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ON A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS*

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The aim of this paper is to obtain sharp results involving coefficient bounds, growth and distortion properties for some classes of analytic and univalent functions with negative coefficients.

1. Introduction and definitions. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk E. We denote by C and S^* the classes of convex and starlike functions, respectively.

A function $f(z)$ analytic in E, is said to be starlike of order β $(0 \le \beta < 1)$ in E if $f(0) = f'(0) - 1 = 0$ and

$$\Re \frac{zf'(z)}{f(z)} > \beta$$

for $z \in E$. The class of such functions is denoted by S^*_β. Clearly, $S^*_0 = S^*$.

A function $f(z)$ analytic in E is said to be close-to-convex of order β $(0 \le \beta < 1)$ in E if there exist a function $g(z) \in S^*$ and a real number γ such that for $z \in E$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\Re e^{i\gamma} \frac{zf'(z)}{g(z)} > \beta.$$

The class of such functions is denoted by K_β.

A function $f(z)$ is said to be close-to-star of order β $(0 \le \beta < 1)$ if there exists a function $g(z) \in S^*$ such that for $z \in E$,

$$\Re \frac{f(z)}{g(z)} > \beta.$$

The class of such functions is denoted by R_β.

A function $f(z)$, analytic in E with $f(0) = f'(0) - 1 = 0$ is said to be quasi-convex if and only if there exists a function $g(z) \in C$ such that for $z \in E$,

$$\Re \frac{(zf'(z))'}{g'(z)} > \beta.$$

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The class of such functions is denoted by C^*_β.

Let T denotes the subclass of S, consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

We denote $T^*_\beta = S^*_\beta \cap T$; $K^*_\beta = K_\beta \cap T$; $R^*_\beta = R_\beta \cap T$; $L^*_\beta = C^*_\beta \cap T$.

It is known that $T = T^*_0 = T^*$ and $f \in T^*_\beta$ if and only if, for $0 \le \beta < 1$

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \le 1 \qquad [1].$$

In [2] Schild considered a subclass of T consisting of polynomials having $|z| < 1$ as a disk of univalence. Schild showed [2] that a necessary and sufficient condition for $f \in T$ is

$$1 - \sum_{n=2}^{\infty} n|a_n| = 0.$$

By means of this result he get better results for certain quantities connected with conformal mapping of univalent functions. Other subclasses of T have been studied by Gupta and Jain [3], [4] and Silverman [1], [5].

In this paper we consider the following subclass $H_{t,\alpha}(\beta)$ of T:

Definition 1.1. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is said to be in* $H_{t,\alpha}(\beta)$ $(0 \le \alpha < 1$, $0 \le \beta < 1$, $0 < t \le 1)$, *if there exists a function* $g \in T^*$, *with*

$$g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$$

such that for $z \in E$

$$\Re \left\{ \frac{tzf'(z) + (1-t)z\,(zf'(z))'}{\alpha g(z) + (1-\alpha) z g'(z)} \right\} > \beta. \tag{1.1}$$

Evidently, $H_{1,1}(\beta) = K^*_\beta$, the class of close-to-convex functions of order β introduced in [6]. Note also that $H_{1,0}(\beta) = R^*_\beta$ and $H_{0,1}(\beta) = L^*_\beta$.

In the sequel we write

$$J_{t,\alpha}(f, g, z_0) = \frac{1}{1-\beta} \left\{ \frac{tz_0 f'(z_0) + (1-t) z_0 (z_o f'(z_0))'}{\alpha g(z_0) + (1-\alpha) z_0 g'(z_0)} - \beta \right\}. \tag{1.2}$$

2. Some results about the class $H_{t,\alpha}(\beta)$.

Lema 2.1. *Let* $f \in H_{t,\alpha}(\beta)$ *be given by* (1.1). *Then,*

$$\min_{|z| \le r < 1} \Re J_{t,\alpha}(f, g, z) = J_{t,\alpha}(f, g, r).$$

The proof is standard.

Theorem 2.1. *Let $f(z) \in H_{t,\alpha}(\beta)$ be given by* (1.1). *Then, for $0 < r < 1$,*

$$t\sum_{n=2}^{\infty}\frac{[n|a_n| - (\alpha + n(1-\alpha))|b_n|]\, r^{n-1}}{1 - \sum_{n=2}^{\infty}(\alpha + n(1-\alpha))|b_n| r^{n-1}} + \tag{2.1}$$
$$+(1-t)\frac{\sum_{n=2}^{\infty}[n^2|a_n| - (\alpha + n(1-\alpha))|b_n|] r^{n-1}}{1 - \sum_{n=2}^{\infty}(\alpha + n(1-\alpha))|b_n| r^{n-1}} < 1 - \beta$$

when $0 \le \alpha \le 1$, $0 \le t \le 1$. The estimate (2.1) *is also sufficient for f to be in $H_{t,\alpha}(\beta)$.*

Proof. If $f \in H_{t,\alpha}(\beta)$, then $J_{t,\alpha}(f, g, r) > 0$ for $0 < r < 1$, which is equivalent to (2.1). If, on the other hand, (2.1) holds for every $r < 1$, then from Lemma 2.1 it follows that $f \in H_{t,\alpha}(\beta)$.

Remark A. If $\sum_{n=2}^{\infty} n|a_n| < 1$, $\sum_{n=2}^{\infty} n|b_n| < 1$ and $\sum_{n=2}^{\infty} n^2|a_n| < \infty$, then $J_{t,\alpha}(f, g, r)$ is continuous at $r = 1$ and (2.1) may be replaced by

$$\sum_{n=2}^{\infty}\left(tn + (1-t)n^2\right)|a_n| - \beta\sum_{n=2}^{\infty}(\alpha + (1-\alpha)n)\,|b_n| \le 1 - \beta. \tag{2.2}$$

Remark B. In [1] it was shown that $\sum_{n=2}^{\infty}|b_n| \le \frac{1}{2}$ for $g \in T^*$, so that

$$\sum_{n=2}^{\infty}\left(tn + (1-t)n^2\right)|a_n| \le 1 - \beta + \frac{\beta(2-\alpha)}{2} = 1 - \frac{\alpha\beta}{2}. \tag{2.3}$$

In fact, (2.3) is a necessary condition for f to be in $H_{t,\alpha}(\beta)$ and we could always take $g(z) = z - \frac{1}{2}z^2$ and (2.3) would also be sufficient.

Theorem 2.2. *Let $f \in H_{t,\alpha}(\beta)$ be given by* (1.1). *Then,*

$$a_n \le A_n = \frac{\alpha\beta + (1-\alpha\beta)n}{n^2\,[t + (1-t)n]}.$$

The result is sharp for every n, with equality for

$$f(z) = z - A_n z^n$$

and $g \in T$, with $g(z) = z - \frac{1}{n}z^n$.

Proof. By (2.2) we have

$$\left(tn + (1-t)n^2\right)|a_n| \le 1 - \beta + \beta\,(\alpha + (1-\alpha)n)\,|b_n| \le \frac{\alpha\beta + (1-\alpha\beta)n}{n}$$

and, hence,

$$|a_n| \le \frac{\alpha\beta + (1-\alpha\beta)n}{n^2\,[t + (1-t)n]}$$

where we have used the fact that $|b_n| \le \frac{1}{n}$ [1].

Theorem 2.3. *If $f \in H_{t,\alpha}(\beta)$, then*

$$r - \frac{(2-\alpha\beta)}{4(2-t)}r^2 \le |f(z)| \le r + \frac{(2-\alpha\beta)}{4(2-t)}r^2, \qquad |z| \le r \tag{2.4}$$

$$1 - \frac{(2-\alpha\beta)}{2(2-t)}r \le |f'(z)| \le 1 + \frac{(2-\alpha\beta)}{2(2-t)}r, \qquad |z| \le r. \tag{2.5}$$

Equality holds in all cases for

$$f(z) = z - \frac{2-\alpha\beta}{4(2-t)}z^2.$$

Proof. From (2.3) we have

$$(4-2t)\sum_{n=2}^{\infty}|a_n| \le \sum_{n=2}^{\infty} n\left\{t+(1-t)n\right\}|a_n| \le \frac{2-\alpha\beta}{2}$$

which gives

$$\sum_{n=2}^{\infty}|a_n| \le \frac{2-\alpha\beta}{4(2-t)}. \tag{2.6}$$

Thus,

$$|f(z)| \le r + \frac{2-\alpha\beta}{4(2-t)}r^2.$$

Also

$$|f(z)| \ge r - \sum_{n=2}^{\infty}|a_n|r^n \ge r - \frac{(2-\alpha\beta)}{4(2-t)}r^2$$

where we have used (2.6). Hence, (2.4) follows.

Further,

$$|f'(z)| \le 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \le 1 + r\sum_{n=2}^{\infty} n|a_n|$$

and

$$|f'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \ge 1 - r\sum_{n=2}^{\infty} n|a_n|.$$

Again from (2.3) we obtain

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{2-\alpha\beta}{2(2-t)}$$

and (2.5) follows.

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ВЪРХУ ЕДИН КЛАС ОТ ЕДНОЛИСТНИ ФУНКЦИИ С ОТРИЦАТЕЛНИ КОЕФИЦИЕНТИ

Донка Пашкулева

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