n)$ ($n \in \{6,7\}$). We shall use this in Section 3 while proving Theorem 1.7.

As for the case $n \geq 8$, the Problem $HC(n)$ is not yet solved. For this case we'll prove in our next Theorem 1.9 that there are no gaps for $LN(n)$ in a big range $F(n)$ defined in the sequel. We first recall from [12, 13, 14] the followings numbers $\alpha_p = \alpha_p(d,n)$ ($n \geq 5$, $0 \leq p \leq n-2$, $n,p \in \mathbf{Z}$):

$$\alpha_p(d,n) := \frac{x_p(x_p-1)}{2}(n+p-1) + x_p(t_p+p) + u_p, \tag{1.5}$$

where

$$x_p = x_p(d,n) := [(d-a_p^n)/(n+p-1)]_* \tag{1.6}$$

$$a_p = a_p^n := [(n-p)/2]_* + 1 \tag{1.7}$$

$$t_p = t_p(d,n) := d-1-x_p(n+p-1) \tag{1.8}$$

$$u_p = u_p(d,n) := [(p-n+1+t_p(d,n))/2]_*. \tag{1.9}$$

We also recall from [14] the following numbers $\alpha''_p = \alpha''_p(d,n)$ ($n \geq 8$, $p = n-\delta+1$, $\delta \in \{2,3,4\}$):

$$\alpha''_{n-\delta+1}(d,n) := G_d^\delta(J_\delta(d,n),n) \tag{1.10}$$

where

$$J_\delta(d,n) := [(d+n-\delta)/(2n-\delta)]_* \tag{1.11}$$

$$G_d^{\delta}(r,n) := r\left(d - nr + \frac{1}{2}\delta(r-1)\right). \tag{1.12}$$

Further, for $n \geq 8$ we define some domains from the (d,g)-plane (of curves from $\mathbf{P}^n$, where d is the degree and g the genus):

$$F'^n_{n-3} : d - n \leq g \leq \begin{cases} \alpha''_{n-3}(d,n) - 1, 2n+1 \leq d \leq 5(n-2) \\ \alpha_{n-3}(d,n) - 1, d \geq 5n - 9 \end{cases} \tag{1.13}$$

$$F^n_p : \alpha_{p+1}(d,n) \leq g \leq \alpha_p(d,n),\ d \geq \frac{1}{3}(9p + 5n + 15) \tag{1.14}$$

for $n/3 \leq p \leq n-4$.

$$F'^n_k : \begin{cases} \alpha_k(d,n), & \text{if} \quad n = 3k \\ \alpha_{k+1}(d,n), & \text{if} \quad n = 3k+1, 3k+2 \end{cases} \leq g \leq A(d,n), d \geq d_2(n), \tag{1.15}$$

where
(1.16)

$$d_2(n) = \begin{cases} \max(10k, 4k + \frac{1}{6}(-3 + (4k-1)\sqrt{24k-33})), & \text{if} \quad n = 3k \\ \max(10k+3, \frac{4k}{4k+1}\left(1 + \sqrt{32k^3 + 16k^2 - 2k - 1}\right)), & \text{if} \quad n = 3k+1 \\ \max(10k+6, 9k+4+(2k+1)\sqrt{48k+6}), & \text{if} \quad n = 3k+2 \end{cases}$$

$$A(d,n) = \begin{cases} \pi_k(d,n), & \text{if} \quad n = 3k, 3k+1 \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & \text{if} \quad n = 3k+2 \end{cases} \tag{1.17}$$

(see (1.1)–(1.4)).

We define now the domain (from (d,g)-plane):

$$F(n) := F'^n_{n-3} \cup \left(\bigcup_{n/3 \leq p \leq n-4} F^n_p \right) \cup F'^n_k \tag{1.18}$$

We are ready now to state our

Theorem 1.9. *If $n \geq 8$ and $(d,g) \in F(n)$, then (d,g) is not a gap for $LN(n)$.*

Remark 1.10. 1) In order to prove Theorem 1.6 it is enough to construct linearly normal curves in the range $d - n \leq g \leq \pi_1(d,n)$ $(n \in \{3,4,5\})$, because everything follows then from Theorem 1.5 and the inequality $\pi_1(d,n) \geq \pi_0(d,n+1)$, $(\forall)n \geq 3$ (this has been already noted in [10]).

2) It is obvious that, both in Theorem 1.6 (see the previous Remark 1.10.1) and in Theorem 1.7 it is proved, in fact, the existence of linearly normal curves in the range $d \geq n$, $d-n \leq g \leq \pi_k(d,n)$, where $k = [n/3]_*$, for $n \in \{3,4,5,6,7\}$.

3) Although the range $F(n)$ (see (1.18)) from Theorem 1.9 looks quite complicated, it is easy to see, using Lemma 2.4 b) from Section 2, that in fact it is

$$d-n \leq g \leq \begin{cases} \alpha_k(d,n), & \text{if} \quad n=3k \\ \alpha_{k+1}(d,n), & \text{if} \quad n=3k+1, 3k+2 \end{cases}$$

for $5d-9 \leq d < d_2(n)$, and

$$d-n \leq g \leq \begin{cases} \pi_k(d,n), & n=3k, 3k+1 \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & n = 3k+2 \end{cases}$$

for $d \geq \max(5n-9, d_2(n))$.

It follows (see Remark 1.10.2) that the domain from the (d,g)-plane covered with linearly normal curves from $\mathbf{P}^n$ by Theorem 1.9 is of the same type as the domain from the (d,g)-plane covered by Theorems 1.6 and 1.7 (for $\mathbf{P}^3, \mathbf{P}^4, \mathbf{P}^5, \mathbf{P}^6, \mathbf{P}^7$) for $d >$ a function of degree $\frac{1}{2}$ in $n (= d_2(n))$ (the perturbation appearing for $n \equiv 2 \pmod 3$ is necessary, as pointed in [2])

We'll give the proofs of the Theorem 1.7 in Section 3 and of the Theorem 1.9 in Section 4, while in Section 2 we'll recall some constructions and results from [12, 2, 3, 4, 10, 13, 14] necessary in Sections 3, 4.

2. Curves in $\mathbf{P}^n$ on some rational surfaces.

First, we recall from [14] the next Theorem 2.1, essentially proved in [11] for $\delta = 2$ and in [3] for $\delta \in \{3,4\}$.

Theorem 2.1 ([11, 3, 14]). *If $\delta \in \{2,3,4\}$, $n \in \mathbf{Z}$, $n \geq 2\delta - 1$, $d, g \in \mathbf{Z}$, $d \geq 2n+1$ and*

$$0 \leq g \leq \begin{cases} \alpha''_{n-\delta+1}(d,n), & \text{if} \quad \delta = 2 \\ \alpha''_{n-\delta+1}(d,n) - 1, & \text{if} \quad \delta \in \{3,4\} \end{cases}$$

(for the definition of $\alpha''_{n-\delta+1}$ see (1.10)), then there is a smooth, irreducible, non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g, lying on a rational surface of degree $2n-\delta$ from $\mathbf{P}^n$. If $\delta \in \{3,4\}$ and $g \geq d-n$ the previous curves can be choosen to be linearly normal.

We further recall the main results from [12]. In this paper we constructed smooth, irreducible, non-degenerate algebraic curves on some rational surfaces, denoted $X_p^n \subset \mathbf{P}^n$, of degree $(n+p-1)$ in $\mathbf{P}^n$, for $p \in \mathbf{Z}$, $k = [n/3]_* < p \leq n-2$, $n \geq 8$. We recall the definition of X_p^n.

If $\Sigma = \{P_0, P_1, \ldots, P_s\} \subset \mathbf{P}^2$ is a set of general points and $S := Bl_\Sigma(\mathbf{P}^2) \to \mathbf{P}^2$ is the blow up of $\mathbf{P}^2$ in Σ, then $Pic(S) = \mathbf{Z} \oplus \mathbf{Z}^s$, having $(l; -e_0, \ldots, -e_s)$ a $\mathbf{Z}$-basis, where l is the preimage of a line from $\mathbf{P}^2$ and e_i is the class of the exceptional divisor E_i for $i = 0, 1, \ldots, s$. The intersection form on $Pic(S)$ is given by: $(l^2) = 1$, $(e_i^2) = -1$, $(\forall) i = \overline{0, s}$, $(l \cdot e_i) = 0$, $(\forall) i = \overline{0, s}$, $(e_i \cdot e_j) = 0$, $(\forall) i \neq j$. If $\mathcal{D} = al - \Sigma b_i e_i \in Pic(S)$, we write $\mathcal{D} = (a; b_0, \ldots, b_s)$. We denote by $[\mathcal{D}] = [a; b_0, \ldots, b_s]$ the associated complete linear system. In [5] is proved that, if $p \in \mathbf{Z}$, $p \geq 1$ and $s \leq 3p$, then the linear system on S $[p+2; p, 1^s]$ is very ample. Take now $n \geq 8$ and $p \in \mathbf{Z}$, $k = [n/3]_* \leq p \leq n-2$. Put $s_p^n := 3p - n + 5$. By the previous results of Gimigliano, we get that $[p+2; p, 1^{s_p^n}]$ is very ample on $S_p^n = Bl_{\Sigma_p^n}(\mathbf{P}^2)$, where $\Sigma_p^n = \{P_0, P_1, \ldots, P_{s_p^n}\} \subset \mathbf{P}^2$ is a set of general points. Put then $X_p^n := \mathrm{Im}\left(\varphi_{[p+2;p,1^{s_p^n}]}(S_p^n)\right)$, where $\varphi_{[\mathcal{D}]}$ is the morphism associated to a base-point-free linear system $[\mathcal{D}]$. Then it easy to see that $X_p^n \subset \mathbf{P}^n$ and $\deg(X_p^n) = n + p - 1$.

We recall now from [12] the following

Theorem 2.2 ([12]). *If $n, p \in \mathbf{Z}$, $n \geq 8$, $n/3 \leq p \leq n-4$, $d, g \in \mathbf{Z}$, $d \geq \frac{2}{3}(3p + n + 9)$ and $\alpha_{p+1}(d-1, n) \leq g \leq \alpha_p(d, n)$ (for the definition of $\alpha_p(d, n)$ see* (1.5)*), then there is a smooth, irreducible algebraic curve $C \subset X_p^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g.*

Remark 2.3. From the proof of Theorem 2.2 it follows immediately that, if $C \subset X_p^n$ is a curve from before and $H = H_p^n$ is a hyperplane section of X_p^n, then the linear system $|C + H|$ is base-point-free of dimension ≥ 2, and the associated morphism $\varphi_{|C+H|}$ satisfies the conditions from the Bertini-type Theorem 5.1 from [9]. So $|C + H|$ contains a smooth, irreducible curve.

Lemma 2.4 ([12, 14]).
a) $x_p(d + (n+p-1), n) = x_p(d, n) + 1;$
$t_p(d + (n+p-1), n) = t_p(d, n);$
$u_p(d + (n+p-1), n) = u_p(d, n);$
$\alpha_p(d + (n+p-1), n) = \alpha_p(d, n) + (d + p - 1).$
b) $\alpha_{p+1}(d-1, n) \leq \alpha_p(d, n)$, $(\forall) d \geq a_p^n + 1$, $d \in \mathbf{Z}$;
$\alpha_{p+1}(d, n) \leq \alpha_p(d, n)$, $(\forall) d \geq a_p^n + n + p$, $d \in \mathbf{Z}$.

We remembered the definitions of the numbers from Lemma 2.4 in (1.5)–(1.9), Section 1.

Let's now recall from [2] the following Theorem 2.5. We need to recall some numbers from [2]. For $n \geq 6$:

$$d_0(n) := \tag{2.1}$$

$$= \begin{cases} \max\left(2n+1, \frac{1}{6}(-k+1+(4k-1)\sqrt{12k-27})\right), & hboxif\ n=3,k\geq 3 \\ 2n+1, & \text{if } n=6 \\ \max(2n+1, -5k+1+4k\sqrt{k}), & \text{if } n=3k+1, k\geq 2 \\ \max(2n+1, \frac{1}{2}(-6k+1+ & \\ \quad +\sqrt{(6k+1)^2+(64k^3+12k^2+k)}\ )), & \text{if } n=3k+2, k\geq 2 \end{cases}$$

$$F_d^{k,n}(r) := \frac{1}{2}\left[d(r-1)+(k-1)r-\frac{n+k-1}{4}r^2\right]+1,\ k=[n/3]_* \tag{2.2}$$

$$s(d,n) := \tag{2.3}$$

$$= \begin{cases} \frac{1}{3}(-4k+1+\sqrt{12d+16k(k+1)-38}), & \text{if } \ n=3k, k\geq 2 \\ -2(2k+1)+2\sqrt{d+2(k-1)+(2k+1)^2}, & \text{if } \ n=3k+1, k\geq 2 \\ -(8k+3)+\sqrt{(8k+3)^2+24k+8d-11}, & \text{if } \ n=3k+2, k\geq 2 \end{cases}$$

$$\varphi(d,n) := F_d^{k,n}(s(d,n)). \tag{2.4}$$

Theorem 2.5 ([2])**.** *Let $n \geq 6$ be an integer and $d, g \in \mathbf{Z}$ such that $d \geq d_0(n)$ and $\varphi(d,n) < g \leq A(d,n)$. Then there is a smooth, irreducible, algebraic curve $C \subset X_p^n$ ($k = [n/3]_*$), non-degenerate in $\mathbf{P}^n$, of degree d and genus g.*

We recall the definitions of the numbers appearing in Theorem 2.5 from (2.1)–(2.4) and (1.17).

Before stating the next Theorem 2.6, some more definitions (see [12]).

For $n \geq 8$, put

$$d_1(n) := \tag{2.5}$$

$$= \begin{cases} \max\left(2n+1, \frac{1}{6}(3+(4k-1)\sqrt{24k-33})\right), & \text{if } n=3k, k\geq 3 \\ \max\left(2n+1, \frac{4k}{4k+1}(-4k+\sqrt{32k^3+16k^2-2k-1})\right), & \text{if } n=3k+1, k\geq 3 \\ \max\left(2n+1, 5k+3+(2k+1)\sqrt{48k+6}\right), & \text{if } n=3k+2, k\geq 2 \end{cases}$$

(2.6)

$$B(d,n) := \begin{cases} \left.\begin{array}{l} \alpha_{k+1}(d,n), \text{ if } n = 3k+1, 3k+2 \\ \alpha_k(d,n), \text{ if } n = 3k \end{array}\right\}, & \text{if } 2n+1 \le d < d_1(n) \\ A(d,n), & \text{if } d \ge d_1(n) \end{cases}$$

(see (1.5), (1.17)).

We consider now the following domain from the (d, g)-plane of curves from $\mathbf{P}^n$.

$$D_1^n : 0 \le g \le \begin{cases} \pi_0(d,n), & n \le d \le 2n \\ B(d,n), & d \ge 2n+1 \end{cases} \tag{2.7}$$

Theorem 2.6 ([12]). *In the domain D_1^n there is no gap for $HC(n)$ (see Section 1).*

The proof comes from Theorems 2.1, 2.2, 2.5, if $d \ge \frac{2}{3}(4n-3)$. For smaller values of d we use some ad-hoc constructions (see [12, 13, 14]).

We end this section with two more theorems from [2].

Theorem 2.7 ([2]). *If $d, g \in \mathbf{Z}$ such that $d \ge 13$ and $\left[\frac{(d-6)^2}{18}\right]_* + 1 \le g \le \pi_2(d, 6)$, then there is a smooth, irreducible curve $C \subset X_2^6$, non-degenerate in $\mathbf{P}^6$, of degree d and genus g.*

Theorem 2.8 ([2]). *If $d, g \in \mathbf{Z}$ such that $d \ge 15$ and $\left[\frac{(d-7)^2}{20}\right]_* + 1 \le g \le \pi_2(d, 7)$, then there is a smooth, irreducible curve $C \subset X_2^7$, non-degenerate in $\mathbf{P}^7$, of degree d and genus g.*

We'll see in the next section that Theorem 1.7 will be a consequence of Theorems 2.1, 2.7 and 2.8, while Theorem 1.9 will follow from Theorems 2.1, 2.2, 2.5 and 4.1 (see Section 4).

Two more Lemmas, necessary in Sections 3, 4.

Lemma 2.9 ([4]). *Let $C \subset \mathbf{P}^n$ be a smooth, irreducible curve and H a hyperplane intersecting C transversally. Let $Z \subset H$ be a curve. We assume:*

i) $h^j(\mathcal{I}_{Z,H}(1)) = 0$, $j = 0, 1$ (here $\mathcal{I}_{Z,H}$ is the ideals-sheaf of Z in H);

ii) $C \cap Z = C \cap H$ (as schemes).

Then $X := C \cup Z$ satisfies: $h^j(\mathcal{I}_X(1)) = 0$, $j = 0, 1$ ($\mathcal{I}_X$ is the ideals-sheaf of X).

Lemma 2.10 ([10]). *Let $X \subset \mathbf{P}^n$ be a regular (i.e. $q = h^1(O_X) = 0$) smooth, irreducible, projective surface. Let $H \subset \mathbf{P}^n$ be a general hyperplane so that $C := H \cap X$ is a smooth, irreducible curve. Then $h^j(\mathcal{I}_{C,H}(1)) = 0$, $j = 0, 1$.*

We recall now the following

Definition 2.11 ([9]). *Let X be a smooth projective variety and $\varphi : X \to \mathbf{P}^N$ a morphisme. For $x \in X$ let $T_{\varphi,x} : T_{X,x} \to T_{\mathbf{P}^n,\varphi(x)}$ be the tangent application. For any $i \in \mathbf{Z}$, $i > 0$ we consider the set $\Sigma_i := \{x \in X | dim\, KerT_{\varphi,x} \geq i\}$. We say that a base point free linear system on X is* **almost very ample** *if its associated morphism φ has the property that $codim(\Sigma_i, X) \geq i$ for any $i \in \mathbf{Z}$, $i > 0$.*

Obviously, a very ample linear system is almost very ample.

Corollary 2.12. *Let $n \in \mathbf{Z}$, $n \geq 5$, $p \in \mathbf{Z}$, $k = [n/3]_* \leq p \leq n-2$. Let $[\mathcal{L}]$ be a (complete) linear system on the surface X_p^n which is base-point free and almost very ample, so, by [9], Théorème 5.1, it contains a smooth curve C. Suppose that $dim(Im\varphi_{[\mathcal{L}]}) = 2$, so C is irreducible. Let H be a general hyperplane section of X_p^n. Then, the general curve $C' \in [\mathcal{L} \otimes \mathcal{O}_{X_p^n}(H)]$ is a smooth, irreducible, linearly normal curve in $\mathbf{P}^n$. Moreover, if $deg(C) = d$ and $g(C) = g$ are the degree and the genus of C, respectively, then $d' = deg(C') = d + (n+p-1)$ and $g' = g(C') = g + (d+p-1)$.*

P r o o f. Immediate from Lemmas 2.9 and 2.10 (see also Remark 2.3). □

3. Linearly normal curves in $\mathbf{P}^6$ and $\mathbf{P}^7$. In this section we prove Theorem 1.7 (see Section 1).

• **For $\mathbf{P}^6$.**

We know from Theorem 2.7 the existence of smooth, irreducible, non-degenerate curves of degree d and genus g for $d \geq 13$ and $[(d-6)^2/18]_* + 1 \leq g \leq \pi_2(d, 6)$ (see Section 2).

These curves belongs to complete, base-point free, almost very ample linear systems on X_2^6, so, by adding to them general hyperplane sections of X_2^6, we get, by Corollary 2.12, linearly normal curves in $\mathbf{P}^6$ of degree d and genus g for

$$[(d^2 - 8d + 79)^2/18]_* + 1 \leq g \leq \pi_2(d, 6),\ d \geq 20 \tag{3.1}$$

From Theorem 2.1 ($n = 6, \delta = 3$) we already know the existence of linearly normal curves of degree d and genus g for

$$d - 6 \leq g \leq \alpha_4''(d, 6) - 1,\ d \geq 13. \tag{3.2}$$

Moreover, $\alpha_4''(d, 6) = G_d^3(J_3(d, 6), 6)$ (see (1.10)–(1.12)), where $J_3(d, 6) = [(d+3)/9]_*$. Let's denote by $r_v^6(d) = (2d-3)/18$ the point where $G_d^3(r, 6)$ achieves its maximum (as function of r). We have $J_3(d,6) \le (d+3)/9 < J_3(d,6)+1$. If $J_3(d,6) < r_v^6(d)$, then $\alpha_4''(d,6) > G_d^3((d-6)/9,6)$ (because $J_3(d,6) > (d-6)/9 = (d+3)/9-1$). If $J_3(d,6) \ge r_v^6(d)$, then $\alpha_4''(d,6) = G_d^3(J_3(d,6),6) = G_d^3(2r_v^6(d) - J_3(d,6),6)$ and $2r_v^6(d) - J_3(d,6) \le r_v^6(d)$. Moreover $J_3(d,6) \le (d+3)/9$, so $2r_v^6(d) - J_3(d,6) \ge (d-6)/9$. We get again $\alpha_4''(d,6) \ge G_d^3((d-6)/9,6)$. It's straight forward now to see that $G_d^3((d-6)/9,6) = (d-6)(d+3)/18 \ge (d^2-8d+79)/18$ for any $d \ge 20$. Using (3.1) and (3.2), it follows that we covered the range from Theorem 1.7 for $d \ge 20$. For the other values of d, we use the fact that the exis tence of linearly normal curves of degree d and genus g is clear if $g = d-6$ (due to the existence of non-special curves), and we construct directly curves belonging to base-point free almost very ample linear systems on X_2^6. We'll get linearly normal curves by Corollary 2.12. We give the linear systems in the following table. The smallest value of g is $\alpha_4''(d,6)$ if this is *not* $d-6$ and $\alpha_4''(d,6)+1$ if $\alpha_4''(d,6) = d-6$ while the bigger value of g is $\pi_2(d,6)$.

d	g	linear systems on X_2^6
13	8	$[7;3,2^4,1]$
14	9	$[7;3,2^3,1^2]$
	10	$[8;4,2^5]$
15	10	$[7;3,2^2,1^3]$
	11	$[8;4,2^4,1]$
16	11	$[7;3,2,1^4]$
	12	$[8;4,2^3,1^2]$
	13	$[9;5,2^5]$
17	13	$[8;4,2^2,1^3]$
	14	$[9;5,2^4,1]$
	15	$[9;4,3,2^4]$

d	g	linear systems on X_2^6
18	15	$[9;5,2^3,1^2]$
	16	$[10;6,2^5]$
	17	$[9;4,2^5]$
19	17	$[10;6,2^4,1]$
	18	$[9;4,2^4,1]$
	19	$[10;5,3,2^4]$
	20	$[11;5,3^5]$

• **For $\mathbf{P}^7$.**

We know from Theorem 2.8 the existence of smooth, irreducible, non-degenerate curves of degree d and genus g for $d \ge 15$ and

$$[(d-7)^2/20]_* + 1 \le g \le \pi_2(d,7)$$

(see Section 2).

These curves belongs to complete, base-point free, almost very ample linear systems on X_2^7, so, by adding to them general hyperplane sections of X_2^7, we get, by Corollary 2.12, linearly normal curves in $\mathbf{P}^7$ of degree d and genus g for

$$[(d-5)^2/20]_* + 4 \le g \le \pi_2(d,7),\ d \ge 23 \tag{3.3}$$

From Theorem 2.1 ($n = 7$, $\delta = 4$) we already know the existence of linearly normal curves of degree d and genus g for

$$(3.4) \qquad d - 7 \leq g \leq \alpha''_4(d, 7) - 1, \ d \geq 15$$

As we did when studied the Problem for $\mathbf{P}^6$, we show that $\alpha''_4(d, 7) \geq (d - 5)^2/20]_* + 4$ for any $d \geq 23$, covering the domain from Theorem 1.7 for $\mathbf{P}^7$ if $d \geq 23$ (use (3.3), (3.4)). As for $\mathbf{P}^6$, everything follows now from the following table (similar to the table for $\mathbf{P}^6$, but now the linear systems are on X_2^7):

d	g	linear systems on X_2^7
15	9	$[7; 3, 2^3, 1]$
16	10	$[7; 3, 2^2, 1^2]$
	11	$[8; 4, 2^4]$
17	11	$[7; 3, 2, 1^3]$
	12	$[8; 4, 2^3, 1]$
18	12	$[7; 3, 1^4]$
	13	$[8; 4, 2^2, 1^2]$
	14	$[9; 5, 2^4]$
19	14	$[8; 4, 2, 1^3]$
	15	$[9; 5, 2^3, 1]$
	16	$[9; 4, 3, 2^3]$

d	g	linear systems on X_2^7
20	16	$[9; 5, 2^2, 1^2]$
	17	$[9; 4, 3, 2^2, 1]$
	18	$[9; 4, 2^4]$
21	18	$[9; 4, 3, 2, 1^2]$
	19	$[9; 4, 2^3, 1]$
	20	$[10; 5, 3, 2^3]$
22	20	$[9; 4, 2^2, 1^2]$
	21	$[10; 5, 3, 2^2, 1]$
	22	$[10; 5, 2^4]$
	23	$[11; 5, 3^4]$

□

Example 3.1. We'll show that the pair $(d, g) = (45, 130)$ *is not* a gap for $HC(6)$ but *it is* a gap for $LN(6)$. Moreover, if we compute $\pi_2(45, 6)$ from (1.1)-(1.4), we get $130 = \pi_2(45, 6) + 1$, so the pair (45,130) is just near the domain from Theorem 1.7 for $n = 6$. It is easy to see that the pair (45,130) is not a gap for $HC(6)$ because there is a (smooth, irreducible, non-degenerate) curve C_1 of degree 45 and genus 130 lying on a rational scroll $S \subset \mathbf{P}^6$ of degree 6 (take $a = 6$, $c = 27$ in Prop.2.7 from [Ci]). Such a curve C_1 cannot lie on surfaces from [Ci], Prop. 2.5 (there are no integers a, c such that $45 = 3a + c$ and $130 = a(a-1)/2 + (a-1)(c-1)$). We prove now that no curve of degree 45 and genus 130 can lie on a Del Pezzo surface of degree 6. Indeed, by [2], Theorem 2.10, it is enough to prove that $E_1(r) := 8(F_1(45, r) - 130)$ cannot be written as the sum of the squares of two integers less than or equal to r, for any positive integer $r \leq 15$, where $F_1(d, r) := \frac{1}{2}[d(r-1) - 3r^2/2] + 1$ (this includes both the case r even and r odd). Doing computations, we get $E_1(r) = -6[(r-15)^2 - 23]$, so $E_1(r) < 0$ for $1 \leq r \leq 10$. Moreover, $E_1(11) = 42$, $E_1(12) = 84$, $E_1(13) = 114$,

$E_1(14) = 132$, $E_1(15) = 138$ and it is immediate that none of these five numbers can be written as the sum of the squares of two integers. We conclude that any curve of degree 45 and genus 130 lies only on a scroll of degree 6, $S \subset \mathbf{P}^6$ (see [2], 2.e., last paragraph), so it is not linearly normal, being a projection from $\mathbf{P}^7$ (because S is a projection from $\mathbf{P}^7$).

Example 3.2. We'll show now that the pair $(d, g) = (150, 1351)$ *is not* a gap for $HC(7)$ but *it is* a gap for $LN(7)$. It is easy to see that $1351 = \pi_2(150, 7) + 1$, so the pair $(150, 1351)$ is near the domain from Theorem 1.7 for $n = 7$. Our pair is not a gap for $HC(7)$ because a (smooth, irreducible, non-degenerate) curve of degree 150 and genus 1351 lies on a rational scroll S of degree 7, $S \subset \mathbf{P}^7$ (take in (b) from [2], p.226, last lines, $a = 15$, $c = 90$). Moreover no curve of degree 150 and genus 1351 can lie on the rational normal scroll of degree 6 from $\mathbf{P}^7$ (there are no integers a, c such that $150 = 3a + c$ and $1351 = (a - 1)(c - 1)$). Because any rational scroll degree 7 from $\mathbf{P}^7$ is a projection from $\mathbf{P}^8$, it follows, using [2], p.226-227, that, in order to prove that the pair (150, 1351) is a gap for $LN(7)$, it is enough to show that there is no curve of degree 150 and genus 1351 lying on the Del Pezzo surface from $\mathbf{P}^7$. As in [2], Theorem 2.10, we have

Lemma 3.3. *There exists a smooth, irreducible curve of degree d and genus g on a general Del Pezzo surface in $\mathbf{P}^7$ if and only if there exists a positive integer r such that $r \leq (2d + 3)/7$ and:*

(i) $2(F_2(d, r) - g)$ is the square of some integer less than or equal to $r/2$, if r is even;

(ii) $8(F_2(d, r) - g)$ is the square of some odd integer less than or equal to r, if r is odd.

Here $F_2(d, r) := \frac{1}{2}[d(r - 1) - 7/4r^2] + 1.$

Now, as in Example 3.1, it is enough to show that $E_2(r) := 8(F_2(150, r) - 1351)$ is not the square of some integer r, $1 \leq r \leq 43$. We have $E_2(r) = -7r^2 + 600r - 11400$. The last figure of the square of some integer can be $1, 4, 5, 6, 9$, so the last figure of $E_2(r)$ can be 2, 3, 5, 7, 8 if r is integer, being 5 iff $r \equiv 5 \pmod{10}$. So $E_2(r)$ may be the square of some integer only if $r \in \{5, 15, 25, 35\}$. But $E_2(35) = 41 \times 5^2$, which is not a square and $E(r) < 0$ if $r \leq 25$. It remains that (150,1351) is a gap for $LN(7)$ although it is not a gap for $HC(7)$.

4. Linearly normal curves in $\mathbf{P}^n$, $n \geq 8$.

We end this paper with the proof of Theorem 1.9 (Section 1). Both in Theorem 2.2 and Theorem 2.5 (Section 2) the necessary curves belongs to complete, base-point free almost very ample linear systems on surfaces X_p^n ($n/3 \le p \le n-4$ in Theorem 2.2 and $p = [n/3]_*$ in Theorem 2.5). So, by Corollary 2.12, adding to them general hyperplane sections, we get linearly normal curves in $\mathbf{P}^n$. By the formula of transformation of the degree and the genus from Corollary 2.12 and Lemma 2.4, we get linearly normal curves of degree d and genus g on X_p^n, $n/3 \le p \le n-4$ for $d \ge (9p+5n+15)/3$ and $\alpha_{p+1}(d,n) \le g \le \alpha_p(d,n)$. On X_k^n ($k = [n/3]_*$), we get from Theorem 2.5 linearly normal curves of degree d and genus g for $d \ge d_2(n)$ and $\alpha_k(d,n) \le g \le A(d,n)$ (see (1.17)) (the function $d_1(n)$ from [12] (see (2.5)) changes in $d_2(n)$, (see (1.16))). Putting together all these linearly normal curves with the linearly normal curves obtained by Corollary 2.12 again, adding general hyperplane sections to the curves from the next Proposition 4.1, from [14], we get exactly the domain from Theorem 1.9, proving in such a way this Theorem (see Theorem 2.1 also, for $2n+1 \le d \le 5(n-2)$).

Proposition 4.1 ([14]). *Let $d, g \in \mathbf{Z}$, $n \in \mathbf{Z}$, $n \ge 7$, $d \ge 3n-5$ and $\alpha''_{n-3}(d,n) \le g \le \alpha_{n-3}(d,n) - 1$. Then there is a smooth, irreducible curve $C \subset X_{n-3}^n$ of degree d and genus g, non-degenerate in $\mathbf{P}^n$, and belonging to a complete, base-point free, almost very ample linear system in X_{n-3}^n.*

Corollary 4.2. *Let be $n \ge 8$ an integer. Then:*
i) There is no gap for $LN(n)$ in the domain

$$d - n \le g \le \begin{cases} \alpha_k(d,n), n = 3k \\ \alpha_{k+1}(d,n), n = 3k+1, 3k+2 \end{cases}, d \ge 5n - 9;$$

ii) There is no gap for $LN(n)$ in the domain

$$d - n \le g \le \begin{cases} \pi_k(d,n), & \text{if} \quad n = 3k \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & \text{if} \quad n = 3k+1, 3k+2 \end{cases}$$

for $d \ge d_2(n) =$ a function of degree $1/2$ in n.

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