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THREE-DIMENSIONAL OPERATIONAL CALCULI FOR NONLOCAL EVOLUTION BOUNDARY VALUE PROBLEMS*

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Direct algebraic operational calculi for functions $u(x,\ y,\ t)$, continuous in a domain of the form $D = [0,\ a] \times [0,\ b] \times [0,\ \infty)$, are proposed. Along with the classical Duhamel convolution, the construction uses also two non-classical convolutions for the operators ∂_x^2 and ∂_y^2. These three one-dimensional convolutions are combined into one three-dimensional convolution $u * v$ in $C(D)$. Instead of J. Mikusiński's approach, based on convolution fractions, we develop systematically an alternative approach, based on the multiplier fractions of the convolution algebra $(C(D),\ *)$.

1. Introductions. Till recently, most of the existing operational calculi were intended to deal with initial value problems. In Gutterman [8], direct operational calculi for functions of several real variables are proposed, applicable for solution of Cauchy problems for linear partial differential equations with constant coefficients. As for mixed problems, i.e. problems, containing both boundary and initial conditions, Gutterman announced that his method is unpractical, and its extension to them would need essentially new ideas and approaches. Not to speak about nonlocal boundary value problems. Here we extend the direct operational calculus approach to nonlocal boundary value problems for functions of one, two, and three real variables. It is intended to cope with BVPs of the form

$$(1)\qquad \begin{cases} P(\partial_t)u = Q(\partial_x^2)u + R(\partial_y^2)u + F(x,y,t), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < t, \\ \partial_t^k u(x,y,0) = f_k(x,y), \quad k = 0,1,\dots \deg P - 1, \\ \partial_x^{2l} u(0,y,t) = \varphi_l(y,t), \quad \Phi_\xi\{\partial_x^{2l} u(\xi,y,t)\} = g_l(y,t), \quad l = 0,1,\dots \deg Q - 1, \\ \partial_y^{2m} u(x,0,t) = \psi_m(x,t), \ \Psi_\eta\{\partial_y^{2m} u(x,\eta,t)\} = h_m(x,t), \ m = 0,1,\dots \deg R - 1. \end{cases}$$

Here P, Q, R are polynomials and Φ and Ψ are supposed to be non-zero linear functionals on $C^1[0,\ a]$ and $C^1[0,\ b]$, correspondingly. These linear functionals have Stieltjes-type representations of the form:

$$(2)\qquad \Phi\{f\} = Af(a) + \int_0^a f'(\xi)d\alpha(\xi), \quad f \in C^1[0,a] \ \text{ and}$$

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(3) $$\Psi\{g\} = Bg(b) + \int\limits_0^b g'(\eta)d\beta(\eta), \quad g \in C^1[0,b],$$

where $\alpha(x)$ and $\beta(y)$ are function with bounded variation, A and B being constants.

For technical reasons, we suppose that

(4) $$\Phi_\xi\{\xi\} = 1 \ \text{ and } \ \Psi_\eta\{\eta\} = 1.$$

These restrictions may be ousted by some non-essential involvements.

Our operational calculi are connected with the right inverse operators L_x and L_y of $\left(\frac{d}{dx}\right)^2$ and $\left(\frac{d}{dy}\right)^2$ in $C[0,\, a]$ and $C[0,\, b]$, correspondingly defined by:

(5) $$L_x f(x) = \int_0^x (x-\xi)f(\xi)d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\},$$

(6) $$L_y g(y) = \int_0^y (y-\eta)g(\eta)d\eta - y\Psi_\eta\left\{\int_0^\eta (\eta-\varsigma)g(\varsigma)d\varsigma\right\}.$$

2. Three one-dimensional convolutions.

2.1. The Duhamel convolution. This is the operation

(7) $$(\varphi \overset{t}{*} \psi)(t) = \int\limits_0^t \varphi(t-\tau)\psi(\tau)d\tau, \quad \varphi, \quad \psi \in C\,[0,\infty).$$

It bears the name of Duhamel, but sometimes it is called either Borel, or Laplace convolution. It is connected with the integration operator

(8) $$l_t\varphi(t) = \int\limits_0^t \varphi(\tau)\,d\tau,$$

since $l_t\varphi(t) = \{1\} \overset{t}{*} \varphi$. To say it differently, l_t is the convolution operator $\{1\}\overset{t}{*}$.

2.2. A family of convolutions in $C[0,a]$ and $C[0,b]$.

Theorem 1 (Dimovski [2], p. 119). *The operations*

(9) $$(f \overset{x}{*} g)(x) = -\frac{1}{2}\tilde{\Phi}_\xi\{h(x,\xi)\} \quad \textit{and} \quad (f \overset{y}{*} g)(y) = -\frac{1}{2}\tilde{\Psi}_\eta\{h(y,\eta)\},$$

with $\tilde{\Phi}_\xi = \Phi_\xi \circ l_\xi$, $\tilde{\Psi}_\eta = \Psi_\eta \circ l_\eta$ *and*

$$h(x,\xi) = \int\limits_x^\xi f(\xi+x-\varsigma)g(\varsigma)d\varsigma - \int\limits_{-x}^\xi f(|\xi-x-\varsigma|)g(|\varsigma|)sgn\varsigma(\xi-x-\varsigma)\,d\varsigma$$

are bilinear, commutative and associative operations in $C[0,a]$ *and* $C[0,b]$, *correspondingly, such that*

$$L_x f = \{x\} \overset{x}{*} f \quad \textit{and} \quad L_y g = \{y\} \overset{y}{*} g.$$

3. A two-dimensional convolution in $C([0,a]\times[0,b])$.

Theorem 2 (Dimovski [3]). *Let u, $v\in C([0,a]\times[0,b])$. Then,*

$$u(x,y) \overset{(x,y)}{*} v(x,y) = \frac{1}{4}\tilde{\Phi}_\xi \tilde{\Psi}_\eta \{h(x,y,\xi,\eta)\}, \tag{10}$$

with

$$h(x,y,\xi,\eta) = \int_x^\xi \int_y^\eta u(\xi+x-\sigma,\eta+y-\tau)\, v(\sigma,\tau)\, d\sigma\, d\tau$$
$$-\int_{-x}^\xi \int_y^\eta u(|\xi-x-\sigma|,\eta+y-\tau)\, v(|\sigma|,\tau)\,\mathrm{sgn}\,(\xi-x-\sigma)\,\sigma\, d\sigma\, d\tau$$
$$-\int_x^\xi \int_{-y}^\eta u(\xi+x-\sigma,|\eta-y-\tau|)\, v(\sigma,|\tau|)\,\mathrm{sgn}\,(\eta-y-\tau)\,\tau\, d\sigma\, d\tau$$
$$+\int_{-x}^\xi \int_{-y}^\eta u(|\xi-x-\sigma|,|\eta-y-\tau|)\, v(|\sigma|,|\tau|)\,\mathrm{sgn}\,(\xi-x-\sigma)(\eta-y-\tau)\,\sigma\tau\, d\sigma\, d\tau$$

is a bilinear, commutative and associative operation in $C([0,a]\times[0,b])$, such that

$$L_x L_y u(x,y,t) = \{x\, y\} \overset{(x,y)}{*} u(x,y,t).$$

4. A three-dimensional convolution in $C(D)$.

Theorem 3. *Let u, $v\in C = C([0,a]\times[0,b]\times[0,\infty)) = C(D)$. Then,*

$$u(x,y,t) \overset{v}{*} (x,y,t) = \int\limits_0^t u(x,y,t-\tau) \overset{(x,y)}{*} v(x,y,\tau)\, d\tau \tag{12}$$

is a bilinear, commutative and associative operation in $C(D)$, such that

$$l_t\, L_x L_y\, u(x,y,t) = \{x\, y\} *\ u(x,y,t). \tag{13}$$

Proof. If $u(x,\,y,\,t) = f_1(x)g_1(y)h_1(t)$ and $v(x,y,t) = f_2(x)g_2(y)h_2(t)$, then we have $u \overset{v}{*} = (f_1 \overset{x}{*} f_2)(g_1 \overset{y}{*} g_2)(h_1 \overset{t}{*} h_2)$. The commutativity and associativity of (12) for product functions follow from the corresponding properties of one-dimensional convolutions $\overset{t}{*}$, $\overset{x}{*}$ and $\overset{y}{*}$. In order to elucidate the transition from the one-dimensional case to the three-dimensional, let us prove (13). First we take $u(x,\,y,\,t) = f(x)\quad g(y)\quad h(t)$ and get

$$l_t\, L_x L_y u(x,y,t) = l_t\, L_x L_y\, [f(x)g(y)h(t)] = \ L_x f(x)\,.\, L_y g(y)\,.\, l_t h(t) =$$
$$= [\{1\} \overset{t}{*} h(t)\,][\{x\} \overset{x}{*} f(x)][\{y\} \overset{y}{*} g(y)] \ = \{x\, y\} * \{f(x)g(y)h(t)\} = \{x\, y\} *\ u(x,y,t).$$

Then, we conclude that (13) is true for linear combinations of products. But every function $u(x,y,t)\in C(D)$ can be approximated by linear combinations of products functions $f(x)g(y)h(t)$ where $f(x)\in C[0,a]$, $g(y)\in C[0,b]$ and $h(t)\in C[0,\infty)$, e.g. by polynomials of the variables x, y, t. Hence, (13) is true for arbitrary $u\in C(D)$ since (12) is a continuous operation.

5. Multipliers of $(C,*)$. Further, we consider the ring of the multipliers of the convolutional algebra $(C,*)$.

Definition 1. *A linear operator $M : C \to C$ is said to be a multiplier of the algebra $(C,*)$, iff the relation*

$$M(u*v) = (Mu)*v$$

holds for all u, $v \in C$.

If $f \in C[0,a]$, then the convolution operator $f \overset{x}{*}$ in $C_x = C[0,a]$ may be considered also as an operator on the space $C(D)$: $(f \overset{x}{*})\{u(x,y,t)\} = \{f(x)\} \overset{x}{*} \{u(x,y,t)\}$, considering the variables y and t as parameters. The same is true for the convolution operators $g \overset{y}{*}$, where $g = g(y) \in C_y = C[0,b]$; $\varphi \overset{t}{*}$, $\varphi = \varphi(t) \in C[0,\infty)$ and $G \overset{x,y}{*}$ where $G = G(x,y) \in C([0,a]\times[0,b])$, correspondingly.

Lemma 1. *The convolution operators*

$$[\varphi]_t = \{\varphi(t)\} \overset{t}{*}, \quad [f]_x = \{f(x)\} \overset{x}{*}, \quad [g]_y = \{g(y)\} \overset{y}{*}, \quad [G]_{x,y} = \{G(x,y)\} \overset{x,y}{*}.$$

are multipliers of the convolution algebra $(C,*)$.

The proof follows the lines of the proof of relation (13) from Theorem 3.

Let us denote the set of the multipliers of $(C,*)$ by $\mathfrak{M}$. As it is well-known, $\mathfrak{M}$ is a commutative ring (see [9]).

Definition. *The elements $[\varphi]_t$, $[f]_x$, $[g]_y$ and $[G]_{x,y}$ of $\mathfrak{M}$ are said to be numerical operator with respect to x, y to y, t to x, t and to t, correspondingly.*

Remark. Here we deviate slightly from the notations, accepted in [6].

5.1. Ring of the multiplier fractions of $(C,*)$. In the commutative ring $\mathfrak{M}$ there are elements, which are not divisors of 0. Indeed, such elements are the multipliers $\{x\} \overset{x}{*} = [x]_x$ and $\{y\} \overset{y}{*} = [y]_y$, i.e. the operators L_x and L_y.

Denote by $\mathfrak{N}$ the set of the non-zero non-divisors of zero on $\mathfrak{M}$. The set $\mathfrak{N}$ is a multiplicative subset on $\mathfrak{M}$, i.e. such that p, $q \in \mathfrak{N}$ implies $pq \in \mathfrak{N}$.

Further, we consider the multiplier fractions of the form $\dfrac{M}{N}$ with $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$. They are introduced in a standard manner, using the well-known method of “localisation” from the general algebra (see [10]).

Let $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$ be the set of multiplier fractions of $(C,*)$. It is a commutative ring, containing the basic field ($\mathbb{R}$ or $\mathbb{C}$), the algebras $(C[0,a], \overset{x}{*})$, $(C[0,b], \overset{y}{*})$, $(C[0,\infty), \overset{t}{*})$, $C([0,a]\times[0,b], \overset{x,y}{*})$, $(C,*)$ and $\mathfrak{M}$, due to the embeddings:

i) $$\mathbb{R} \hookrightarrow \mathcal{M} \;\text{ or }\; \mathbb{C} \hookrightarrow \mathcal{M} \;\text{ by }\; \alpha \mapsto \frac{\alpha L_x}{L_x},$$

ii) $$(C[0,a], \overset{x}{*}) \hookrightarrow \mathcal{M} \;\text{ and }\; (C[0,b], \overset{y}{*}) \hookrightarrow \mathcal{M} \;\text{ by }\; f(x) \mapsto \frac{(L_x f)\overset{x}{*}}{L_x} \;\text{ and }\; g \mapsto \frac{(L_y g)\overset{y}{*}}{L_y},$$

iii) $$(C[0,\infty), \overset{t}{*}) \hookrightarrow \mathcal{M} \;\text{ by }\; \varphi \mapsto \frac{[l_t\,\varphi]_t}{l_t} = \frac{(l_t \varphi)\overset{t}{*}}{l_t},$$

iv) $\quad (C([0,a]\times[0,b]), \overset{x,y}{*}) \hookrightarrow \mathcal{M}$ by $G \mapsto \dfrac{[L_x G]_{x,y}}{L_x} = \dfrac{[L_y G]_{x,y}}{L_y} = \dfrac{(L_x L_y G)\overset{x,y}{*}}{L_x L_y}$,

v) $\quad (C([0,a]\times[0,b]\times[0,\infty)), *) \hookrightarrow \mathcal{M}$ by $u \mapsto \dfrac{(l_t L_x L_y u)*}{l_t L_x L_y} = \dfrac{u*}{1}$,

where 1 is the unit of $\mathcal{M}$.

Further, we consider all numbers, functions, multiplier and multiplier fractions as elements of a *single algebraic system*: the ring $\mathcal{M}$ of the multiplier fractions.

5.2. Elements of $\mathcal{M}$. In the ring $\mathcal{M}$ we introduce the algebraic inverses $s = \dfrac{1}{l_t}$, $S_x = \dfrac{1}{L_x}$ and $S_y = \dfrac{1}{L_y}$ of the multipliers l_t , L_x and L_y, correspondingly.

Let $E(\lambda) = \Phi_\xi \left\{\dfrac{\sin\lambda\xi}{\lambda}\right\}$ and $F(\mu) = \Psi_\eta\left\{\frac{\sin\mu\,\eta}{\mu}\right\}$ be the sine-indicatrices of the functionals Φ and Ψ.

Theorem 4. *If $E(\lambda)\neq 0$ and $F(\mu)\neq 0$, then $S_x+\lambda^2$ and $S_y+\mu^2$ are non-divisors of zero in $\mathcal{M}$, and*

$$\frac{1}{S_x+\lambda^2} = \left\{\frac{\sin\lambda x}{\lambda E(\lambda)}\right\}\overset{x}{*} \quad\text{and}\quad \frac{1}{S_y+\mu^2} = \left\{\frac{\sin\mu y}{\mu E(\mu)}\right\}\overset{y}{*}.$$

For a proof see [2].

The elements S_x, S_y, s are connected with the differential operators, $\dfrac{\partial^2}{\partial x^2}$, $\dfrac{\partial^2}{\partial y^2}$ and $\dfrac{\partial}{\partial t}$, but are not identical with them.

Theorem 5. *Let $u\in C(D)$ and u_{xx}, u_{yy} and u_t exist and continuous in D. Then,*

$$u_{xx} = S_x u + S_x\{(x\Phi_\xi\{1\}-1)u(0,y,t) - x\Phi_\xi\{u(\xi,y,t)\}\}, \tag{15}$$

$$u_{yy} = S_y u + S_y\{\,(y\,\Psi_\eta\{1\}-1)u(x,0,t) - y\,\Psi_\eta\{u(x,\eta,t)\}\}, \tag{16}$$

$$u_t = s\,u - [u(x,y,0)]_{x,y}. \tag{17}$$

Proof. Relation (17) is similar to one in Mikusiński [11]. As for (15) and (16), they can be proved in one and the same way. Let us prove (15). It is easy to verify the identity

$$L_x\{u_{xx}\} = u(x,y,t) + (x\Phi_\xi\{1\}-1)\;u(0,y,t) - x\;\Phi_\xi\{u(\xi,y,t)\}.$$

It remains simply to multiply this relation by S_x. □

Relations (15)–(17) allow to algebraise our BVP, i.e. to reduces (1) to the single algebraic equation in $\mathcal{M}$:

$$[P(s) - Q(S_x) - R(S_y)]u = \tilde{F}$$

where $\tilde{F}$ is a known element of $\mathcal{M}$.

The problem of uniqueness of the solution of (1) reduces to the algebraic problem, whether $P(s)-Q(S_x)-R(S_y)$ is a divisor of zero or not.

Example. For definiteness, let us consider the general nonlocal BVP for the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy} + F(x, y, t), \quad 0 < t, \quad 0 < x < a, \quad 0 < y < b, \tag{18}$$

determined by the initial and boundary conditions

$$\begin{aligned} u(x, y, 0) = f(x, y), \quad & u(0, y, t) = 0, \quad \Phi_\xi\{u(\xi, y, t)\} = p(y, t), \\ & u(x, 0, t) = 0, \quad \Psi_\eta\{u(x, \eta, t)\} = q(x, t). \end{aligned} \tag{19}$$

Using (15)–(17), the BVP (18)–(19) reduces to the following algebraic equation in $\mathcal{M}$:

$$(s - S_x - S_y)u = [f(x, y)]_{x,y} - [p(y, t)]_{y,t} - [q(x, t)]_{x,t} + F(x, y, t). \tag{20}$$

The problems of uniqueness of the solution of BVP (18)–(19) in $C(D)$ reduces to the problem of uniqueness of the solution of (20) in $\mathcal{M}$ i.e. to the question, when the elements $s - S_x - S_y$ is a non-divisor of zero in $\mathcal{M}$. The following lemma supplies a necessary condition.

Lemma 3. *Let $a \in \operatorname{supp}\Phi$ and $b \in \operatorname{supp}\Psi$. Then, the element $s - S_x - S_y$ is a non-divisor of zero in $\mathcal{M}$.*

The proof follows the lines of the proof for uniqueness in the one-dimensional problem (see [4])

$$\begin{aligned} & u_t = u_{xx} + F(x, y, t), \quad 0 < t, \quad 0 < x < a, \quad 0 < y < b \\ & u(x, 0) = f(x), \\ & u(0, t) = 0, \quad \Phi_\xi\{u(\xi, t)\} = p(t) \end{aligned}$$

with essential used of a theorem of N. Bozhinov [1].

Theorem 6. *Boundary value problem* (18)–(19) *has a unique solution, provided a* $\in \operatorname{supp}\Phi$ *and* $b \in \operatorname{supp}\Psi$.

Proof. Assume the contrary, i.e. that there are two different solutions u_1 and u_2 of (18)–(19). Then, $u = u_1 - u_2$ satisfy the homogeneous BVP

$$\begin{aligned} & u_t = u_{xx} + u_{yy}, \quad 0 < t, \quad 0 < x < a, \quad 0 < y < b, \\ & u(x, y, 0) = 0, \qquad u(0, y, t) = 0, \quad \Phi_\xi\{u(\xi, y, t)\} = 0, \\ & \qquad\qquad\qquad\quad u(x, 0, t) = 0, \quad \Psi_\eta\{u(x, \eta, t)\} = 0. \end{aligned}$$

By (15)–(17) the problem reduces to the equation

$$(s - S_x - S_y)u = 0$$

in $\mathcal{M}$. But, by Lemma 3, $s - S_x - S_y$ is a non-divisor of zero in $\mathcal{M}$. Hence $u \equiv 0$, i.e. $u_1 \equiv u_2$. □

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ТРИМЕРНИ ОПЕРАЦИОННИ СМЯТАНИЯ ЗА НЕЛОКАЛНИ ЕВОЛЮЦИОННИ ГРАНИЧНИ ЗАДАЧИ

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Построени са директни операционни смятания за функции $u(x, y, t)$, непрекъснати в област от вида $D = [0, a] \times [0, b] \times [0, \infty)$. Наред с класическата дюамелова конволюция, построението използва и две некласически конволюции за операторите ∂_x^2 и ∂_y^2. Тези три едномерни конволюции се комбинират в една тримерна конволюция $u * v$ в $C(D)$. Вместо подхода на Я. Микусински, основаващ се на конволюционни частни, се развива алтернативен подход с използване на мултипликаторните частни на конволюционната алгебра $(C(D), *)$.