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# TIME-DEPENDENT DIFFERENTIAL INCLUSIONS AND VIABILITY*

Luis Marco, José Alberto Murillo

This paper is devoted to study the existence of viable solutions for nonautonomous higher order differential inclusions. Two cases are considered, according to the properties of the set-valued maps on the right-hand side. Firstly, upper semicontinuity is assumed and both necessary and sufficient conditions are given by means of higher order tangent sets. Later, almost upper semicontinuous case is investigated, and similar results are stated. Finally, these results are used to solve a multivalued differential inequality.

**Keywords**: viable solutions, differential inclusions, high order, nonautonomous, higher order tangent sets, tangential conditions, multivalued differential inequalities

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## 1 Introduction

Let $F : \mathbb{R} \times X \to 2^X$ be a nontrivial set-valued map, with $X$ a finite dimensional vector space and let $K \subseteq X$ a nonempty set. A function $x(\cdot)$ satisfying the differential inclusion $x'(t) \in F(t, x(t))$, a.e., is called viable in $K$, if $x(t) \in K$ holds.

The concept of viability appeared in the framework of differential inclusions in [26] and [6] under the name of weak invariance and admissibility and also in [18]. In the autonomous case, i.e. when an autonomous differential inclusion is considered, the more general concept of monotone trajectory was studied in [1] (for convex viability domains and Marchaud multivalued maps), [10] (for compact viability domains and continuous maps) and [19] (for locally compact viability domains and upper semicontinuous correspondences).

The existence of viable solutions to a nonautonomous or time-dependent differential inclusion was investigated in [19], [2] and [12] (for upper semicontinuous set-valued maps) and also in [20], [25] and [13] (for Carathéodory multivalued maps). The more general

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case in which the viable set changes along the time, and therefore the viability condition becomes $x(t) \in K(t)$, is considered in [19] and [2] (for upper semicontinuous set-valued maps), [25] (for Carathéodory multivalued maps, by assuming that $K(t)$ is convex), [7] (for almost upper semicontinuous maps) and also in [16], [21] and [17] (for Carathéodory multivalued maps), where some applications are provided. We refer to [4] for more detailed historical notes and references.

The viability problem for second order differential inclusions was first investigated by Cornet and Haddad in [11]. They noted that finding solutions of the autonomous viability problem:

$$\left.\begin{array}{l} x''(t) \in F(x(t), x'(t)) \\ x(0) = x_0, \qquad x'(0) = v_0 \\ x(t) \in K \end{array}\right\} \tag{1.1}$$

is equivalent to the following viability problem of first order,

$$\left.\begin{array}{ll} x'(t) = u(t), & u'(t) \in F(x(t), u(t)) \\ x(0) = x_0, & u(0) = v_0 \\ (x(t), u(t)) \in \mathcal{G}(T_K) & \end{array}\right\} \tag{1.2}$$

where $T_K(x)$ is Bouligand's tangent cone to $K$ at $x$, and $\mathcal{G}(T_K)$ is its graph. However, one of the main assumptions of the viability theorems for first order differential inclusions is that the viability set must be locally compact, and this is not in general satisfied by $\mathcal{G}(T_K)$. So if we want to use this way to solve (1.1), we must assume that $\mathcal{G}(T_K)$ is locally compact. Hence, this approach is very seldomly applicable.

In the above mentioned paper, Cornet and Haddad stated the existence of viable solutions of a differential inclusion of second order by imposing restrictions on initial conditions by means of Dubovickii-Miljutin and Clarke's tangent cones. In [5], Auslender and Mechler first established a necessary condition by using the set of second order tangents of $K$ introduced by Ben-Tal and later they gave a sufficient condition valid for all initial states by introducing the notion of the second order interior tangent set. Finally, in [22] and [23] the authors studied the viability problem of $n^{th}$ order in the autonomous case, and we noted that the viability condition on the solution of a differential inclusion of $n^{th}$ order involves an underlying viability condition on the solution and its derivatives up to order $n-1$. To describe it we introduced a class of higher order tangent sets. We also stated necessary and sufficient conditions ensuring the existence of viable solutions to an autonomous differential inclusion of $n^{th}$ order.

In this paper the time-dependent viability problem of $n^{th}$ order is analysed. Section 2 presents some preliminaries and sets up notation and terminology. In Section 3 we consider the case in which the set-valued map is upper semicontinuous. Firstly, we assume that $\mathcal{G}(A_K^{(n-1)})$ is locally compact, and under this hypothesis both local and global viability theorems are stated. Later we study the general case, and we give necessary and sufficient conditions ensuring the existence of viable solutions. Section 4 is devoted to analyse the almost upper semicontinuous case. In this section we also suppose that the set-valued map is integrably bounded, and when $\mathcal{G}(A_K^{(n-1)})$ is assumed to be closed, we obtain a viability result similar to that in the preceding section. In the general case, we impose regularity assumptions on the growth of the set-valued map to getting viable solutions. Finally, in Section 5 we consider a multivalued differential inequality of higher order and we solve it by using previous results.

# 2 Preliminaries

Let us first recall some notions and notation, a detailed discussion of these concepts can be found, e.g. in [4], [3] or [14]. Let $X$, $Y$ be metric spaces. The domain of a set-valued map $F : X \to 2^Y$, is the set $D(F) = \{x \in X : F(x) \neq \emptyset\}$, and it is said to be nontrivial if $D(F) \neq \emptyset$. The graph of $F$ is the set $\mathcal{G}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. A set-valued map is said to be upper semicontinuous (u.s.c. for short) on $\Omega \subseteq X$ if $F^{-1}(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in $\Omega$ for all closed set $C \subseteq Y$. In the case $F$ defined from $\mathbb{R} \times X$ to $2^Y$, we call it almost u.s.c. on $I \times \Omega$, $I$ being a compact interval, if for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subseteq I$ with $\mu(I \setminus I_\varepsilon) \leq \varepsilon$ such that $F|_{I_\varepsilon \times \Omega}$ is u.s.c. with nonempty values; here $\mu$ denotes the Lebesgue measure. If $I$ is not compact, $F$ is called almost u.s.c. on $I \times \Omega$ if it satisfies this property on $J \times \Omega$, for all compact $J \subseteq I$.

For a family of sets $\{S_\sigma\}_{\sigma \in \Sigma}$, the upper limit in the Kuratowski sense is the set:

$$\limsup_{\sigma \in \Sigma} S_\sigma = \{x \in X : \liminf_{\sigma \in \Sigma} d(x, S_\sigma) = 0\}$$

The contingent derivative of a set-valued map $F$ at $(x_0, y_0) \in \mathcal{G}(F)$, denoted by $DF(x_0, y_0)$, is given by means of its graph:

$$\mathcal{G}(DF(x_0, y_0)) = \limsup_{h \to 0^+} \frac{\mathcal{G}(F) - (x_0, y_0)}{h}.$$

Given a nonempty set $K \subseteq X$ and $x_0, x_1, \ldots, x_{n-1} \in X$, the $n^{th}$ order tangent set of $K$ at $(x_0, \ldots, x_{n-1})$, denoted by $A_K^{(n)}(x_0, \ldots, x_{n-1})$ is defined as follows:

$$A_K^{(n)}(x_0, \ldots, x_{n-1}) = \limsup_{h \to 0^+} \frac{n!}{h^n} \left( K - x_0 - h\, x_1 - \cdots - \frac{h^{n-1}}{(n-1)!}\, x_{n-1} \right).$$

An important property of these sets is that $A_K^{(n)}(x_0, \ldots, x_{n-1}) \neq \emptyset$, implies $x_{n-1}$ belongs to $A_K^{(n-1)}(x_0, \ldots, x_{n-2})$. The $n^{th}$ order interior tangent set of $K$ at $(x_0, \ldots, x_{n-1})$, denoted by $AI_K^{(n)}(x_0, \ldots, x_{n-1})$ is the set of points $y \in X$ such that there are $\varepsilon > 0$ and $\alpha > 0$ satisfying:

$$\sum_{j=0}^{n-1} \frac{h^j}{j!}\, x_j + \frac{h^n}{n!}\, (y + \varepsilon \mathcal{U}_X) \subseteq K, \qquad 0 \leq h \leq \alpha$$

where $\mathcal{U}_X$ is the unit ball in $X$. We refer to [22], [23] or [24] for more information on these higher order tangent sets.

In the remainder of the paper, $X$ will be a finite dimensional vector space, $K \subseteq X$ a nonempty closed set and $F : \mathbb{R} \times X^n \to 2^X$ will be a nontrivial set-valued map with convex compact values. The problem that we shall consider is the following time-dependent viability problem of $n^{th}$ order:

(2.1) $$x^{(n)}(t) \in F(t, x(t), x'(t), \ldots, x^{n-1}(t))$$

$$(2.2) \qquad x(0) = x_0,\, x'(0) = v_1,\, \ldots,\, x^{(n-1)}(0) = v_{n-1}$$

$$(2.3) \qquad x(t) \in K.$$

A solution of (2) on an interval $[0,T]$ is a function $\varphi$ possessing absolutely continuous derivatives up to order $n-1$, i.e. $\varphi \in W^{n,1}(0,T;X)$, such that it is solution of (2.1) viable in $K$ (condition (2.3)) and satisfying initial conditions (2.2).

# 3 Upper semicontinuous case

In this section we will make the assumption that $F$ is u.s.c. Under this hypothesis, an analysis similar to that in the proof of Proposition 4.1. in [23] shows that the next proposition holds

**Proposition 3.1** *Let $\varphi$ be a solution of* (2) *on* $[0,T]$, *then*

$$(\varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in \mathcal{G}(A_K^{(n-1)})$$

*for each* $t \in [0,T[$.

From this result it follows that the inclusion $(x_0, v_1, \ldots, v_{n-1}) \in \mathcal{G}(A_K^{(n-1)})$ must be satisfied by the initial conditions in (2.2). Therefore we will assume in the remainder of this section the next "compatibility condition" on $F$,

$$(3.1) \qquad [0,\delta[ \times \mathcal{G}(A_K^{(n-1)}) \subseteq D(F)$$

for some $0 < \delta$. Furthermore, we will suppose that $F$ is u.s.c. on $[0,\delta[ \times \mathcal{G}(A_K^{(n-1)})$.

We now distinguish two situations in accordance with the topological properties of the graph of the $n^{th}$ order tangent set of $K$.

## 3.1 $\mathcal{G}(A_K^{(n-1)})$ locally compact

Since we are assuming that $F$ is u.s.c. on $[0,\delta[ \times \mathcal{G}(A_K^{(n-1)})$ and $\mathcal{G}(A_K^{(n-1)})$ is locally compact, we can use Theorem 4.1. in [23] to get solutions to the autonomous problem,

$$(3.2) \qquad \begin{aligned} & y^{(n)}(t) \in G(y(t), y'(t), \ldots, y^{(n-1)}(t)) \\ & y(0) = (0, x_0), \quad y'(0) = (1, v_1), \quad y^{(i)}(0) = (0, v_i), \quad 2 \le i \le n-1 \\ & y(t) \in [0,\delta[ \times K \end{aligned}$$

where $G(y_1, \ldots, y_n) = \{0\} \times F(y_1, \pi_X(y_2), \ldots, \pi_X(y_n))$; $\pi_X : \mathbb{R} \times X \to X$ being the projector onto $X$. Hence (2) has a solution for each initial condition $(x_0, v_1, \ldots, v_{n-1})$ in $\mathcal{G}(A_K^{(n-1)})$ iff the tangential condition,

$$(3.3) \qquad G(y, \omega_1, \ldots, \omega_{n-1}) \cap DA^{(n-1)}_{[0,\delta[\times K}(y, \omega_1, \ldots, \omega_{n-1})[\omega_1, \ldots, \omega_{n-1}] \neq \emptyset$$

holds for all $(y, \omega_1, \ldots, \omega_{n-1}) \in \mathcal{G}(A^{(n-1)}_{[0,\delta[\times K})$. But the next technical lemma allow us to rewrite (3.3) in terms of the set-valued map $F$ and the contingent derivative of $A_K^{(n-1)}$, and Theorem 3.1 is established.

**Lemma 3.1** *Let $C$ be a nonempty subset in $X$ and let $\eta > 0$, then*

$$A^{(n)}_{[0,\eta[\times C}(y_1,\dots,y_n) = \begin{cases} \mathbb{R}_+ \times A^{(n)}_C(\pi_X(y_1),\dots,\pi_X(y_n)) & \text{if } \pi_{\mathbb{R}}(y_i) = 0 \\ \mathbb{R} \times A^{(n)}_C(\pi_X(y_1),\dots,\pi_X(y_n)) & \text{otherwise} \end{cases}$$

**Theorem 3.1** (Local viability) *Under the hypotheses of this section,* (2) *has a solution for each initial condition in $\mathcal{G}(A^{(n-1)}_K)$ if and only if,*

$$F(t,x,u_1,\dots,u_{n-1}) \cap DA^{(n-1)}_K(x,u_1,\dots,u_{n-1})[u_1,\dots,u_{n-1}] \neq \emptyset \tag{3.4}$$

*holds for all $(t,x,u_1,\dots,u_{n-1}) \in [0,\delta[ \times \mathcal{G}(A^{(n-1)}_K)$.*

*Moreover, given $(x_0,v_1,\dots,v_{n-1}) \in \mathcal{G}(A^{(n-1)}_K)$, there are $\eta > 0$ and $T_0 > 0$, such that* (2) *has a solution on $[0,T_0]$ for each initial condition*

$$(x,u_1,\dots,u_{n-1}) \in \mathcal{G}(A^{(n-1)}_K) \cap ((x_0,v_1,\dots,v_{n-1}) + \eta\mathcal{U}_{X^n})$$

If we make the stronger assumption: $\mathcal{G}(A^{(n-1)}_K)$ is closed, then we can state a global viability theorem. It comes from the next lemma, which is a generalization of Theorem 3.3.4 in [4].

**Lemma 3.2** *Let $\mathcal{G}(A^{(n-1)}_K)$ be closed. Suppose* (3.4) *holds. If $\varphi$ is a solution of* (2) *on $[0,T[$ ($T < \delta$) such that,*

$$\limsup_{t\to T^-} \sum_{j=0}^{n-1} \|\varphi^{(j)}(t)\| < +\infty \tag{3.5}$$

*then $\varphi$ and its derivatives up to order $n-1$ can be extended to the full interval $[0,T]$.*

**Theorem 3.2** (Global viability) *Under the assumptions of Theorem* 3.1*, if moreover $\mathcal{G}(A^{(n-1)}_K)$ is closed and $F$ is bounded on $[0,\delta[ \times \mathcal{G}(A^{(n-1)}_K)$, then every solution of* (2) *can be extended to the full interval $[0,\delta[$.*

Proof. Let $\varphi$ be a solution of (2). By classical arguments (Zorn's Lemma) there exists a maximal solution extending $\varphi$ (again denoted by $\varphi$ for simplicity) defined on $[0,T[$. Suppose, contrary to our claim, that $T < \delta$. Then (3.5) holds by boundedness of $F$. Hence, $\varphi$ and its derivatives up to order $n-1$ can be extended to $T$ (Lemma 3.2). Furthermore, the problem:

$$\begin{aligned} & x^{(n)}(t) \in F(t+T, x(t), x'(t), \dots, x^{(n-1)}(t)) \\ & x^{(i)}(0) = \varphi^{(i)}(T), \quad 0 \le i \le n-1 \\ & x(t) \in K \end{aligned}$$

has a solution $\phi$ on an interval $[0,\gamma[$, because the assumptions of Theorem 3.1 are satisfied. Finally, the function:

$$\begin{array}{rccl} \tilde{\varphi}: & [0,T+\gamma[ & \longrightarrow & X \\ & t & \rightsquigarrow & \tilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in [0,T] \\ \phi(t-T), & t \in ]T,T+\gamma[ \end{cases} \end{array}$$

extends $\varphi$ and is a solution of (2), a contradiction. $\square$

**Note 3.1** *The preceding theorem remains true if boundedness of $F$ is replaced by integrably boundedness, i.e. if we assume the following growth condition on $F$:*

$$F(t,y) \subseteq \alpha(t)\,(1+\|y\|)\,\mathcal{U}_X \tag{3.6}$$

*for all $(t,y) \in [0,\delta[\, \times \mathcal{G}(A_K^{(n-1)})$, with $\alpha \in L^1(0,\delta)$. Under this hypothesis, if $\varphi$ is a solution of* (2)*, by using Bellman's Inequality we obtain,*

$$\|\psi(t)\| \leq (1+\|y_0\|)\exp\left(\sqrt{2}\int_0^t \alpha(s)\,ds\right) - 1 \tag{3.7}$$

*where $\psi(t) = (\varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t))$ and $y_0 = (x_0, v_1, \ldots, v_{n-1})$. Hence,* (3.5) *holds for all $0 < T < \delta$. Theorem* 3.2 *also remains true under the weaker hypothesis,*

$$F(t,y) \cap \alpha(t)\,(1+\|y\|)\,\mathcal{U}_X \neq \emptyset \tag{3.8}$$

*for all $(t,y) \in [0,\delta[\, \times \mathcal{G}(A_K^{(n-1)})$. (See Section* 4.*)*

## 3.2 General case

In the preceding we have assumed locally compactness on $\mathcal{G}(A_K^{(n-1)})$. Unfortunately, this is not usually true. Furthermore, if this condition fails tangential condition (3.4) does not imply the existence of viable solutions, even in the single-valued case, as the next example shows.

**Example 3.1** *Let $[a,b] \subseteq \mathbb{R}$ ($b > 1$). Obviously, $\mathcal{G}(T_{[a,b]})$ is not locally compact, because:*

$$T_{[a,b]}(x) = \begin{cases} [0,+\infty[ & \text{if } x = a \\ \mathbb{R} & \text{if } x \in\, ]a,b[ \\ ]-\infty,0] & \text{if } x = b \end{cases}$$

*Let us consider the problem,*

$$\begin{array}{l} x'' = x + t \\ x(0) = b \qquad x'(0) = 0 \\ x \in [a,b] \end{array}$$

*It is easy to see that* (3.4) *is satisfied, because $(u, t+x) \in T_{\mathcal{G}(T_{[a,b]})}(x,u)$ for all $(x,u)$ in $\mathcal{G}(T_{[a,b]})$. However, the solution of the initial value problem, $\varphi(t) = \dfrac{b+1}{2}e^t + \dfrac{b-1}{2}e^{-t} - t$ is not viable in $[a,b]$.*

The aim of this section is to state both necessary and sufficient conditions ensuring the existence of solutions of (2). To get such conditions we introduce the following set-valued map,

**Definition 3.1** *Let $\varphi$ be a solution of* (2) *on $[0,T]$. We define the set-valued map:*

$$\begin{array}{rccl} \Lambda(\varphi,\cdot): & [0,T[ & \longrightarrow & 2^X \\ & t & \rightsquigarrow & \Lambda(\varphi,t) = \limsup\limits_{h\to 0^+} \dfrac{n}{h^n}\displaystyle\int_t^{t+h} (t+h-s)^{n-1}\,\varphi^{(n)}(s)\,ds \end{array}$$

*That is,* $\Lambda(\varphi, t)$ *is the set of limit points of* $\frac{n}{h^n}\int_t^{t+h}(t+h-s)^{n-1}\,\varphi^{(n)}(s)\,ds$*, letting* $h \to 0^+$*.*

The study of $\Lambda(\varphi, \cdot)$ will allow us to give a necessary condition for (2).

**Theorem 3.3** *Let* $\varphi$ *be a solution of* (2) *on* $[0,T]$*. Then for each* $t \in [0,T[$*,* $\Lambda(\varphi,t)$ *is nonempty and compact. Moreover, the inclusion:*

$$\Lambda(\varphi,t) \subseteq F(t,\psi(t)) \cap A_K^{(n)}(\psi(t)) \tag{3.9}$$

*holds, here* $\psi(t) = (\varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t))$*.*

Proof. Since $F$ is u.s.c. having convex compact values and $\varphi \in W^{n,1}(0,T;X)$, there is $\beta > 0$ such that $F(t,\psi(t)) \subseteq \beta\mathcal{U}_X$. Hence, for each $t \in [0,T[$:

$$\frac{n!}{h^n}\int_t^{t+h}(t+h-s)^{n-1}\,\varphi^{(n)}(s)\,ds \in \beta\mathcal{U}_X$$

which implies that $\Lambda(\varphi,t)$ is nonempty (by compactness of $\mathcal{U}_X$) and bounded. It is also closed from properties of upper limits (see e.g. [3]). On the other hand, by its very definition, $\Lambda(\varphi,t) \subseteq A_K^{(n)}(\psi(t))$. Let $\varepsilon > 0$ and $t \in [0,T[$. Since $F$ is u.s.c., there is $\gamma > 0$ such that $F(s,\psi(s)) \subseteq F(t,\psi(t)) + \varepsilon\mathcal{U}_X$ if $0 < |s-t| < \gamma$. Therefore, for $0 < h$ small enough we have:

$$\frac{n!}{h^n}\int_t^{t+h}(t+h-s)^{n-1}\,\varphi^{(n)}(s)\,ds \in F(t,\psi(t)) + \varepsilon\mathcal{U}_X$$

by using Lemma 5.1 in [23], and letting $h \to 0^+$ we complete the proof. □

**Corollary 3.1** (Necessary condition) *If there is a solution of* (2) *for the initial condition* $(x_0, v_1, \ldots, v_{n-1}) \in \mathcal{G}(A_K^{(n-1)})$*, then:*

$$F(0,x_0,v_1,\ldots,v_{n-1}) \cap A_K^{(n)}(x_0,v_1,\ldots,v_{n-1}) \neq \emptyset. \tag{3.10}$$

Finally, the sufficient condition is stated in the next theorem, which proof is similar to that of Theorem 6.1 in [23]. So we will not give it.

**Theorem 3.4** (Sufficient condition) *Let us suppose that* $K = L \cap M$ *with* $\mathcal{G}(A_L^{(n-1)})$ *closed and* $F$ *u.s.c. on* $[0,\delta[\, \times \mathcal{G}(A_L^{(n-1)})$*. Let us suppose that the tangential condition,*

$$F(t,x,u_1,\ldots,u_{n-1}) \cap DA_L^{(n-1)}(x,u_1,\ldots,u_{n-1})[u_1,\ldots,u_{n-1}] \neq \emptyset \tag{3.11}$$

*holds for all* $(t,x,u_1,\ldots,u_{n-1}) \in [0,\delta[\, \times \mathcal{G}(A_L^{(n-1)})$*. Then* (2) *has a solution for each initial condition* $(x_0,v_1,\ldots,v_{n-1}) \in \mathcal{G}(A_K^{(n-1)})$ *satisfying:*

$$F(0,x_0,\omega_0) \cap A_L^{(n)}(x_0,\omega_0) \subseteq AI_M^{(n)}(x_0,\omega_0) \tag{3.12}$$

*where* $\omega_0 = (v_1,\ldots,v_{n-1})$*.*

In this general case is not possible to obtain a global viability result like Theorem 3.2, but a procedure similar to that in the proof of this Theorem can be used to state the next result.

**Theorem 3.5** *Under the hypotheses of Theorem 3.4, if moreover $F$ is bounded or integrably bounded on $[0,\delta[\times\mathcal{G}(A_L^{(n-1)})$ and*

$$F(t,x,\omega)\cap A_L^{(n)}(x,\omega)\subseteq AI_M^{(n)}(x,\omega)$$

*holds, for all $(t,x,u_1,\ldots,u_{n-1})\in[0,\delta[\times\mathcal{G}(A_K^{(n-1)})$, here $\omega=(u_1,\ldots,u_{n-1})$, then given a maximal solution $\varphi$ of (2) defined on $[0,\mathrm{T}[$ either $T=\delta$ or $(\varphi(T),\varphi'(T),\ldots,\varphi^{(n-1)}(T))\notin\mathcal{G}(A_K^{(n-1)})$.*

# 4 Almost upper semicontinuous case

In many problems, the set-valued map $F$ is only measurable in $t$, and no longer upper semicontinuous. This situation arises, for instance when we consider variational inclusions obtained by linearization of a differential inclusion, even in the autonomous case (see [3, Chap. 10]) or in the study of multivalued differential inequalities (see [15]). So we will devote this section to investigate the existence of solutions of (2) by assuming that $F$ is only almost upper semicontinuous.

Notice that in this context our assumption on $F$ is equivalent to the Carathéodory property, as a consequence of Scorzà-Dragoni theorem for set-valued maps (see e.g. [14, Prop. 5.1]). This is no longer true when $K$ changes along the time (see [7] for a counterexample).

We will make another assumption on $F$: it will be integrably bounded (or (3.8) will be satisfied). Under these hypotheses, the statement of Proposition 3.1 remains true, because given $\varphi$ a solution of (2) on $[0,T]$:

$$\begin{aligned}\frac{d(\sum_{j=0}^{n-1}\frac{h^j}{j!}\varphi^{(j)}(t);\,K)}{h^{n-1}/(n-1)!}&\le\frac{1}{h^{n-1}}\int_t^{t+h}(t+h-s)^{n-1}\,\|\varphi^{(n)}(s)\|\,ds\\&\le\frac{1}{h^{n-1}}\int_t^{t+h}(t+h-s)^{n-1}\,\alpha(s)\,(1+\|\psi(s)\|)\,ds\\&\le(1+\|\psi\|_\infty)\int_t^{t+h}\alpha(s)\,ds\end{aligned}$$

here $\psi$ as in (3.7) and $\|\psi\|_\infty=\sup_{t\in[0,T]}\|\psi(t)\|$ being. Hence, letting $h\to0^+$ we have the desired result.

From now we shall suppose that there is $\delta>0$ such that $F$ is almost u.s.c. and integrably bounded (or satisfying (3.8)) on $[0,\delta[\times\mathcal{G}(A_K^{(n-1)})$.

The first theorem in this section states that, as in the u.s.c., tangential condition (3.4) implies that (2) has a solution.

**Theorem 4.1** *Under the hypotheses of this section, if $\mathcal{G}(A_K^{(n-1)})$ is closed and (3.4) is satisfied for all $(t,x,u_1,\ldots,u_{n-1})\in([0,\delta[\setminus N)\times\mathcal{G}(A_K^{(n-1)})$, $N\subseteq[0,\delta[$ being a null set, then (2) has a solution on $[0,\delta[$.*

Proof. The existence of a solution of (2) comes from [4, Theorem 11.7.1], because (2) is equivalent to,

$$\left.\begin{array}{l} y'(t) \in \tilde{F}(t, y(t)) \\ y(0) = (x_0, v_1, \ldots, v_{n-1}) \\ y(t) \in \mathcal{G}(A_K^{(n-1)}) \end{array}\right\}$$

where $\tilde{F}(t, y) = \{(y_2, \ldots, y_{n-1})\} \times F(t, y)$, and tangential condition

$$\tilde{F}(t, y) \cap T_{\mathcal{G}(A_K^{(n-1)})}(y) \neq \emptyset$$

can be rewritten as (3.4). □

**Remark 4.1** *It is possible to give a different proof, which does not depend on first order viability theorem. To get it we consider the family of u.s.c. problems,*

$$\left.\begin{array}{l} x^{(n)}(t) \in F_h(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \\ x(0) = x_0, \;\; x'(0) = v_1, \;\; \ldots, \;\; x^{(n-1)}(0) = v_{n-1} \\ x(t) \in K \end{array}\right\} \tag{4.1}$$

*where* $F_h(t, y) = \dfrac{n}{h^n} \displaystyle\int_t^{t+h} (t+h-s)^{n-1} F(s, y)\, ds$, *and* $h > 0$. *Then we apply Theorem* 3.2 *to have a solution of* (4.1) *on* $[0, \delta[$, *and finally a solution of* (2) *on* $[0, \delta[$ *is obtained as a limit of solutions of the problems* (4.1), *letting* $h \to 0^+$.

In this case, when $\mathcal{G}(A_K^{(n-1)})$ is not closed, $\Lambda(\varphi, t)$ can be empty as the next example shows.

**Example 4.1** *Let us consider the almost u.s.c. set-valued map with convex compact values,*

$$F(t, x, y) = \begin{cases} [-t^{-a}, t^{-a}], & 0 < t < 1 \\ \{0\}, & t = 0 \end{cases}$$

*being* $0 < a < 1$. *Obviously, $F$ is integrably bounded, taking:*

$$\alpha(t) = \begin{cases} 2t^{-a}, & 0 < t < 1 \\ 0, & t = 0 \end{cases}$$

*It is easy to check that* $\varphi(t) = \frac{t^{2-a}}{(1-a)(2-a)}$ *is a solution of,*

$$\begin{array}{l} x''(t) \in F(t, x(t), x'(t)) \\ x(0) = 0, \;\; x'(0) = 0 \\ x(t) \in [0, 2] \end{array}$$

*however,* $\Lambda(\varphi, 0) = \emptyset$.

Nevertheless, it is immediate to show that $\Lambda(\varphi, t)$ is not empty and compact if $t$ is a Lebesgue point of $\alpha$, i.e. if $\displaystyle\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \alpha(s)\, ds = \alpha(t)$. Moreover, if $\Lambda(\varphi, t)$ is nonempty, then it is contained in $A_K^{(n)}(\psi(t))$, here $\psi$ as in Theorem 3.3, and if $F(\cdot, \psi)$ is u.s.c. at $t$, then (3.9) holds. But, $F(\cdot, \psi)$ is measurable and Lusin's Theorem (see e.g.

[8]) states that it is continuous at almost every point in $[0,\delta[$. Therefore, Theorem 3.3 is satisfied "almost everywhere", and making assumptions on $\alpha$ we can state the next necessary condition.

**Theorem 4.2** (Necessary condition) *Let us assume that $\alpha$ is continuous at zero, then if* (2) *has a solution* (3.10) *is satisfied.*

We close this section with a sufficient condition.

**Theorem 4.3** (Sufficient condition) *Let us suppose that $K = L \cap M$ with $\mathcal{G}(A_L^{(n-1)})$ closed and $F$ almost u.s.c. and integrably bounded (or satisfying* (3.8)*) on $[0,\delta[\times\mathcal{G}(A_L^{(n-1)})$, with $\alpha$ continuous at zero. Let us suppose that the tangential condition,*

$$F(t,x,u_1,\dots,u_{n-1})\cap DA_L^{(n-1)}(x,u_1,\dots,u_{n-1})[u_1,\dots,u_{n-1}]\neq\emptyset \tag{4.2}$$

*holds for all $(t,x,u_1,\dots,u_{n-1})\in([0,\delta[\setminus N)\times\mathcal{G}(A_L^{(n-1)})$, $N\subseteq[0,\delta[$ being a null set. Then* (2) *has a solution for each initial condition $(x_0,v_1,\dots,v_{n-1})$ in $\mathcal{G}(A_K^{(n-1)})$ satisfying* (3.12).

Proof. By Theorem 4.1 there exists a solution $\varphi$ of (2.1)-(2.2) on $[0,\delta[$, viable in $L$. Since $\alpha$ is continuous at zero, (3.10) is satisfied, and in a procedure similar to that in the proof of Theorem 6.1 in [23], we find $0<T<\delta$, such that $\varphi$ is viable in $K$ on $[0,T]$. □

# 5 Multivalued differential inequalities of higher order

In this section we shall apply the previous results to find a solution of a multivalued differential inequality of higher order. Let $F: X^n\to 2^X$ be an u.s.c. set-valued map having nonempty convex compact values and linear growth, i.e. there exists a positive constant $\beta$ such that

$$F(y)\subseteq\beta\,(1+\|y\|)\,\mathcal{U}_X,\qquad y\in X^n$$

A set-valued map satisfying that properties is usually called a Marchaud map (see e.g. [4]). We look for a solution of the initial value problem:

$$\begin{aligned}&x^{(n)}(t)\in F(x(t),x'(t),\dots,x^{(n-1)}(t))\\&x(0)=x_0,\ \ x'(0)=v_1,\ \ \dots,\ \ x^{(n-1)}(0)=v_{n-1}\end{aligned}\tag{5.1}$$

such that,

$$\omega(t)\le x(t)\tag{5.2}$$

here $\omega\in W^{n,1}(0,\delta;X)$ satisfying $\omega(0)\le x_0$ and $\omega^{(j)}(0)\le v_j$, $1\le j\le n-1$, and $\le$ refers to the partial ordering given by $X^+=\{x\in X:\pi_i(x)\ge 0\ \ \forall\, i\}$. Obviously, if we define $y(t)=x(t)-\omega(t)$, we can rewrite (5.1)–(5.2) as the following time-dependent viability problem:

$$
(5.3) \qquad \begin{aligned}
& y^{(n)}(t) \in \tilde{F}(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \\
& y(0) = x_0 - \omega(0), \quad y^{(j)}(0) = v_j - \omega^{(j)}(0), \quad 1 \le j \le n-1 \\
& y(t) \in X^+
\end{aligned}
$$

where $\tilde{F}(t, y_1, \ldots, y_n) = F(y_1 + \omega(t), y_2 + \omega'(t), \ldots, y_n + \omega^{(n-1)}(t)) - \omega^{(n)}(t)$.

Since $F$ is u.s.c. and $\omega \in W^{n,1}$, then $\tilde{F}$ is almost u.s.c. It is also integrably bounded because,

$$\tilde{F}(t, y) \subseteq \beta \,(1 + \sqrt{2}\phi(t))\,(1 + \|y\|)$$

here $\phi(t) = \|(\omega(t), \omega'(t), \ldots, \omega^{(n)}(t))\|$, and so $\phi \in L^1(0, \delta)$.

From Section 6 in [23] we can compute the $(n-1)^{th}$ order tangent set of $X^+$, and we clearly show that $\mathcal{G}(A_{X^+}^{(n-1)})$ is not locally compact. Therefore, we are under the hypotheses of Theorem 4.3 taking $L = X$ and $M = X^+$ and assuming that $\omega^{(n)}$ is continuous at zero. Then (5.1)–(5.2) has a solution if the initial condition satisfies,

$$\tilde{F}(0, y_0, y_1, \ldots, y_{n-1}) \subseteq AI_{X^+}^{(n)}(y_0, y_1, \ldots, y_{n-1})$$

where $y_0 = x_0 - \omega(0)$ and $y_j = v_j - \omega^{(j)}(0)$, $1 \le j \le n-1$. Finally, this expression can be rewritten by using Corollary 6.1 in [23] as,

$$\pi_i(\omega^{(n)}(0)) < \inf_{y \in F(x_0, v_1, \ldots, v_{n-1})} \pi_i(y)$$

for all $i$ such that $\pi_i(x_0 - \omega(0)) = \pi_i(v_j - \omega^{(j)}(0)) = 0$, $1 \le j \le n-1$.

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*Luis Marco*
*Departament de Matemàtica Aplicada*
*Universitat de València*
*Doctor Moliner, 50*
*46100 Burjassot, Spain*
`e-mail: Luis.Marco@uv.es`

*José Alberto Murillo*
*Departamento de Matemática Aplicada*
*Universidad de Murcia*
*Paseo de Alfonso XIII, 44,*
*30203 Cartagena, Spain*
`e-mail: murilloa@plc.um.es`
`Alberto.Murillo@uv.es`