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SECOND ORDER OPTIMALITY CONDITIONS IN NONSMOOTH UNCONSTRAINED OPTIMIZATION

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Second order sufficient and necessary conditions are given for a nonsmooth function f defined in a Banach space to attain a minimum at a point in the interior of its domain. At the beginning sufficient conditions in terms of Riemann type derivatives are introduced. The considered examples suggest improvements to gain more efficiency. Consequently second order conditions based on generalized Riemann type finite difference are proved and their efficiency is shown. On this ground a generalized second order derivative is defined.

Keywords: second order optimality conditions, nonsmooth optimization, first and second order directional derivatives, Riemann type derivatives.

AMS subject classification: 49J52, 90C30.

1 Introduction

Many of the concepts in nonsmooth analysis have their origin in classical analysis. In particular first and second order conditions in nonsmooth optimization are usually constructed as analogues of known results from classical optimization theory. Much has been done for the first order case. Following J.-P. Penot [16] we say that the challenge to introduce sensible notions of second order derivatives for functions which do not even have a first order derivative is certainly appealing. This is actually the intention of this paper. We introduce second order conditions on the ground of Riemann type derivatives. In Section 2 the lower second order Riemann type derivative is defined using some type of $\liminf$ instead of the usual limit from the classical case. By symmetry the upper derivative is constructed replacing $\liminf$ by $\limsup$. Section 3 states sufficient conditions in terms of the introduced derivatives in order that a point x_0 be a minimizer of the function f. The assumption for f is that it is defined on a finite dimensional Banach space X and takes values in the extended real line $\overline{R}$ (the finite dimension of X is needed only for the sufficient conditions, because the proof uses the compactness of the unit sphere in X).

The assumption for x_0 is that it is an interior point for the domain $\operatorname{dom} f$. This demand comes to avoid the eventual appearance of indefinite differences of the type $\infty - \infty$ in the definition of the second order derivative. In principle this restriction can be removed, but since this possibility is not considered here, we find the presented theory appropriate only for unconstrained problems. Identity (2) used in the proof can be considered as a discrete analogue of the second order Taylor expansion of f near x_0, that is the derivatives in the usual Taylor expansion formula are replaced by finite differences. In Section 4 we apply on examples the proved sufficient conditions. The "oscillating" function in Example 3 is used as a test for the "efficiency" of these conditions. Unfortunately the test fails. The fixed midpoint in the Riemann type derivative has bad influence. In Section 5 we improve the sufficient conditions using generalized Riemann type finite differences and show on the test example their efficiency. Section 6 states necessary conditions corresponding to the sufficient ones from Section 3 (Theorem 3) and Section 5 (Theorem 4). Theorem 3 illustrates the usages of the upper second order derivatives and makes evident the "gap" between the necessary and sufficient conditions based on Riemann type derivatives, while this gap does not occur in Theorem 4 based on generalized Riemann type finite differences.

Obviously two questions fall immediately into mind: to describe the relation with other second order derivatives and to compare the obtained conditions with known second order conditions. The second question will not be discussed, since we think this discussion must be postponed till the theory appropriates to treat constrained problems, that is to avoid the restriction of x_0 being only interior point for the domain of f.

Second order generalized derivatives and second order condition in nonsmooth optimization are considered by many authors. In 1965 A. Dubovitskij and A. Milyutin [9] define second order objects in variational calculus. In 1974 V. Demyanov and A. Pevnyj [8] studying parametric problems of mathematical programming introduce some generalized second order derivative. In 1978 K.-H. Hoffmann and H. Kornstaedt [11] consider higher order necessary conditions in abstract mathematical programming. Thus the topic has more than 20 years old history. This subject is studied by A. Ben-Tal [1] and A. Ben-Tal and J. Zowe [2]. They develop the concept of so-called (exact) parabolic derivatives being intensively studied and modified, say allowing acceleration by taking limits along "horns" rather than along parabolic curves (approximate parabolic derivatives), see e. g. J.-P. Penot [16] and elsewhere. Another tool in nonsmooth optimization are the epi-derivatives introduced for the convex case by J.-B Hiriart-Urruty [10] and developed for the nonconvex case by R. T. Rockafellar [19], A. D. Ioffe [12], M. Studniarski [20] and others.

Roughly speaking the classical analysis suggests two possible approaches to a direct definition of second order directional derivatives for nonsmooth functions. One of them comes from solving the Taylor expansion formula with respect to the second order term

$$f''(x_0, u, u) = \lim_{t \to +0} \frac{2}{t^2} (f(x_0 + tu) - f(x_0) - f'(x_0, u)t).$$

Under suitable definition of the first order directional derivative the above equality suggests a definition for the second order derivative. The cited derivative of V. Demyanov and A. Pevnyj is of this type. This relation inherits directly the approach of G. Peano, whose work [15] initiates the known concept of Peano derivative. First order Clarke generalized directional derivatives [6] are used by R. Chaney in several papers, say [4] or [5], to study second order sufficient conditions in nonsmooth optimization. W. Oettli and P. H. Sach [14] utilize this idea to abstract spaces and introduce the concept of prederivative. Applications to nonlinear programming and composite problems and relations to other concepts are given in R. Poliquin and R. T. Rockafellar [17].

The other possibility comes from a direct representation of the second order derivative by finite differences, an illustration is for instance

$$f''(x_0, u, v) = \lim_{t \to +0} \lim_{s \to +0} \frac{1}{ts} \left(f(x_0 + tu + ts) - f(x_0 + tu) - f(x_0 + ts) + f(x_0) \right).$$

A similar equality is used by R. Cominetti and R. Correa [7]. The generalized derivative of Michel—Penot [13] is of this type. Such is also the introduced by X. Yang and V. Jeyakumar [21] derivative for $\mathcal{C}^{1,1}$ functions, i. e. differentiable functions with Lipschitz first order derivative.

This brief and far not complete review on second order generalized derivatives aims to classify to some extend the introduced here second order derivatives of Riemann type and second order conditions. They contribute to derivatives and conditions expressed by finite differences. Let us however mention that the boundary between generalized derivatives based on Taylor expansion formula and those based on finite differences is rather fuzzy. For instance W. L. Chan, L. R. Huang and K. F. Ng [3] prove a Taylor expansion formula for some derivatives of Michel—Penot and Cominetti–Correa type. As for the name Riemann type derivative used in the present paper, it is motivated by the underlying finite difference identical with that of the second order derivative introduced by B. Riemann [18] in his famous study on convergence of trigonometric series.

2 Directional first and second order derivatives

Throughout this paper X denotes a Banach space, B is the unit ball and S is the unit sphere in X. We write $\|\cdot\|$ for the norm in X. The sets of real and integer numbers are denoted by R and Z respectively and R_+ stands for the nonnegative reals.

Let $f : X \to R \cup \{+\infty\}$ be a given function. We consider the minimization problem

$$f(x) \longrightarrow \min, \quad x \in X. \tag{1}$$

Remind that f is said to possess a local minimum at $x_0 \in X$ if there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0)$ for all $x \in U$. If this inequality is strict for $x \in U \setminus \{x_0\}$ then the local minimum is said to be strict.

Further we suggest second order necessary and sufficient conditions in terms of second order directional derivatives. Let x_0 be an interior point for the domain $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $u \in X$. We consider the following first and second order directional derivatives:

$$\begin{aligned}
f'_-(x_0,u) &= \liminf_{s\to+0,v\to u} \frac{f(x_0+sv)-f(x_0)}{s}, \\
f'_+(x_0,u) &= \limsup_{s\to+0,v\to u} \frac{f(x_0+su)-f(x_0)}{s}, \\
f''_-(x_0,u) &= \liminf_{s\to+0,v\to u} \frac{f(x_0+2sv)-2f(x_0+sv)+f(x_0)}{s^2}, \\
f''_+(x_0,u) &= \limsup_{s\to+0,v\to u} \frac{f(x_0+2su)-2f(x_0+su)+f(x_0)}{s^2}.
\end{aligned}$$

The limits $\liminf$ and $\limsup$ in the above equalities are considered as taking values in $R\cup\{-\infty\}\cup\{+\infty\}$. In the case the $\liminf$ in the definition of the second order derivatives can be substituted by $\lim$ we get the so-called Riemann (-Hadamard) derivatives. For this reason we say that the second order derivatives defined here are of Riemann type.

The definition of the second order derivatives is obtained by using only one direction u. In fact a second direction u "emerges" since the second finite difference is the difference of the first difference. We prefer however for simplicity to write $f''_-(x_0,u)$ instead of say $f''_-(x_0;u,u)$.

The derivatives could be defined with the above formula in the case x_0 is not necessarily an interior point of $\operatorname{dom} f$. The general case must be handled with more care because of the possibility the function f to take infinite values. To avoid the difficulties we confine to the case of an interior point x_0. Let us mention that to treat constrained and composite optimization problems one must get rid of this restriction.

3 Sufficient conditions, Riemann type derivatives

In this section we derive the following sufficient optimality condition, which is well known – as a necessary condition – when it is expressed through the first order derivative usually referred to as the Hadamard contingent lower epi-derivative:

Theorem 1 *Let X be a finite dimensional Banach space and $f : X \to R\cup\{+\infty\}$. Suppose that for some $x_0 \in \operatorname{int}\operatorname{dom} f$ and for each $u \in X\setminus\{0\}$ one of the following two conditions holds:*

(a) $f'_-(x_0,u) > 0$,

(b) $f'_-(x_0,u) = 0$ and $f''_-(x_0,u) > 0$,

then f possesses a strict local minimum at x_0.

Proof. We show first that for each $u \in X\setminus\{0\}$ there exists a neighborhood $U = U(u)$ of u and a positive number $\delta = \delta(u)$ such that $f(x_0+tv) > f(x_0)$ for all $t \in (0,\delta)$ and $v \in U(u)$. This is true even not assuming that X is finite dimensional. We prove this statement separately for the cases (a) and (b).

If (a) holds then from the lim inf definition there exists $\delta_1 > 0$ and $\alpha_1 > 0$ such that

$$\frac{f(x_0 + sv) - f(x_0)}{s} > 0$$

and hence $f(x_0 + sv) > f(x_0)$ for all $s \in (0, \delta_1)$ and $v \in u + \alpha_1 B$. In this case $\delta = \delta_1$ and $U = u + \alpha_1 B$ satisfy the assertion.

Let (b) hold. Then there exist $\delta_2 > 0$ and $\alpha_2 > 0$ such that if $0 < s < \delta_2$ and $v \in u + \alpha_2 B$ one has

$$\frac{f(x_0 + 2sv) - 2f(x_0 + sv) + f(x_0)}{s^2} > \frac{1}{2} f''_-(x_0, u).$$

We use the following identity

$$f(x_0 + tv) = f(x_0) + \frac{f(x_0 + \frac{t}{2^n} v) - f(x_0)}{\frac{t}{2^n}}\, t$$

$$+\frac{1}{2} \sum_{i=1}^{n} \frac{1}{2^i}\, \frac{f(x_0 + 2\frac{t}{2^i} v) - 2f(x_0 + \frac{t}{2^i} v) + f(x_0)}{\left(\frac{t}{2^i}\right)^2}\, t^2. \tag{2}$$

This equality resembles the Taylor expansion formula of second order for $f(x_0 + tv)$ with derivatives replaced by finite differences. Fix $t \in (0, \delta_2)$ and $v \in u + \alpha_2 B$. Since $f'_-(x_0, v) \geq 0$, there exists $\delta_3 > 0$ such that for $0 < s < \delta_3$ it holds

$$\frac{f(x_0 + sv) - f(x_0)}{s} \geq -\frac{1}{16} f''_-(x_0, u)\, t.$$

Now choose and fix the positive integer n such that $0 < t/2^n < \delta_3$. Then from (2) we have

$$f(x_0 + tv) \geq f(x_0) - \frac{1}{16} f''_-(x_0, u)\, t\, t + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2^i}\, \frac{1}{2} f''_-(x_0, u)\, t^2$$

$$= f(x_0) - \frac{1}{16} f''_-(x_0, u)\, t^2 + \frac{1}{4}(1 - \frac{1}{2^n}) f''_-(x_0, u)\, t^2$$

$$\geq f(x_0) - \frac{1}{16} f''_-(x_0, u)\, t^2 + \frac{1}{8} f''_-(x_0, u)\, t^2 = f(x_0) + \frac{1}{16} f''_-(x_0, u)\, t^2 > f(x_0).$$

Therefore $f(x_0 + tv) > f(x_0)$ for $t \in (0, \delta_2)$ and $v \in u + \alpha_2 B$. In this case $\delta = \delta_2$ and $U = u + \alpha_2 B$ satisfy the assertion.

The proof is completed by a routine compactness argument. Obviously $S \subset \bigcup\{U(u) \mid u \in S\}$. Since S is compact as the unit sphere in a finite dimensional Banach space, we have $S \subset U(u_1) \cup U(u_2) \cup \cdots \cup U(u_n)$ for a finite set of directions $\{u_1, \cdots, u_n\}$. Put $\delta = \min\{\delta(u_1), \ldots, \delta(u_n)\}$. Then $\delta > 0$. If $t \in (0, \delta)$ and $u \in S$ we have $f(x_0 + tu) > f(x_0)$. $\square$

4 Examples

If $f(x)$ has some monotonic properties around x_0 then usually optimality is checked by first order conditions. To test a second order theory it is natural to involve nonsmooth functions f having some oscillatory (i. e. nonmonotonic) behavior near x_0. With the following examples we intend to illustrate and show the necessity of an improvement of Theorem 1 for the sake of its "applicability". For this reason we confine to the one-dimensional case $f : R \to \overline{R}$.

Example 1 *The function*

$$f : R \longrightarrow R, \quad f(x) = \begin{cases} |x| + \frac{1}{2} x \sin \frac{1}{x} & , \quad x \neq 0, \\ 0 & , \quad x = 0, \end{cases}$$

obviously has a strict minimum at $x = 0$, since for $x \neq 0$ it holds $f(x) \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x| > 0 = f(0)$. In this case for $u \in R \setminus \{0\}$ we have $f'_-(0, u) = \frac{1}{2}|u|$ and therefore the minimum can be recognized already from the first order condition in Theorem 1.

The next Example 2 involves essentially second order conditions. To construct it we need a function described in the following Lemma.

Lemma 1 *Let $a > 0$ and $\varphi_0 : [a, 2a) \to R$ be an arbitrary function. Then the function $\varphi : R_+ \to R$ defined by*

$$\varphi(x) = \begin{cases} 2^{-n} \varphi_0(2^n x), & a \leq 2^n x < 2a, \\ 0, & x = 0, \end{cases}$$

is an extension of φ_0 and satisfies the conditions:

(a) For arbitrary $u > 0$ it holds $\varphi'_-(0, u) = u\, m_\varphi$, $\varphi'_+(0, u) = u\, M_\varphi$, where $m_\varphi = \inf\limits_{x \in [a,2a)} \dfrac{\varphi(x)}{x}$, $M_\varphi = \sup\limits_{x \in [a,2a)} \dfrac{\varphi(x)}{x}$;

(b) For arbitrary $u > 0$ it holds $\varphi''_-(0, u) = \varphi''_+(0, u) = 0$.

Proof. If $x > 0$ then there exists a unique $n \in Z$ satisfying $a \leq 2^n x < 2a$, namely n is the integer for which $\log_2 \frac{a}{x} \leq n < \log_2 \frac{a}{x} + 1$. Therefore $\varphi : R_+ \to R$ is correctly defined and it is obviously an extension of φ_0.

(a) Let $\varepsilon > 0$ and suppose that $v > 0$. If $a \leq x < 2a$ then $s := 2^{-n} \frac{x}{v} < \varepsilon$ for some $n \in Z$ and

$$\frac{1}{s}(\varphi(sv) - \varphi(0)) = v \frac{\varphi(sv)}{sv} = v \frac{\varphi(2^{-n}x)}{2^{-n}x} = v \frac{\varphi_0(x)}{x}.$$

On the other hand for arbitrary $0 < s < \varepsilon$ there exist $n \in Z$ such that $x := 2^n sv \in [a, 2a)$. Therefore

$$\varphi'_-(0, u) = \liminf_{s \to +0, v \to u} \frac{1}{s}(\varphi(sv) - \varphi_(0)) = \lim_{v \to u} \inf_{x \in [a,2a)} \left(v \frac{\varphi_0(x)}{x} \right) = m_\varphi\, u.$$

Similarly $\varphi'_+(0, u) = M_\varphi\, u$.

(b) The equality $\varphi(2^n x) = 2^n\varphi(x)$ is true for arbitrary x. Indeed, if $a \le 2^m x < 2a$, $m \in Z$, then also $a \le 2^{m-n}\, 2^n x < 2a$, whence

$$2^n\varphi(x) = 2^{n-m}\varphi_0(2^m x) = 2^{n-m}\varphi_0(2^{m-n}\, 2^n x) = \varphi(2^n x).$$

In particular

$$\frac{1}{s^2}\left(\varphi(2sv) - 2\varphi(sv) + \varphi(0)\right) = \frac{1}{s^2}\left(2\varphi(sv) - 2\varphi(sv) + \varphi(0)\right) = 0,$$

and consequently $\varphi''_-(0,u) = \varphi''_+(0,u) = 0$. $\square$

Example 2 *Let $f : R \to R$ be defined by $f(x) = \alpha|x| + \beta x^2 + \varphi(|x|)$, $\alpha, \beta \in R$, where φ is the function from Lemma 1. Then for $u \ne 0$ an easy calculation gives*

$$f'_-(0,u) = (\alpha + m_\varphi)\,|u|, \quad f'_+(0,u) = (\alpha + M_\varphi)\,|u|, \quad f''_-(0,u) = f''_+(0,u) = 2\beta u^2.$$

Therefore according to Theorem 1 the function $f(x)$ attains a strict minimum at $x_0 = 0$ if one of the following conditions hold: $\alpha + m_\varphi > 0$, or $\alpha + m_\varphi = 0$ and $\beta > 0$.

Example 2 is somewhat artificial. The next function is “more natural”.

Example 3 *The function $f : R \longrightarrow R$ defined for some $\kappa > 0$ by*

$$f(x) = \begin{cases} |x|\left(1 - \sin\frac{1}{|x|}\right) + \kappa x^2, & x \ne 0, \\ 0, & x = 0, \end{cases}$$

has obviously a strict minimum at $x_0 = 0$.

We use this function as a test for “applicability” of the sufficient second order optimality conditions. We have

$$f'_-(0,u) = \liminf_{s\to+0,v\to u}\left(|v|\left(1 - \sin\frac{1}{s\,|v|}\right) + \kappa s v^2\right) = 0.$$

Therefore sufficient second order conditions should use essentially the second order derivative. The second order derivative is however

$$f''_-(0,u) = \liminf_{s\to+0,v\to u}\left(\frac{2}{s}\,|v| \sin\frac{1}{2s\,|v|}\left(2\cos\frac{1}{2s\,|v|} - 1\right) + 2\kappa v^2\right) = -\infty.$$

Hence the sufficient conditions from Theorem 1 do not hold.

5 Sufficient conditions, general case

Wishing to get sufficient condition which work on examples like Example 3 we generalize Theorem 1 in this section.

Theorem 2 *Let X be a finite dimensional Banach space and $f : X \to R \cup \{+\infty\}$. Suppose that for some $x_0 \in \operatorname{int}\operatorname{dom} f$ and for each $u \in X \setminus \{0\}$ one of the following two conditions holds:*

(*a*) $f'_-(x_0,u) > 0$,

(*b*) $f'_-(x_0,u) = 0$ *and there exist real numbers* $\delta > 0$, $\alpha > 0$, $0 < q < 1$, $\varepsilon > 0$, *possibly depending on* u, *such that to each* $t \in (0,\delta)$ *and* $v \in u + \alpha B$ *a number* $\gamma = \gamma(t,v) \in (0,q)$ *can be chosen for which*

$$\Gamma(t,\gamma,x_0,v) := \frac{1}{1-\gamma}\, f(x_0+tv) - \frac{1}{\gamma(1-\gamma)}\, f(x_0+t\gamma v) + \frac{1}{\gamma}\, f(x_0) \ge \varepsilon\, t^2$$

Then f *attains a strict local minimum at* x_0.

Proof. We show that for each $u \in X \setminus \{0\}$ there exist positive numbers $\delta_0 = \delta_0(u)$ and $\alpha_0 = \alpha_0(u)$ such that $f(x_0+tv) > f(x_0)$ for all $t \in (0,\delta_0)$ and $v \in u + \alpha_0 B$, true even without the assumption that X is finite dimensional. In advance we restrict δ_0 and α_0 so that the considered values $f(x_0+tv)$ are finite, which is possible since x_0 is in the interior of $\operatorname{dom} f$.

In case (a) the reasonings repeat those of Theorem 1.

Suppose that case (b) takes place. We use the following identity

$$\begin{aligned} f(x_0+tv) &= f(x_0) + \frac{1}{\lambda_n t}\,(f(x_0+\lambda_n tv) - f(x_0))\,t + \sum_{i=1}^{n}(1-\gamma_i)\lambda_{i-1}\frac{1}{(\lambda_{i-1}t)^2}\times \\ &\times\left(\frac{1}{1-\gamma_i}\, f(x_0+\lambda_{i-1}tv) - \frac{1}{\gamma_i(1-\gamma_i)} f(x_0+\lambda_i tv) + \frac{1}{\gamma_i} f(x_0)\right) t^2, \end{aligned} \tag{3}$$

where $\lambda_0 = 1$, $\lambda_i = \gamma_1\,\gamma_2,\dots\gamma_i$, $i = 1,2,\dots$, with $\gamma_i \in (0,1)$. To derive (3) one simplifies the sum $\sum_{i=1}^{n}((1-\gamma_i)/\lambda_{i-1}))\,\Gamma(\lambda_{i-1}t,\gamma_i,x_0,v)$. Equality (2) is a particular case of (3) with $\gamma_i = 1/2$, $i = 1,\dots,n$.

Let $\delta > 0$, $\alpha > 0$, $0 < q < 1$, $\varepsilon > 0$ be chosen according to case (b) of the theorem. Put $\delta_0 = \delta$, $\alpha_0 = \alpha$. Now we chose and fix $t \in (0,\delta_0)$, $v \in u + \alpha_0 B$. We construct by induction the sequence γ_n, $n = 1,\dots$. Put $\lambda_0 = 1$. Suppose that $\gamma_n \in (0,q)$ is defined for $n = 1,\dots,i-1$ and $\lambda_{i-1} = \gamma_1\dots\gamma_{i-1}$. Then $\lambda_{i-1}t \in (0,\delta)$ and we may choose $\gamma_i = \gamma(\lambda_{i-1}t,v)$. That is $\gamma_i \in (0,q)$ and $\Gamma(\lambda_{i-1}t,\gamma_i,x_0,v) \ge \varepsilon\,(\lambda_{i-1}t)^2$. Let us mention that we have in fact $0 < \lambda_i < q^i$, $i = 1,2,\dots$.

Since $f'_-(x_0,v) \ge 0$ therefore we may choose $\delta_1 > 0$ and $\alpha_1 > 0$ so that for arbitrary $s \in (0,\delta_1)$ it holds

$$\frac{f(x_0+sv) - f(x_0)}{s} > -\frac{1}{2}\,\varepsilon(1-q)\,t.$$

Choose and fix n such that $0 < \lambda_n t < \delta_1$. Then (3) gives the estimation

$$\begin{aligned} f(x_0+tv) &> f(x_0) - \frac{1}{2}\,\varepsilon(1-q)\,t\,t + \sum_{i=1}^{n}(1-\gamma_i)\lambda_{i-1}\;\varepsilon t^2 \\ &= f(x_0) - \frac{1}{2}\,(1-q)\,\varepsilon t^2 + (1-\lambda_n)\;\varepsilon t^2 \end{aligned}$$

$$\geq \quad f(x_0) - \frac{1}{2}(1-q)\,\varepsilon t^2 + (1-q^n)\;\varepsilon t^2$$

$$\geq \quad f(x_0) + \frac{1}{2}(1-q)\,\varepsilon t^2 > f(x_0).$$

which had to be shown.

The proof is completed by repeating the compactness argument from the final part of Theorem 1. □

Now we show that Theorem 2 can establish that $x_0 = 0$ is the strict minimum in Example 3. We saw in the previous section that $f'_-(0,u) = 0$ for each $u \neq 0$. Therefore we are in case (b) of Theorem 2. Let $u \neq 0$ be arbitrary. Choose $\alpha = \frac{1}{2}|u|$. For $t > 0$, $0 < \gamma < 1$ and $v \neq 0$ we have

$$\Gamma(t,\gamma,0,v) = \frac{1}{1-\gamma}\left(\sin\frac{1}{\gamma t|v|} - \sin\frac{1}{t|v|}\right) t|v| + \kappa t^2 v^2.$$

Let q, $0 < q < 1$, and $\delta > 0$ be arbitrary. Fix $t \in (0,\delta)$ and $v \in (u-\alpha, u+\alpha)$. Remark that then $|v| \geq |u|/2$. Choose $\gamma = \gamma(t,v) \in (0,q)$ so that $\sin(1/\gamma t|v|) = 1$. That is $\gamma = 2/((4n+1)\pi t|v|)$ where n is taken so that $\gamma \in (0,q)$. Then we have

$$\Gamma(t,\gamma,0,v) = \frac{1}{1-\gamma}\left(1 - \sin\frac{1}{t|v|}\right) t|v| + \kappa t^2 v^2 \geq \kappa t^2 v^2 \geq \frac{1}{4}\,\kappa u^2\, t^2.$$

Therefore case (b) of Theorem 2 is satisfied with $\varepsilon = \frac{1}{4}\,\kappa u^2$.

6 Necessary conditions

In classical analysis the study of optimization problems starts with first order necessary conditions. Second order usually appears with the study of sufficient conditions. Nevertheless second order necessary conditions are valuable as a test to what extend the sufficient conditions are "good". They are "good" if the necessary conditions look similarly to the sufficient ones and the gap between them is not too big. We establish first necessary conditions in Riemann type derivatives.

Theorem 3 *Let X be a Banach space and $f : X \longrightarrow R \cup \{+\infty\}$. If f possesses a local minimum at $x_0 \in \operatorname{int}\operatorname{dom} f$ then the following two conditions hold for each $u \in X$:*
(a) $f'_-(x_0,u) \geq 0$ and moreover $f'_+(x_0,u) \geq 0$,
(b) if $f'_+(x_0,u) = 0$ for some $u \in X$ then $f''_+(x_0,u) \geq 0$.

Proof.
(a) Since x_0 is a minimizer, we get $f'_+(x_0,u) \geq f'_-(x_0,u) = \liminf\limits_{s\to+0,v\to u} \frac{1}{s}(f(x_0+sv) - f(x_0)) \geq 0$.
(b) Let $f'_+(x_0,u) = 0$ and let $\varepsilon > 0$ is an arbitrary positive number. Let also $\beta > 0$. Choose and fix the numbers $t > 0$ and $\alpha_0 > 0$ such that:
i) $f(x_0+tu) - f(x_0) \geq 0$,

ii) $\frac{1}{s^2}(f(x_0+2sv)-2f(x_0+sv)+f(x_0)) \le f''_+(x_0,u)+\beta$ for $0<s<t$, $v \in u+\alpha_0 B$. In particular the above inequality is true for $0<s<t$, $v=u$. If $f'_+(x_0,u)=0$, we can choose $\delta_1>0$ and $\alpha_1>0$ such that $\frac{1}{s}(f(x_0+sv)-f(x_0)\le \varepsilon t$ for $0<s<\delta_1$ and $v\in u+\alpha_1 B$. In particular this inequality is satisfied for $v=u$. Choose and fix n such that $0<\frac{t}{2^n}<\delta_1$. Then

$$0 \le f(x_0+tu)-f(x_0) = \frac{f(x_0+\frac{t}{2^n}u)-f(x_0)}{\frac{t}{2^n}}\,t$$

$$+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{2^i}\frac{f(x_0+2\frac{t}{2^i}u)-2f(x_0+\frac{t}{2^i}u)+f(x_0)}{\left(\frac{t}{2^i}\right)^2}\,t^2$$

$$\le \varepsilon t\,t+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{2^i}\,(f''_+(x_0,u)+\beta)\,t^2$$

$$\le \left(\varepsilon+\frac{1}{2}(1-\frac{1}{2^n})(f''_+(x_0,u)+\beta)\right)\,t^2 < \left(\varepsilon+f''_+(x_0,u)+\beta\right)\,t^2.$$

We get from here $f''_+(x_0,u)>-\varepsilon-\beta$ and since ε and β can be arbitrary small, therefore $f''_+(x_0,u)\ge 0$. $\square$

The strength of the necessary conditions is in rejecting the suspect that a nonminimizer is a minimizer. As a test example consider the function f from Example 3 with $\kappa<0$. Obviously x_0 is then not a minimizer for f. At the same time $f'_+(0,u)=2|u|$ and therefore condition (b) cannot be used to reject the "suspect" on x_0. This example points out the "gap" between the first order and the second order part of Theorem 3. The upper derivative $f'_+(x_0,u)$ is not necessary zero, while $f'_-(x_0,u)$ vanishes.

We state now necessary conditions in terms of Theorem 2.

Theorem 4 *Let X be a Banach space and $f: X \longrightarrow R\cup\{+\infty\}$. If f possesses a local minimum at $x_0 \in \operatorname{int}\operatorname{dom} f$, then the following two conditions hold for each $u\in X\setminus\{0\}$:*

(a) $f'_-(x_0,u)\ge 0$,

(b) if $f'_-(x_0,u)=0$ is satisfied for some $u\in X$ then for all sufficiently small t it holds $\limsup\limits_{\gamma\to+0,v\to u}\dfrac{1}{t^2}\Gamma(t,\gamma,x_0,v)\ge 0.$

Proof. Case (a) is the same as in Theorem 3.

(b) Let $f'_-(x_0,u)=0$ and let $\varepsilon>0$ is an arbitrary positive number. Suppose that $\delta_0>0$ is such that $f(x)\ge f(x_0)$ for $\|x-x_0\|\le\delta_0$. Let U be a bounded neighborhood of u and fix $t>0$ such that $f(x_0+sv)\ge f(x_0)$ for $0<s\le t$, $v\in U$. Since $f'_-(x_0,u)=0$, there exist a sequence $\gamma_n=\gamma_n(\varepsilon)\to+0$ satisfying $0<\gamma_n<q<1$ for some number q, and a sequence $v_n=v_n(\varepsilon)\to u$, satisfying $v_n\in U$, such that $\frac{1}{\gamma_n t}(f(x_0+\gamma_n t v_n)-f(x_0))<$

$\varepsilon(1-q)$. Then

$$0 \le f(x_0 + tv_n) - f(x_0) = \frac{1}{\gamma_n t}\left(f(x_0 + \gamma_n t v_n) - f(x_0)\right) t$$

$$+ (1-\gamma_n)\left(\frac{1}{1-\gamma_n} f(x_0 + tv_n) \; - \; \frac{1}{\gamma_n(1-\gamma_n)} f(x_0 + \gamma_n t v_n) \; + \; \frac{1}{\gamma_n} f(x_0)\right)$$

$$\le (1-q)\varepsilon t^2 + (1-\gamma_n)\Gamma(t, \gamma_n, x_0, v_n) \le (1-\gamma_n)(\varepsilon t^2 + \Gamma(t, \gamma_n, x_0, v_n)).$$

Therefore $\frac{1}{t^2}\Gamma(t, \gamma_n, x_0, v_n) \ge -\varepsilon$. Since $\varepsilon > 0$ is arbitrary, for sufficiently small t it holds $\limsup\limits_{\gamma \to +0, v \to u} \dfrac{1}{t}\Gamma(t, \gamma, x_0, v) \ge 0$. $\square$

Consider Example 3 with $\kappa < 0$. The function f obviously has not $x = 0$ as a minimizer though the satisfaction of the first order condition $f'_-(0, u) = 0$ ($u \ne 0$ arbitrary) makes this point suspicious. Fix in this case $u \ne 0$ and choose $t_n \to +0$ such that $\sin \frac{1}{t_n |u|} = 1$. Then we have

$$\lim_{\gamma \to +0, v \to u} \frac{1}{t_n^2} = \frac{1}{t_n}|u|\left(1 - \sin\frac{1}{t_n|u|}\right) + \kappa u^2 = \kappa u^2 < 0.$$

Therefore $x_0 = 0$ is not a minimizer, the "suspect" is rejected using the second order part of Theorem 4

Condition (b) of Theorem 4 obviously could be used to introduce a new type second order derivative

$$f''_-(x_0, u) := \liminf_{t \to +0} \limsup_{\gamma \to +0, v \to u} \frac{1}{t^2}\Gamma(t, \gamma, x_0, v),$$

which could be referred as the a second order generalized Riemann type derivative. Having in mind this derivative, then condition (b) of Theorem 4 can be obviously reformulated (b^*) *if* $f'_-(x_0, u) = 0$ *is satisfied for some* $u \in X$ *then* $f''_-(x_0, u) \ge 0$.

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