ABOUT TOPOLOGICAL GROUPS AND THE BAIRE
PROPERTY IN REMAINDERS∗

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In this paper we study the remainders with Baire property of topological groups.

1. Introduction. By a space we understand a Tychonoff topological space. We use
the terminology from [11]. The present paper is a continuation of the articles [1, 2], which
contain the definitions of an o-homogeneous space, a fan-complete space, a q-complete
space, a sieve-complete space, an lo-homogeneous space, a do-homogeneous space, a co-
homogeneous space.

A remainder of a space X is the subspace Y \ X of a Tychonoff extension Y of X.
The space Y is an extension of X if X is a dense subspace of Y.

In this article we consider what kind of remainders can have a space.

Problem A. Let P be a property and Y be an extension of a space X. Under which
conditions the remainder Y \ X has the property P?

In [3, 4, 5, 6, 7, 8] the Problem A was examined for topological groups. Some results
for rectifiable spaces were obtained in [9].

A particular case of the Problem A is the next question

Question A. Under which conditions the Stone-Čech compactification βX of a space
X is the Stone-Čech compactification of the remainder βX \ X?

In [4], Theorem 1.1, A. V. Arhangel’skii has proved: If a topological group G is a dense
subspace of the Čech-complete space X and G is not Čech-complete, then the subspace
Y = X \ G is dense in X and has the Baire property. We establish that analogous result
is true for a more large classes of spaces.

One of the first remarkable results concerning Problem A was obtained by E. Čech,
M. Henriksen and J. R. Isbel. Theorem of E. Čech affirms that for any space X the
character χ(x, βX) of any point x ∈ βX \ X is uncountable. Theorem of M. Henriksen
and J. R. Isbel affirms that the remainder βX \ X is a Lindelöf space if and only if the
space X is of countable type (see [11, 12]).

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2. **On remainders of topological groups.** Let $G$ be a topological group. Let $\rho G$ be the Raikov completion of a topological group $G$. If the topological group $G$ is densely fan-complete, then the Raikov completion $\rho G$ is a paracompact Čech-complete space.

A space $X$ is called a **paracompact $p$-space** if it admits a perfect mapping onto a metrizable space. A feathered group is a topological group whose underlying space is a paracompact $p$-space. A topological group is a feathered group if and only if it is a space of pointwise countable type (see [10]).

In [9] it was proved that a topological group $G$ is densely fan-complete if and only if it is fan-complete and, consequently, $G$ is a dense $G_δ$-subspace of some pseudocompact space. Any topological group is an $o$-homogeneous space. Thus from Theorem 4.4 in [2] it follows:

**Corollary 2.1.** Let $Y$ be a densely fan-complete extension of a topological group $G$. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or $G$ is a fan-complete space.

In [9] it was proved that a topological group $G$ is densely $q$-complete if and only if it is $q$-complete and there exists a countably compact subgroup $H$ such that the quotient space $G/H$ is metrizable and the projection $\pi : G \to G/H$ is open and closed. Moreover, a $q$-complete topological group is a $G_δ$-subset in some countably compact extension. Thus from Theorem 4.5 in [2] it follows:

**Corollary 2.2.** Let $Y$ be a densely $q$-complete extension of a topological group $G$. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or $G$ is a $q$-complete space.

From Theorem 4.5 in [2] it follows:

**Corollary 2.3.** Let $Y$ be a densely sieve-complete extension of a topological group $G$. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or $G$ is a paracompact Čech-complete space.

**Corollary 2.4** (A. V. Arhangel’skii [4]). Let $Y$ be a Čech-complete extension of a topological group $G$. Then, either the remainder $Z = Y \setminus G$ has the Baire property, or $G$ is a paracompact Čech-complete space.

**Example 2.5.** Let $\xi \in \beta \omega \setminus \omega$ and $X = \{\xi\} \cup \omega$. We put $L = \mathbb{R}^X$ and $B = C_p(X) \subseteq L$. D. J. Lutzer and R. A. McCoy [13] proved that the space $B$ is not complete metrizable and has the Baire property. Since $B$ is not Čech-complete, from Corollary 2.3 (or from Theorem 1.1 of [4]) it follows that $Y = L \setminus B$ is a dense subspace of $L$ and has the Baire property. Thus the linear space $L$ is a complete metrizable extension of the spaces $B, Y$ with the Baire property, $B$ is a linear subspace and it is not complete metrizable.

3. **Embedding into remainders of topological groups.** The following fact is a generalization of Theorem 2.18 from [8].

**Theorem 3.1.** Let $Y$ be a space. Then there exist a compact abelian group $A$ and a dense subgroup $B$ of $A$ such that:

1. $X = A \setminus B$ is a pseudocompact space.
2. $Y$ is a closed subspace of the space $X$.
3. $A$ is a compactification of the space $X$.

**Proof.** There exists a compact space $K$ such that $Y$ is a nowhere dense subspace of the space $K$ and the subspace $\Phi = K \setminus \text{cl}_K Y$ is not paracompact and has not $G_δ$-points.

Fix a point $e \in \Phi \subseteq K$. Then, there exists a compact abelian group $A$ with the properties:

1. $K$ is a subspace of the space $A$ and $e$ is the unity of the group $A$. 

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2. For any continuous mapping $f : K \to H$ into a compact abelian group $H$ for which $f(e)$ is the unity of $H$ there exists a continuous homomorphism $\overline{f} : A \to H$ such that $f = \overline{f}|B$.

3. The group $G$ algebraically generated by the set $K$ in $A$ is dense in $A$.

Put $Z = K \setminus Y$ and denote by $B$ the subgroup of $A$ algebraically generated by the set $Z$. Now, put $X = A \setminus B$. Let $F_1 = K \cup \{ x^{-1} : x \in K \}$ and $F_n = F_1^n$ for each $n \in \mathbb{N}$. Then, $G = \bigcup \{ F_n : n \in \mathbb{N} \}$ and each set $F_n$ is nowhere dense in $A$.

By construction, the set $B$ is dense in $A$. Thus each set $B_n = F_n \cap B$ is nowhere dense in $B$.

Claim 1. The subspace $Y$ is closed in $X$.

The set $K$ is compact and $Y = K \cap X$.

Claim 2. The group $B$ is not locally compact.

If $B$ is locally compact, then $B$ is open-and-closed in $A$, that is a contradiction.

Claim 3. The space $A$ is a compactification of the space $X$.

This assertion follows from Claim 2.

Claim 4. The space $X$ is not Lindelöf.

Assume that the space $X$ is Lindelöf. Since $A$ is a compactification of the space $B$ and $X = A \setminus B$, by virtue of Theorem of M. Henriksen and J. R. Isbel the space $B$ is of countable type (see [11, 12]). Then, in $B$ there exists a compact subgroup $C$ of countable character such that $C \cap K \subseteq \Phi$. The natural projection $g : A \to A/C$ is an open-and-closed continuous homomorphism and $g^{-1}(g(B)) = B$. Thus $A$, $X$ and $Y$ are Lindelöf $p$-spaces. Since $C$ is compact, there exist $n \in \mathbb{N}$ and an open non-empty subset $V$ of $C$ such that $V \subseteq C \cap F_n$. Then we can assume that $C = V \subseteq F_n$. In this case $C \cap K = \{ e \}$ and the space $K$ has a countable base at the point $e$, a contradiction. Claim is proved.

Claim 5. The space $X$ is pseudocompact.

Since $A$ is not a paracompact $p$-space, the set $X$ is $G_\delta$-dense in $A$ and $X$ is pseudocompact [8, 9]. The proof is complete.

REFERENCES


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ОТНОСНО ТОПОЛОГИЧНИ ГРУПИ И СВОЙСТВОТО НА БЕР В ПРИРАСТА

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