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ON ROOT ARRANGEMENTS OF POLYNOMIAL-LIKE FUNCTIONS AND THEIR DERIVATIVES

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Abstract. A real polynomial P of degree n in one real variable is *hyperbolic* if its roots are all real. Denote by $x_k^{(i)}$ the roots of $P^{(i)}$, $k = 1, \ldots, n-i$, $i = 0, \ldots, n-1$. Then one has $\forall i < j$, $x_k^{(i)} \leq x_k^{(j)} \leq x_{k+j-i}^{(i)}$ and $((x_k^{(i)} = x_k^{(i+1)})$ or $(x_{k+1}^{(i)} = x_k^{(i+1)})) \Rightarrow (x_k^{(i)} = x_k^{(i+1)} = x_{k+1}^{(i)})$. For $n \geq 4$ not all arrangements of $n(n+1)/2$ real numbers $x_k^{(i)}$ compatible with these two conditions are realizable by the roots of hyperbolic polynomials of degree n and of their derivatives. We show that for $n = 4$ they are realizable either by hyperbolic polynomials of degree 4 or by non-hyperbolic polynomials of degree 6 whose fourth derivatives never vanish (these are a particular case of the so-called hyperbolic polynomial-like functions of degree 4).

1. Introduction.

1.1. Hyperbolic polynomials and polynomial-like functions. Consider the polynomial $P(x, a) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbf{R}$. Call it *(strictly)*

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hyperbolic if all its roots are real (real and distinct). It is clear that if P is (strictly) hyperbolic, then such are P', ..., $P^{(n-1)}$ as well. Examples of hyperbolic polynomials are the ones of all known orthogonal families (e.g. the Legendre, Laguerre, Hermite, Tchebyshev polynomials).

Notation 1. *Denote by* $x_1 \leq \ldots \leq x_n$ *the roots of* P *and by* $x_1^{(k)} \leq \ldots \leq x_{n-k}^{(k)}$ *the ones of* $P^{(k)}$. *We set* $x_j^{(0)} = x_j$. *In the examples we never go beyond degree* 5 *and to avoid double indices we use also the notation* f_j, s_j, t_j, l_j *for the roots respectively of* P', P'', P''', $P^{(4)}$. *The letters are chosen to match "first", "second", "third" and "last".*

Definition 2. *Call* arrangement *(or* configuration*) of the roots of* P, P', ..., $P^{(n-1)}$ *the complete system of strict inequalities and equalities that hold for these roots. We assume that the roots are arranged in a chain in which any two roots occupying consecutive positions are connected with a sign* $<$ *or* $=$. *An arrangement is called* non-degenerate *if there are no equalities between any two of the roots, i.e. no equalities of the form* $x_i^{(j)} = x_q^{(r)}$ *for any indices* i, j, q, r.

Definition 3. *Arrangements are also defined by means of* configuration vectors (CV). *On a CV the positions of the roots of* P, P', P'', P''', $P^{(4)}$ *are denoted by* 0, f, s, t, l *and coinciding roots are put in square brackets. E.g. the CV*

$$([0f0], s, f, t, l, 0, s, f, t, s, [0f0]) \quad (\textit{for} \;\; n = 5)$$

indicates that $x_1 = f_1 = x_2 < s_1 < f_2 < t_1 < l_1 < x_3 < s_2 < f_3 < t_2 < s_3 < x_4 = f_4 = x_5$.

The classical Rolle theorem implies that the roots of P and of its derivatives satisfy the following inequalities:

$$(1) \qquad \forall\; i < j, \quad x_k^{(i)} \leq x_k^{(j)} \leq x_{k+j-i}^{(i)}$$

One has also the self-evident condition:

$$(2) \qquad ((x_k^{(i)} = x_k^{(i+1)}) \;\text{ or }\; (x_{k+1}^{(i)} = x_k^{(i+1)})) \Rightarrow (x_k^{(i)} = x_k^{(i+1)} = x_{k+1}^{(i)})$$

In papers [4], [2] and [3] we dealt with the question given $n(n+1)/2$ real numbers $x_j^{(k)}$, $k = 0, \ldots, n-1$, $j = 1, \ldots, n-k$, satisfying conditions (1) and (2), which of these arrangements (called *a priori admissible*) can be realized by the roots of hyperbolic polynomials of degree n and of their derivatives. We showed there that for $n \geq 4$ not all (and even not all non-degenerate) arrangements can be realized by hyperbolic polynomials. In the present paper we continue this work and we suggest a class of objects (called *polynomial-like functions*, see Definition 10) by which one should replace hyperbolic polynomials in order to realize all a priori admissible arrangements. We show (see Theorem 13) that for $n = 4$ polynomial-like functions realize all such arrangements and that one can choose them to be either hyperbolic polynomials of degree 4 or non-hyperbolic polynomials of degree 6.

1.2. The results. The following lemma results from (1) and (2), see Lemma 4.2 in [4]:

Lemma 4. *A root of multiplicity $m < k$ of a hyperbolic polynomial P is at most a simple root of $P^{(k)}$.*

Remark 5. A result of R.M. Thrall (see [6]) says that for arbitrary $n \in \mathbf{N}^*$ there are exactly $\binom{n+1}{2}!\frac{1!2!\ldots(n-1)!}{1!3!\ldots(2n-1)!}$ possible non-degenerate arrangements of the roots $x_j^{(i)}$ which are compatible with (1). We call arrangements compatible with (1) and (2) also *a priori admissible*.

Remark 6. For $n = 1, 2$ or 3 conditions (1) and (2) together are necessary and sufficient for an arrangement to be realized by the roots of a hyperbolic polynomial.

For $n = 2$ there are two a priori admissible arrangements: $(0, f, 0)$ (non-degenerate) and $([0f0])$ (degenerate). For $n = 3$ there are two a priori admissible non-degenerate arrangements: $(0, f, s, 0, f, 0)$ and $(0, f, 0, s, f, 0)$. All four arrangements are realized by hyperbolic polynomials of degrees respectively 2 and 3 (and all a priori admissible degenerate arrangements for $n = 3$ as well).

For $n = 4$ there are 12 non-degenerate a priori admissible arrangements (see Remark 5) out of which only 10 are realized by hyperbolic polynomials, see [1], [2], [3] or [4]; see also Fig. 1 (explanations concerning the figure are given at the beginning of Subsection 2.1). The two missing arrangements are

$$(A_1)\ :\ (0, f, 0, s, t, f, 0, s, f, 0) \quad \text{and} \quad (A_2)\ :\ (0, f, s, 0, f, t, s, 0, f, 0).$$

For $n = 5$ only 116 out of the 286 a priori admissible non-degenerate arrangements are realized by hyperbolic polynomials, see [2]. It is intuitively clear that for larger n this proportion is to drop even more dramatically because a hyperbolic polynomial has only n coefficients while the number of roots of the polynomial and of all its derivatives equals $n(n+1)/2$. Therefore if one wants to realize all a priori admissible arrangements (or at least the non-degenerate ones) one should try to do it by means of a class larger than the one of hyperbolic polynomials.

For $n = 4$ one has the following

Lemma 7. *Arrangements* (A_1) *and* (A_2) *can be realized by non-hyperbolic polynomials of degree* 6 *which are obtained as analytic perturbations of the polynomial* $P_1 = x^4 - x^2 + 5/36$.

The lemmas from this subsection (except Lemma 4) are proved in the next one.

Remark 8. The polynomial P_1 from Lemma 7 is the *Gegenbauer polynomial* of degree 4, i.e. the unique monic polynomial of the form $x^n - x^{n-2} + \ldots$, $n \geq 3$, which is divisible by its second derivative. The Gegenbauer polynomial is hyperbolic; it is even or odd together with n, see [4], [2] or [3]. The polynomial P_1 and its derivatives of order ≤ 3 realize the following arrangement:

$$(A_B) \ : \ (0, f, [0s], [ft], [0s], f, 0)$$

For $n \geq 5$ perturbations of hyperbolic polynomials are insufficient to realize all a priori admissible non-degenerate arrangements:

Lemma 9. *The following non-degenerate arrangement (which is a priori admissible) cannot be realized by a hyperbolic polynomial of degree* 5 *or by an analytic perturbation of such a polynomial:*

$$(A_3) \ : \ (0, f, s, 0, f, t, l, 0, s, f, t, 0, s, f, 0) \tag{3}$$

This is why to realize all a priori admissible arrangements we introduce a new object:

Definition 10. *A* polynomial-like function (PLF) *of degree* n *is a* C^∞*-smooth real-valued function whose* n*-th derivative never vanishes (we assume that it is everywhere positive). The notion of a PLF was introduced in paper* [5] *(whose authors call a PLF a* pseudopolynomial*). It is clear that a PLF of degree* n *has* $\leq n$ *real roots and that its* k*-th derivative has* $\leq n-k$ *real roots counted with the multiplicities. In case of equality the PLF is called* hyperbolic.

Remark 11. When perturbing analytically a hyperbolic polynomial P of degree n, we use perturbations of the form $P + \varepsilon Q$ where $\varepsilon > 0$ is small and

Q is a monic polynomial of degree $n+2k$, $k \in \mathbf{N}^*$. As $P^{(n)} > 0$ is a constant and as $Q^{(n)}$ is of degree $2k$, with positive leading coefficient, one has $Q^{(n)} \geq 0$ for $|x|$ large enough and one has $(P + \varepsilon Q)^{(n)} > 0$ for all x if $\varepsilon > 0$ is small enough. Hence, such a perturbation of a hyperbolic polynomial of degree n is a PLF of degree n.

The present paper is the first step in an effort to answer the following

Problem 12 (B. Z. Shapiro)**.** *Is it true or not that hyperbolic PLFs of degree n realize all a priori admissible arrangements?*

We prove (see Section 2) the following

Theorem 13. *For $n = 4$ all a priori admissible arrangements (degenerate or not) are realizable by hyperbolic polynomials of degree* 4 *or by non-hyperbolic polynomials of degree* 6 *which are hyperbolic PLFs of degree* 4*. Exactly eight of the degenerate arrangements cannot be realized by perturbations of hyperbolic polynomials of degree* 4*.*

Remarks 14. 1) One can ask the question: What conditions except (1) and (2) have to be imposed upon the $n(n+1)/2$ real numbers $x_j^{(k)}$, $k = 0, \ldots, n-1$, $j = 1, \ldots, n-k$, so that they should be roots of a PLF of degree n and of its derivatives. (The question concerns not only the arrangement defined by the numbers but the choice of the numbers themselves.) For $n = 3$ an exhaustive answer is given in [5] which is a system of linear and quadratic inequalities. For $n \geq 4$ the question seems to be still open.

2) If a non-degenerate arrangement can be realized by a hyperbolic PLF f of degree n, then it can be realized by a polynomial P (in general, not hyperbolic and of degree $> n$) which is a hyperbolic PLF of degree n. To this end one has to approximate $f^{(n)}$ by a polynomial $Q = P^{(n)}$ and leave the same constants of integration to obtain f, P respectively from $f^{(n)}$, $P^{(n)}$ – the roots of the polynomials P, P', …, $P^{(n-1)}$ will be close to the respective roots of f, f', …, $f^{(n-1)}$, they will still be all distinct and will define the same arrangement.

3) It would be interesting to know for what minimal number $C(n)$ (resp. $C^0(n)$) all (resp. all non-degenerate) a priori admissible arrangements can be realized by polynomials of degree $C(n)$ (resp. $C^0(n)$) which are PLFs of degree n. For $n = 4$ one has $C(n) = C^0(n) = 6$, see Theorem 13 (and Remark 6, Lemma 7, Remark 11 about (A_1) and (A_2)). It is clear that $C^0(n) \leq C(n)$ and that n, $C(n)$ and $C^0(n)$ are simultaneously even or odd.

1.3. Proofs of the lemmas.

Proof of Lemma 7. 1^0. Recall that $P_1(x) = (x^2 - 1/6)(x^2 - 5/6)$, $P_1''(x) = 12(x^2 - 1/6)$. Consider the one-parameter deformation (of the polynomial P) $\tilde{P}(x,\varepsilon) = P_1(x)+\varepsilon P_2(x)$, $\varepsilon \in (\mathbf{R}, 0)$, where $P_2(x) = x^3(x^2-1/6)(x-1) = x^6 - x^5 - x^4/6 + x^3/6$. One has $P_2''(x) = 30x^4 - 20x^3 - 2x^2 + x$, $P_2'''(x) = 120x^3 - 60x^2 - 4x + 1$.

As $\deg P_2 = 6$, the polynomial P_2 takes only positive values for $x \ll -1$ and for $x \gg 1$. For $\varepsilon > 0$ small enough $\tilde{P}$ has real roots close to x_1 and x_4 and $\tilde{P}(x_2) = \tilde{P}(x_3) = 0$. For such values of ε the polynomial $\tilde{P}$ (resp. $\tilde{P}'$, $\tilde{P}''$, $\tilde{P}'''$) has only 4 (resp. 3, 2, 1) real roots. For $k = 0, \ldots, 3$ the real roots of $\tilde{P}^{(k)}$ are close to the ones of $P^{(k)}$. The polynomial $\tilde{P}$ is a hyperbolic PLF of degree 4, see Remark 11.

2^0. One has $P_2(\pm 1/\sqrt{6}) = 0$ and $\tilde{P}(\pm 1/\sqrt{6}) = 0$; $P_2'(0) = 0$ and $\tilde{P}'(0) = 0$. Next, $P_2''(-1/\sqrt{6}) = \dfrac{1}{2} + \dfrac{7}{3\sqrt{6}} > 0$, $P_2''(1/\sqrt{6}) = \dfrac{1}{2} - \dfrac{7}{3\sqrt{6}} < 0$ and $P_2'''(0) = 1 > 0$.

Hence, for $\tilde{P}$ one has $x_2 < s_1$, $x_3 < s_2$ and $t_1 < f_2$ (when $\varepsilon > 0$ is small enough). This implies that the real roots of $\tilde{P}$ and of its derivatives realize arrangement (A_1).

To realize arrangement (A_2) one can consider in the same way the deformation $P_1(x) + \varepsilon P_2(-x)$ for $\varepsilon > 0$ close to 0. $\square$

Proof of Lemma 9. 1^0. Suppose that arrangement (A_3) is realizable by a monic hyperbolic polynomial P of degree 5. Then the roots of P', P'', P''' and $P^{(4)}$ define the following a priori admissible arrangement: $(f, s, f, t, l, s, f, t, s, f)$ $(*)$ (just forget the roots of P in (3)). Shifting the order of the derivatives by 1 (i.e. changing in the last arrangement f, s, t, l respectively to 0, f, s, t), this implies that the following arrangement is realizable by the roots of a hyperbolic polynomial of degree 4: $(0, f, 0, s, t, f, 0, s, f, 0)$ $(**)$. This is arrangement (A_1) from Remark 6 which is not realizable by a hyperbolic polynomial.

2^0. Suppose that arrangement (3) is realizable by the perturbation of a monic hyperbolic polynomial of degree 5 different from x^5 (the case of x^5 is considered in 4^0). Show that this must be a strictly hyperbolic polynomial P.

Indeed, one cannot have $x_2 = x_3$ because this will imply $x_2 = t_1 = l_1 = x_3$; however, $t_1 = l_1$ implies that P''' has a double root $t_1 = t_2$ which is possible only for x^5, see Lemma 4.

The root x_1 of P cannot coincide with x_2, because then they will equal also s_1 which implies that this will be a root of multiplicity ≥ 3. Hence, these roots must equal also x_3, which (see the lines above) is impossible.

One cannot have $x_3 = x_4$ because this implies that $x_3 = s_2 = x_4$, hence, the root x_3 of P is of multiplicity at least 3 and given that $x_2 < x_3$, one must have $x_3 = x_4 = x_5 = t_2$. Hence, the root x_3 is of multiplicity ≥ 4 which by $x_2 < x_3$ is impossible.

Finally, $x_4 = x_5$ implies $x_4 = x_5 = s_3$ which means that x_4 is of multiplicity ≥ 3, i.e. $x_3 = x_4 = x_5$. It was shown above that this is impossible.

3^0. The polynomial P from 2^0 being strictly hyperbolic, so is its derivative $Q = P'$ as well. Hence, the arrangement defined by the roots of Q and its derivatives is either $(**)$ (i.e. (A_1) which is impossible, see Remark 6; here once again we shift the order of the derivatives by 1) or is obtained from $(**)$ by replacing certain inequalities between roots by the corresponding equalities.

The polynomial Q being strictly hyperbolic, only equalities of the form $s_i = x_j$ or $t_1 = f_k$ are possible, see Lemma 4. More exactly – of the form $s_1 = x_2$, $s_2 = x_3$ and/or $t_1 = f_2$. Any two of them imply the third one, see [4] or just look at Fig. 1. One can deduce from that figure that if one has $s_1 = x_2$, $s_2 > x_3$ or $s_1 > x_2$, $s_2 = x_3$, then one has $t_1 > f_2$; and that if one has $s_1 > x_2$, $t_1 = f_2$, then one has $s_2 < x_3$.

Hence, up to rescaling of the x-axis the polynomial Q must be the polynomial $P_1 = x^4 - x^2 + 5/36$, see Remark 8, and it defines arrangement (A_B) from Remark 8. This means that up to rescaling of the x-axis one has $P(x) = x^5 - 5x^3/3 + 25x/36$.

One has $P'' = 20x^3 - 10x = 20x(x - 1/\sqrt{2})(x + 1/\sqrt{2})$. Thus $P(1/\sqrt{2}) = 1/9\sqrt{2} > 0$ and one has $x_3 < s_3 < x_4$. By perturbing such a polynomial one cannot obtain arrangement (3) in which one has $x_4 < s_3$. This means that arrangement (3) cannot be obtained by analytically perturbing a monic hyperbolic polynomial different from x^5.

4^0. Suppose that the family of functions $f(x, \varepsilon)$ ($f(x, 0) = x^5$), analytic both in $x \in \mathbf{C}$ and $\varepsilon \in (\mathbf{C}, 0)$, realizes arrangement (3) for some $\varepsilon \neq 0$. Set $f = \sum_{j=0}^{\infty} a_j(\varepsilon)x^j$, $a_j \in \mathcal{O}_\varepsilon$, $a_5(0) = 1$, $a_j(0) = 0$ for $j \neq 5$. Without loss of generality we assume that $a_5(\varepsilon) \equiv 1$. One cannot have $a_j \equiv 0$ for $j = 0, \ldots, 4$ because for such a deformation of x^5 one does not get arrangement (3).

Denote by v_j the valuation of a_j, i.e. one has $a_j = \varepsilon^{v_j} b_j(\varepsilon)$, $b_j(0) \neq 0$. We set $v_j = \infty$ if $a_j \equiv 0$.

Set $\varepsilon = \eta^{120}$ and $g(x, \eta) = f(x, \eta^{120})$. Hence, for $j = 0, \ldots, 4$ one has $a_j = \eta^{120 v_j} b_j(\eta^{120})$, $v_j \in \mathbf{N}$, with at least one non-zero v_j.

Denote by l the smallest of the non-zero numbers $120 v_j/(5 - j)$, $j = 0, \ldots, 4$ (notice that they are integers), and by j_0 one of the indices $j \leq 4$ for which this minimum is attained. Set $x \mapsto \eta^l x$. This change transforms g into $h(x, \eta) =$

$\eta^{5l}h_1(x,\eta)$ where h_1 is analytic in (x,η) and $h_1(x,0)$ is a monic polynomial of degree 5 which is not x^5 (it contains also the monomial x^{j_0}). Indeed, the monomial $\eta^{120v_j}x^j$ (resp. x^5) becomes $\eta^{120v_j+jl}x^j$ (resp. $\eta^{5l}x^5$). For $j=j_0$ one has $120v_j+jl=5l$ while for the other indices $j\leq 4$ one has $120v_j+jl\geq 5l$. For $j>5$ one has $120v_j+jl>jl>5l$.

Hence, the function h/η^{5l} is an analytic deformation of a (necessarily hyperbolic) monic polynomial of degree 5 different from x^5 and realizes arrangement (3) which by 2^0-3^0 is impossible. $\square$

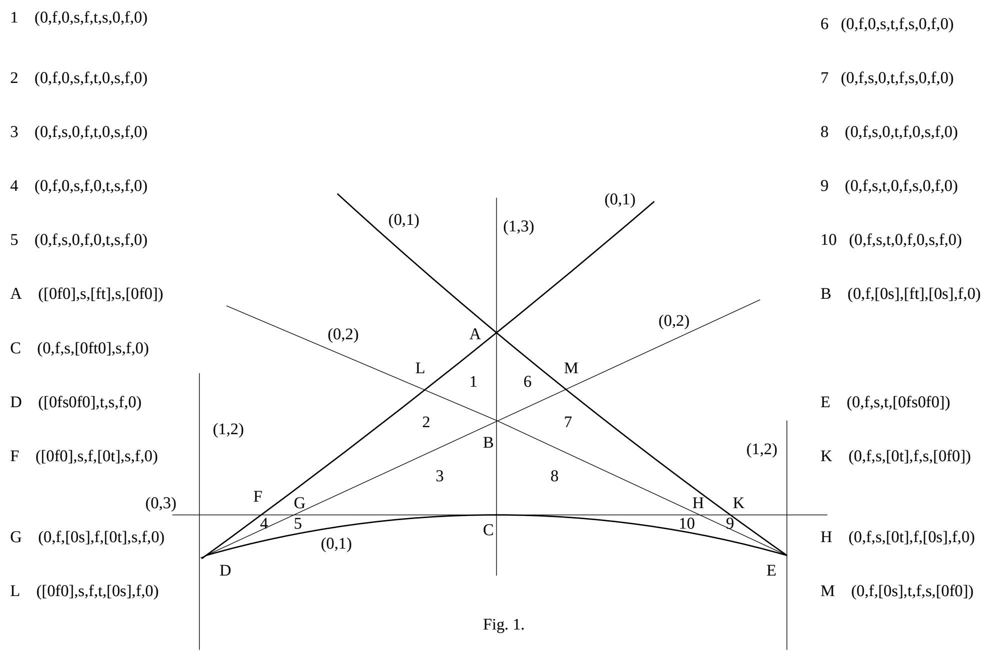

Fig. 1

2. Proof of Theorem 13.

2.1. Plan and basic ideas of the proof. Fig. 1 represents the *hyperbolicity domain* of the family of polynomials $P=x^4-x^2+ax+b$, i.e. the set of values of the coefficients $a,b\in\mathbf{R}$ for which the polynomial is hyperbolic. (One can always normalize the first three coefficients into $1,0,-1$ by a shift of the origin, a change of the scope of the x-axis and by multiplying the polynomial by a non-zero number.) This domain is a curvilinear triangle divided by the *discriminant sets* into 10 open subdomains on each of which one and the same non-degenerate arrangement is realized. The discriminant sets are the sets $D(i,j) := \{(a,b) \in \mathbf{R}^2 | \mathrm{Res}(P^{(i)}, P^{(j)}) = 0\}$, $0 \leq i,j \leq 3$, $i \neq j$.

The boundaries of the domains consist of arcs on which exactly one equality of the form $x_k^{(l)} = x_\mu^{(\nu)}$ takes place and of points at which two or more such equalities take place. (We call them further just *equalities* for short.) On each arc and at each point a degenerate arrangement is realized. For the domains and for the points the arrangements are indicated beside the figure. From them the ones corresponding to the arcs can be deduced immediately.

Definition 15. *Two or more equalities are called* independent *if they are linearly independent as linear equalities and if none of them results from the others due to condition* (2). *The domains, arcs and points on each of which one and the same arrangement of the roots of P and its derivatives is realized are called the* strata *of the hyperbolicity domain. The points A and B are the only* overdetermined strata, *i.e. strata on which more than two independent equalities hold, see* [4], [2] *or* [3]. *Further we denote the arrangement realized at a point X by (A_X).*

Lemma 16. 1) *No a priori admissible arrangement has four independent equalities.*

2) *There are exactly four a priori admissible arrangements with three independent equalities. Two of them (namely,*

$$(A_A) \ : \ ([0f0], s, [ft], s, [0f0]) \quad \text{and} \quad (A_B) \ : \ (0, f, [0s], [ft], [0s], f, 0)$$

see Fig. 1*) are realizable by hyperbolic polynomials. The other two are*

$$(A') \ : \ ([0f0], s, [ft], [0s], f, 0) \quad \text{and} \quad (A'') \ : \ (0, f, [0s], [ft], s, [0f0]).$$

3) *The arrangements (A') and (A'') are realizable by non-hyperbolic polynomials of degree* 6 *which are hyperbolic PLFs of degree* 4.

4) *These two arrangements cannot be realized by analytic deformations (perturbations) of hyperbolic polynomials of degree* 4.

The lemmas from this subsection are proved in the next one.

Remarks 17. 1) In general one expects overdetermined strata not to be present (to define a stratum which is a point on Fig. 1 one needs two, not three independent equalities). Their presence is explained by a symmetry, i.e. by the fact that at these strata P is even. This explains why the other two arrangements with three independent equalities (A') and (A'') are not realizable by hyperbolic polynomials; to realize them one needs an additional parameter.

2) Geometrically overdetermined strata are points on Fig. 1 where three sheats of discriminant sets defined by three independent equalities meet at one point. This is the case of the points A and B but not of C where the three equalities $x_2 = x_3 = f_2 = t_1$ are not independent (the second results from the first one).

3) One can think of the analytic perturbations of hyperbolic polynomials (in particular, of the Gegenbauer polynomial) in the following way. Adding the perturbation parameter ε results in perturbing the discriminant sets which intersect at the point B, see Fig. 1. If one fixes a small non-zero value of ε, then they do not intersect at one point and the set $D(1,3)$ moves, say, to the left of the self-intersection point of the set $D(0,2)$ (which we still call point B). The small triangle with vertex at B and formed by the sets $D(1,3)$ and $D(0,2)$ will be a domain in which arrangement (A_1) is realized. One can easily see which degenerate arrangements are realized on the vertices and on the sides of this triangle. If the set $D(1,3)$ moves to the right, then in the small triangle to the right of the point B arrangement (A_2) will be realized.

To realize arrangement (A') (resp. (A'')) the set $D(1,3)$ must move to the left (resp. to the right) till it passes through the point L (resp. M), see Fig. 1. This is not a small perturbation but a global deformation which illustrates geometrically part 4) of the lemma.

Lemma 18. *All a priori admissible arrangements with exactly two independent equalities and different from (A_D), (A_C), (A_E), (A_F), (A_G), (A_H), (A_K), (A_L) and (A_M) can be obtained by perturbing analytically the polynomials realizing arrangements (A_A), (A_B), (A') or (A''). The perturbed polynomials are of degree 6; they are hyperbolic PLFs of degree 4.*

Remark 19. When perturbing a non-hyperbolic polynomial P of degree 6 which is a hyperbolic PLF of degree 4 we add εQ, $\varepsilon \in (\mathbf{R}, 0)$ where Q is a real polynomial of degree < 6. Therefore the perturbed polynomial is still a hyperbolic PLF of degree 4 for ε small enough.

Lemma 20. *All a priori admissible arrangements with exactly one equality and not realizable by hyperbolic polynomials of degree 4 can be obtained by perturbing analytically polynomials of degree 6 which are hyperbolic PLFs of de-*

gree 4. *The perturbed polynomials are also of degree* 6 *and hyperbolic PLFs of degree* 4.

All non-degenerate arrangements are realizable by hyperbolic polynomials of degree 4 (see Fig. 1) or by polynomials of degree 6 which are perturbations of degree 4 hyperbolic ones, see Lemma 7 and Remark 11. From the degenerate arrangements only (A'), (A'') and six of the arrangements obtained by perturbing these two cannot be realized by perturbations of hyperbolic polynomials, see Remarks 23 and 24. Therefore Lemmas 18 and 20 and Remark 19 finish the proof of the theorem. □

Remark 21. Some of the arrangements from Lemma 20 are in fact obtained by perturbing first arrangements (A_A), (A_B), (A') or (A'') (see Lemma 18) and then again perturbing one of the newly obtained arrangements with exactly two independent equalities. Giving directly a one-parameter perturbation when starting from (A_A), (A_B), (A') or (A'') is possible but technically more difficult.

2.2. Proofs of the lemmas.

Proof of Lemma 16. 1^0. Parts 1) and 2) of the lemma are to be checked directly. Part 1) implies part 4) – if an arrangement with three independent equalities can be realized as a perturbation of another arrangement, then the latter should have at least four independent equalities which by part 1) is impossible.

2^0. Prove part 3). Look for a real polynomial $f = x^6+ax^5+bx^4+cx^3+dx^2$ which realizes arrangement (A') (the root $x_1 = x_2$ is at 0). Impose the conditions

$$f(1) = f''(1) = 0 \ \ , \ \ f'(w) = f'''(w) = 0 \tag{4}$$

We are looking for $w \in (0,1)$ for which arrangement (A') is realized. This means that the following condition must be added to system (4):

$$5a^2 - 12b < 0 \tag{5}$$

It results from $f^{(4)} = 24(15x^2 + 5ax + b)$ having no real roots (f must be a PLF of degree 4).

3^0. Set $\Delta = 40w^4 - 125w^3 + 132w^2 - 56w + 8$. System (4) has the following solution (found with the help of MAPLE):

$$\begin{aligned} a &= -2(56w^5 - 135w^4 + 60w^3 + 56w^2 - 48w + 9)/\Delta \\ b &= 2(40w^6 - 243w^4 + 315w^3 - 120w^2 + 5)/\Delta \\ c &= -4w(50w^5 - 126w^4 + 70w^3 + 40w^2 - 45w + 10)/\Delta \\ d &= w^2(120w^4 - 392w^3 + 456w^2 - 225w + 40)/\Delta \end{aligned}$$

One checks with the help of MAPLE that for $w = 0.55$ one has (5). Hence, this value of w gives a polynomial f which realizes arrangement (A'). For (A'') such a polynomial is $f(-x)$. $\square$

Remark 22. It should be noted that condition (5) holds on a very narrow interval w.r.t. w – $(0.54855..., 0.55249...)$. Therefore using MAPLE here was indispensible. The author acknowledges the help of M. Elkadi who performed the computation with MAPLE.

Proof of Lemma 18. 1^0. We let the reader check oneself that all arrangements with exactly two independent equalities are either (A_D), (A_C), (A_E), (A_F), (A_G), (A_H), (A_K), (A_L), (A_M) or are obtained from (A_A), (A_B), (A'), (A'') by replacing an equality $x_j^{(k)} = x_\mu^\nu$ by an inequality $x_j^{(k)} < x_\mu^\nu$ or $x_j^{(k)} > x_\mu^\nu$.

Perturbing (A_A).

2^0. Arrangement (A_A) is realized by the polynomial $R = (x^2-1)^2$. Consider its perturbation $\tilde{R}(x,\varepsilon) := R(x)+\varepsilon R_1(x)$ where $R_1(x) = (x-1)^2(x+1)^4$. The polynomials R_1' and R_1''' (resp. R' and R''') have only $x = -1$ (resp. $x = 0$) as common zero (to be checked directly). Hence, for almost all $\varepsilon \neq 0$ the polynomials $\tilde{R}'$ and $\tilde{R}'''$ have no root in common.

For $\varepsilon = \varepsilon_0 > 0$ small enough the polynomial $\tilde{R}$ is a hyperbolic PLF of degree 4. (Indeed, it can have no real roots different from ± 1 because both R and R_1 are positive.) It has double roots at ± 1. Hence, it realizes one of the arrangements

$$(A''') \ : \ ([0f0], s, f, t, s, [0f0]) \ \text{ or } \ (A^{(4)}) \ : \ ([0f0], s, t, f, s, [0f0]).$$

The other one of the two is realized by the polynomial $R(x) + \varepsilon_0 R_1(-x)$.

3^0. Arrangement (A_A) is realized also by the polynomial $V_1 := x^2(x-2)^2$. (In this case one has $f_2 = t_1 = 1$.) There exists a real polynomial $V_2 = x^6 + ax^5 + bx^4 + cx^3 + dx^2$ (hence, having a double root at 0) for which one has $V_2'(1) = V_2'''(1) = 0$, $V_2(2) = V_2(3) = 0$ – these conditions are a linear system with unknown variables a, b, c, d and with non-zero determinant. Hence, the polynomial $V(x,\varepsilon) := V_1(x) + \varepsilon V_2(x)$ for $\varepsilon > 0$ small enough is a hyperbolic PLF of degree 4 which realizes arrangement $(A^{(5)}) \ : \ ([0f0], s, [ft], s, 0, f, 0)$. The polynomial $V(-x,\varepsilon)$ realizes arrangement $(A^{(6)}) \ : \ (0, f, 0, s, [ft], s, [0f0])$.

Perturbing (A_B).

4^0. Arrangement (A_B) is realized by a hyperbolic polynomial R_2 with $x_2 = s_1 = -1$, $x_3 = s_2 = 1$. Hence, $R_2 = x^4 - 6x^2 + 5$. The arrangements

$$(0, f, [0s], f, t, [0s], f, 0) \ \text{ and } \ (0, f, [0s], t, f, [0s], f, 0)$$

are realized by polynomials of the kind $R_2 + \varepsilon_0 R_1(\pm x)$ with the same meaning of R_1 and ε_0 as in 2^0.

5^0. To realize arrangements

$$(A^{(7)}) \ : \ (0, f, 0, s, [tf], [0s], f, 0) \ \text{ and } \ (A^{(8)}) \ : \ (0, f, s, 0, [tf], [0s], f, 0)$$

perturb R_2 by a polynomial Φ of degree 6 such that $\Phi'(0) = \Phi'''(0) = 0$ and $\Phi(1) = \Phi''(1) = 0$. The first two of these conditions imply that Φ is of the form $x^6 + \alpha x^5 + \beta x^4 + \gamma x^2 + \delta$.

The root of the polynomial $R_2 + \varepsilon\Phi$ which is close to $x_2 = -1$ equals $x_2 - \varepsilon(\Phi(-1)/R_2'(-1)) + o(\varepsilon)$. One has $\Phi(-1) - 1 = -\alpha + \beta + \gamma + \delta := \mathcal{L}$. The root of the polynomial $R_2'' + \varepsilon\Phi''$ which is close to $s_1 = -1$ equals $s_1 - \varepsilon(\Phi''(-1)/R_2'''(-1)) + o(\varepsilon)$. One has $\Phi''(-1) - 30 = -20\alpha + 12\beta + 2\gamma := \mathcal{M}$.

Show that the linear form $\mathcal{L}/R_2'(-1) - \mathcal{M}/R_2'''(-1)$ (with arguments α, β, γ, δ) is not a linear combination of the linear forms $\Phi(1) - 1 = \alpha + \beta + \gamma + \delta =: \mathcal{U}$ and $\Phi''(1) - 30 = 20\alpha + 12\beta + 2\gamma =: \mathcal{V}$. This will imply that one can choose α, β, γ, δ such that for $\varepsilon \neq 0$ small enough the roots x_2 and s_1 are different; by choosing the sign of ε one can obtain either $x_2 < s_1$ or $x_2 > s_1$.

Indeed, if $\mathcal{L}/R_2'(-1) - \mathcal{M}/R_2'''(-1) = q\mathcal{U} + r\mathcal{V}$, then one must have $q = 1/R_2'(-1)$ (compare the coefficients before δ) and $r = -1/R_2'''(-1)$ (compare the coefficients before γ). But then the coefficients before α to the left and to the right must equal respectively $-1/R_2'(-1) + 20/R_2'''(-1)$ and $1/R_2'(-1) - 20/R_2'''(-1)$. Hence, the equality is possible only if $-1/R_2'(-1) + 20/R_2'''(-1) = 0$. This, however, is not the case – one has $R_2'(-1) = 8$ and $R_2'''(-1) = -24$.

6^0. Similarly to 5^0 one realizes the arrangements

$$(0, f, [0s], [tf], 0, s, f, 0) \ \text{ and } \ (0, f, [0s], [tf], s, 0, f, 0)$$

by means of the polynomial $R_2(x) + \varepsilon\Phi(-x)$.

Perturbing (A').

7^0. Recall that arrangement $(A^{(7)})$ was already obtained in 5^0. Therefore there remains to realize the arrangements obtained by destroying one of the conditions $f_2 = t_1$ and $x_3 = s_2$.

Suppose that arrangement (A') is realized by a polynomial f of degree 6 (which is a PLF of degree 4) such that there hold the conditions $f(0) = f'(0) = 0$ and (4), see the proof of Lemma 16. Recall that we chose $w = 0.55$ in that proof. Consider a real polynomial F of degree 5 of the form $ax^5 + bx^4 + cx^3 + dx^2$. The system of conditions

$$F(1) = \alpha, \quad F''(1) = \beta, \quad F'(w) = \gamma, \quad F'''(w) = \delta$$

is a system of linear equations with unknown variables a, b, c, d. The determinant of this system is non-zero for $w = 0.55$ (computation performed by means of MAPLE).

Hence, one can obtain such polynomials F for which one has $(\alpha, \beta, \gamma, \delta) = (0, \pm 1, 0, 0)$ or $(0, 0, 0, \pm 1)$. In the first case the perturbation $f + \varepsilon F$ preserves the equalities $x_1 = x_2$, $f_2 = t_1$ and destroys $x_3 = s_2$; the latter equality is replaced by $x_3 < s_2$ (we obtain arrangement $(A^{(9)})$: $([0f0], s, [tf], 0, s, f, 0))$ or $x_3 > s_2$ (this is arrangement $(A^{(5)})$) according to the choice of the sign of ε. In the second case the equality $f_2 = t_1$ is replaced by $f_2 < t_1$ (this is arrangement (A_L)) or $f_2 > t_1$ (this gives arrangement $(A^{(10)})$: $([0f0], s, t, f, [0s], f, 0))$ according to the sign of ε while $x_1 = x_2$ and $x_3 = s_2$ are preserved.

8^0. When *perturbing arrangement* (A'') the reasoning is completely analogous. □

Remark 23. Exactly the following six arrangements with two or three independent equalities are not realizable by hyperbolic polynomials of degree 4 or by their perturbations: (A'), (A''), $(A^{(9)})$, $(A^{(10)})$ and the analogs of $(A^{(9)})$, $(A^{(10)})$ when perturbing (A'') instead of (A'). This can be deduced from the proof of Lemma 18.

Proof of Lemma 20. 1^0. Any a priori admissible arrangement (A^*) with only one equality can be obtained from one with two equalities (say, (A^{**})) by replacing one of them by an inequality. We show that in all possible cases one can perturb analytically the polynomial realizing arrangement (A^{**}) to obtain arrangement (A^*).

It suffices to consider the case when the equalities are of one of the three types:

$$1)\ x_i = x_{i+1}; \qquad 2)\ x_i = s_j; \qquad 3)\ f_2 = t_1.$$

We do not mention $x_i = f_i$ and $x_{i+1} = f_i$ in which case an equality of type 1) occurs. We assume that there is no triple root $x_i = x_{i+1} = x_{i+2}$ in which case the arrangements are realized by hyperbolic polynomials.

2^0. If there are two equalities of type 1), then these are $x_1 = x_2$ and $x_3 = x_4$, otherwise there is a triple root; one can perturb the polynomial realizing the arrangement with the two equalities by adding $\varepsilon(x - x_1)^2(x - x_3)$ to destroy $x_3 = x_4$ or $\varepsilon(x - x_1)(x - x_3)^2$ to destroy $x_1 = x_2$. We leave the details in this proof (including the sign of ε) for the reader.

3^0. If there is an equality of type 1) and one of type 2), then these are either $x_1 = x_2$ and $x_3 = s_2$ or $x_3 = x_4$ and $x_2 = s_1$ (otherwise there must be a triple root). We consider only the first case, the second one can be treated by analogy with the first one. One can perturb the polynomial realizing such an arrangement by adding $\varepsilon(x-x_1)^2(x-x_3)$ to destroy $x_3 = s_1$ or $\varepsilon(x-x_3)^3(x-x_1)$ to destroy $x_1 = x_2$.

4^0. If there is an equality of type 1) and one of type 3), then the first is either $x_1 = x_2$ or $x_2 = x_3$ or $x_3 = x_4$. The last case is treated by analogy with the first one. In the second case the arrangements are realized by hyperbolic polynomials.

In the first case one perturbs the polynomial of degree 6 which realizes the arrangement by adding $\varepsilon(x-x_1)^2$ to destroy $f_2 = t_1$ or $\varepsilon(x-f_2)^4(x-x_1)$ to destroy $x_1 = x_2$.

5^0. If there are two equalities of type 2), then these are $x_2 = s_1$ and $x_3 = s_2$. One adds $\varepsilon(x-x_2)^3(x-x_3)$ to destroy $x_3 = s_2$ or $\varepsilon(x-x_2)(x-x_3)^3$ to destroy $x_2 = s_1$.

6^0. If there is one equality of type 2) (say, $x_2 = s_1$) and one of type 3), then one adds $\varepsilon Q(x)$ where $Q(x) = (x-x_2)^3(x-\alpha)$ to destroy $f_2 = t_1$; here α is chosen such that $Q'(f_2) = 0$. To destroy $x_2 = s_1$ one adds $\varepsilon(x-f_2)^4(x-x_2)$.

7^0. In all cases the added polynomials are of degree < 6 and for $\varepsilon \neq 0$ small enough the perturbed polynomial is still a hyperbolic PLF of degree 4, see Remark 19. □

Remark 24. Consider the arrangements with exactly one equality obtained by perturbing $(A^{(9)})$, $(A^{(10)})$ or their analogs in the sense of Remark 23. Only such arrangements with one equality can happen not to be among the ones obtained from hyperbolic polynomials or from their analytic perturbations. For $(A^{(9)})$ (and for $(A^{(10)})$ as well) this is only $([0f0], s, t, f, 0, s, f, 0)$. For their analogs this is only $(0, f, s, 0, f, t, s, [0f0])$. This makes two arrangements.

REFERENCES

[1] B. Anderson. Polynomial root dragging. *Amer. Math. Monthly* **100** (1993), 864–866.

[2] V. P. Kostov. Discriminant sets of families of hyperbolic polynomials of degree 4 and 5. *Serdica Math. J.* **28** (2002), 117–152.

[3] V. P. Kostov. Root configurations of hyperbolic polynomials of degree 3, 4 and 5. *Funct. Anal. Appl.* **36**, *4* (2002), 311–314 (translation from Russian: *Funkcional. Anal. i Prilozhen.* **36**, *4* (2002), 71–74).

[4] V. P. Kostov, B. Z. Shapiro. On arrangements of roots for a real hyperbolic polynomial and its derivatives. *Bull. Sci. Math.* **126**, *1* (January 2002), 45–60.

[5] B. Shapiro, M. Shapiro. This strange and mysterious Rolle theorem. Electronic preprint math.CA/0302215.

[6] R. M. Thrall. A combinatorial problem. *Michigan Math. J.* **1** (1952), 81–88.

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