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Pliska Stud. Math. Bulgar. **20** (2011), 109–120

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MAXIMAL NUMBER OF SUCCESSORS IN A NGINAR(1) PROCESS

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In this paper we obtain upper and lower bounds for normalized sequences of maxima, associated with a stationary integer-valued autoregressive process of first order with geometric marginals (NGINAR(1) process). These processes are a special case of AR(1) processes and strictly stationary ergodic Markov chains.

1. Introduction

Correlated statistical data expressed in terms of counts taken sequentially in time are often used in practice. Some examples include the total claim amount in an insurance company or the number of survivals in a population at a fixed moment in time. Mathematical models for such data with different marginal distributions have been recently investigated.

In 1970 Anderson shows that under linear normalizations the geometric distribution does not belong to max-domain of attraction of any max-stable law. He discusses independent and identically distributed (i.i.d.) random variables (rv's)

*The first author is partially supported by grant RD–05-333/2010 (RD-07-1028) of Shumen University, Bulgaria.

2000 *Mathematics Subject Classification:* 60J80, 60J20, 60J10, 60G10, 60G70, 60F99.

Key words: Markov chains; NGINAR process; Branching processes; Processes of maxima; Stationary sequences.

with distribution function (d.f.) F, whose support consists of all sufficiently large integers, and obtains that for some $t > 0$,

$$\lim_{n\to\infty} \frac{1 - F(n)}{1 - F(n+1)} = e^t. \tag{1}$$

if and only if there exist c_n, such that for all $x \in \mathbb{R}$,

$$e^{-e^{-t(x-1)}} \le \liminf_{n\to\infty} F^n(x + c_n) \le \limsup_{n\to\infty} F^n(x + c_n) \le e^{-e^{-tx}}. \tag{2}$$

In 1983 Leadbetter, Lindgren and Rootzen discuss the problem of estimating the extrema of a stationary sequence and find the following two conditions that guarantee we do not go far away from the i.i.d. case.

Condition $D(u_n)$**:** For any integers r, s, n and $1 \le i_1 < \dots < i_r < j_1 < \dots < j_s \le n$, such that $j_1 - i_r \ge l(n)$ we have

$$|\mathrm{P}(\max_{i\in A_1\cup A_2} X_i \le u_n) - \mathrm{P}(\max_{i\in A_1} X_i \le u_n)\mathrm{P}(\max_{i\in A_2} X_i \le u_n)| \le \alpha_{n,l(n)},$$

where $A_1 = \{i_1, i_2, \dots i_r\}$, $A_2 = \{j_1, j_2, \dots j_s\}$ and $\alpha_{n,l(n)} \to 0$ as $n \to \infty$ for some sequence $l(n) = o(n)$.

Condition $D'(u_n)$**:**

$$\limsup_{n\to\infty} n \sum_{j=2}^{[\frac{n}{k}]} \mathrm{P}(X_1 > u_n, X_j > u_n) \to 0, \quad as\ k \to \infty.$$

In 1992 McCormick and Park generalize Anderson's result for stationary sequence. They define continuous function:

$$F_c(x) = \begin{cases} -\log(1 - F(x)), & x \in \mathbb{N} \\ F([x]) + (x - [x])(F([x+1]) - F([x])), & x \in \mathbb{R} \end{cases}$$

and prove that for a stationary sequence $X_1, X_2, \dots$, with d.f. F, whose support consists of all sufficiently large integers, conditions $D(c_n + x)$ and $D'(c_n + x)$ together guarantee that

$$e^{-e^{-t(x-1)}} \le \liminf_{n\to\infty} \mathrm{P}(M_n < x + c_n) \le \limsup_{n\to\infty} \mathrm{P}(M_n < x + c_n) \le e^{-e^{-tx}}, \tag{3}$$

where $M_n = \max(X_1, X_2, \dots, X_n)$ and $1 - F_c(c_n) = n^{-1}$.

Integer valued autoregressive process of first order (INAR(1)) was introduced by Al-Osh and Alzaid (1987). It is based on binomial thinning operator. This model was generalized to INAR(d) by Du and Li (1991). It is useful tool for modeling discrete-time dependent counting processes. McCormick and Park (1992) and Hall and Mireira (2006) study the behaviour of extreme value generated by such models and obtain bounds for the distribution of the maximum based on negative binomial autoregressive process.

Ristič et al. (2009) define and investigate a new first-order integer-valued autoregressive process with geometric marginals denoted by NGINAR(1). The problem of scalar multiplication for discrete rv's is resolved by the use of the negative binomial thinning operator. These processes are a special case of strictly stationary counting processes, AR(1) processes and ergodic Markov chains, i.e. they are aperiodic and positive recurrent.

In 1995, Dion, Gauthier and Latour (1995) define the generalized integer-valued autoregressive time series of order d as a functional of the multiple branching processes with immigration. In the case $d = 1$ the two models are the same. The fact that the NGINAR(1) process is a special case of the generalized integer-valued autoregressive time series of order d, allows us to consider this process as a special case of a branching process with immigration.

In this paper we investigate maximal number of successors of the NGINAR(1) process, i.e. we deal with normalized maxima of correlated identically geometrically distributed rv's. In order to obtain upper and lower bounds for the d.f. of the normalized maxima we use direct calculations. We do not use the standard approaches discussed above because it is difficult to verify conditions $D(u_n)$ and $D'(u_n)$.

The next section starts with a description of the NGINAR(1) model. Then we obtain some properties of these processes and give the main result of the paper, namely the upper and lower bounds for the maximal number of successors of NGINAR(1) process. Finally we present the dependence of these bounds on the model parameters.

Along the paper $\stackrel{\mathrm{d}}{=}$ stands for the equality in distribution and $\stackrel{\mathrm{w}}{\to}$ for the weak convergence. We suppose that all considered random elements are defined on the same probability space, $(\Omega, \mathcal{A}, \mathrm{P})$.

2. Asymptotic behaviour of maxima

Ristič et al. (2009) define a stationary process $\{X_n : n = 0, 1, \dots\}$ by

$$X_0 = 1,\ X_n = \alpha * X_{n-1} + \varepsilon_n, \quad n = 1, 2, \dots \tag{4}$$

where the operator $*$ is called negative binomial thinning operator. It is defined by

$$\alpha * X = \sum_{i=1}^{X} W_i, \text{ with } \sum_{i=1}^{0} = 0,$$

where W_i are independent, geometrically distributed rv's with parameter

$$p_\alpha = \frac{1}{1+\alpha}, \ \alpha \in [0,1).$$

The process X_n is assumed to have geometric marginals with parameter

$$p_\mu = \frac{1}{(1+\mu)}, \ \mu > 0$$

and $\{\varepsilon_n, n = 1, 2, \dots\}$ are i.i.d. random variables, independent of $\{X_{n-k} : k = 0, 1, \dots, n\}$ and $\{W_i : i = 1, 2, \dots\}$.

These assumptions mean that

$$\mathrm{P}(X_n = k) = (1 - p_\mu)^k p_\mu, \quad k = 0, 1, \dots$$

(briefly $X_n \sim Ge(p_\mu)$) and the random variables $\{\varepsilon_n : n = 1, 2, \dots\}$ have probability mass function

$$\mathrm{P}(\varepsilon_n = j) = \frac{(1-p_\mu)p_\alpha - 1 + p_\alpha}{p_\alpha - p_\mu} p_\mu (1-p_\mu)^j + \frac{(1-p_\mu)(1-p_\alpha)}{p_\alpha - p_\mu} p_\alpha (1-p_\alpha)^j, \qquad j = 0, 1, \dots.$$

The latter implies that

$$\varepsilon_n \stackrel{\mathrm{d}}{=} W I\{A\} + X I\{\overline{A}\},$$

where W and X are independent random variables, $W \sim Ge(p_\alpha)$, $X \sim Ge(p_\mu)$, A is an event with probability $\mathrm{P}(A) = \dfrac{(1-p_\mu)(1-p_\alpha)}{p_\alpha - p_\mu}$, and $I\{A\}$ is the indicator of the event A. The process is well defined for $\alpha < \dfrac{\mu}{1+\mu}$.

Let $\{W_{n,i}, i = 1, \dots, n-1, n \ge 1\}$ be a triangular array of i.i.d. geometrically distributed rv's with parameter p_α. Then, the relations

$$X_0 = 1, \ \ X_n = \sum_{i=1}^{X_{n-1}} W_{n,i} + \ \varepsilon_n, \quad n = 1, 2, \dots. \tag{5}$$

suggest the following branching process interpretation of this model. The random variable X_n could be considered as the number of individuals in the n-th generation. Then μ represents the average number of these individuals. $W_{n,i}$ is the number of successors of the i-th individual in the n-th generation. So, the constant α is the average number of these successors and ε_n is the number of immigrations in the n-th generation. The extinction probability of the sequence is given by $\mathrm{P}(X_n = 0) = p_\mu$.

If $\alpha = 0$, then $X_n \stackrel{\mathrm{d}}{=} \varepsilon_n, n = 1, 2, \dots$, and all these rv's are geometrically distributed with parameter p_μ. In this case the process is a pure immigration process.

If $\alpha = 1 - p_\mu$, then $\varepsilon_n \sim Ge(p_\alpha)$ $n = 1, 2, \dots$ and $X_n \sim Ge\left(\dfrac{p_\alpha p_\mu}{1 - p_\alpha(1 - p_\mu)}\right)$, but they are not independent. The authors call this process NGINAR(1) process.

First we obtain some preliminary results formulated in the following theorem. Denote by $q_\mu = 1 - p_\mu$, $q_\alpha = 1 - p_\alpha$ and by $NBi(n; p)$ the negative binomial distribution with parameters n and p.

Theorem 1. *Let $\{X_n : n = 0, 1, \dots\}$ be NGINAR(1) process and $\xi_y \sim NBi([y] + 1; p_\alpha)$. For $y \geq 0$*

1.
$$\Phi(s, y) = \mathbb{E}(s^{X_2} | X_1 \leq y) = \mathbb{E}s^{X_1} \times \frac{\left(1 - q_\mu^{[y]+1} \mathbb{E}s^{\xi_y}\right)}{1 - q_\mu^{[y]+1}};$$

2.
$$\mathrm{P}(X_2 = k | X_1 \leq y) = \frac{\mathrm{P}(X_1 = k) - q_\mu^{[y]+1} \mathrm{P}(X_1 + \xi_y = k)}{1 - q_\mu^{[y]+1}}, \quad k = 0, 1, \dots;$$

3.
$$\mathrm{P}(X_2 \leq y | X_1 \leq y) = 1 - \frac{q_\mu^{[y]+1}}{1 - q_\mu^{[y]+1}} \mathrm{P}(X_1 + \xi_y \leq [y]).$$

P r o o f. 1. For the probability generating function Φ we obtain consequently:

$$\Phi(s, y) = \mathbb{E}(s^{X_2} | X_1 \leq y)) = \sum_{j=0}^{\infty} s^j \mathrm{P}(X_2 = j | X_1 \leq y) =$$

$$= \sum_{j=0}^{\infty} s^j \frac{\mathrm{P}(X_2 = j, X_1 \leq y)}{\mathrm{P}(X_1 \leq y)} = \sum_{j=0}^{\infty} \sum_{r=0}^{[y]} s^j \frac{\mathrm{P}(X_2 = j, X_1 = r)}{\mathrm{P}(X_1 \leq y)} =$$

$$= \sum_{r=0}^{[y]} \sum_{j=0}^{\infty} s^j \frac{\mathrm{P}(X_2 = j | X_1 = r)\mathrm{P}(X_1 = r)}{\mathrm{P}(X_1 \le y)}.$$

From the model definition we have

$$\Phi(s, y) = \sum_{r=0}^{[y]} \sum_{j=0}^{\infty} s^j \frac{\mathrm{P}(X_1 = r)}{\mathrm{P}(X_1 \le y)}.$$

$$\cdot \left(\mathrm{P}(\sum_{k=1}^{r+1} W_{2,k} = j)\mathrm{P}(A) + \mathrm{P}(\sum_{k=1}^{r} W_{2,k} + X = j)\mathrm{P}(\bar{A}) \right) =$$

$$= \sum_{r=0}^{[y]} \frac{\mathrm{P}(X_1 = r)}{\mathrm{P}(X_1 \le y)} \left((\mathbb{E}s^W)^{r+1}\mathrm{P}(A) + (\mathbb{E}s^W)^r \mathbb{E}s^X \mathrm{P}(\bar{A}) \right) =$$

$$= \frac{\mathbb{E}s^W \mathrm{P}(A) + \mathbb{E}s^X \mathrm{P}(\bar{A})}{\mathrm{P}(X_1 \le y)} \sum_{r=0}^{[y]} (\mathbb{E}s^W)^r \mathrm{P}(X_1 = r) =$$

$$= \frac{\mathbb{E}s^\varepsilon}{\mathrm{P}(X_1 \le y)} \sum_{r=0}^{[y]} (\mathbb{E}s^W)^r q_\mu^r p_\mu = \frac{\mathbb{E}s^\varepsilon}{\mathrm{P}(X_1 \le y)} p_\mu \frac{1 - (\mathbb{E}s^W q_\mu)^{[y]+1}}{1 - \mathbb{E}s^W q_\mu}.$$

In the last expression we substitute respectively

$$\mathbb{E}s^\varepsilon = \frac{1 - p_\alpha q_\mu - q_\alpha s}{(1 - q_\mu s)(1 - q_\alpha s)},$$

$$\mathrm{P}(X_1 \le y) = 1 - q_\mu^{[y]+1},$$

$$\mathbb{E}s^W = \frac{p_\alpha}{1 - q_\alpha s},$$

and

$$(\mathbb{E}s^W)^{[y]+1} = \mathbb{E}s^{\xi_y}.$$

In this way we get

$$\Phi(s, y) = \frac{\frac{1 - p_\alpha q_\mu - q_\alpha s}{(1 - q_\mu s)(1 - q_\alpha s)}}{1 - q_\mu^{[y]+1}} \times p_\mu \times \frac{1 - q_\mu^{[y]+1} \mathbb{E}s^{\xi_y}}{1 - \frac{q_\mu p_\alpha}{1 - q_\alpha s}} = \frac{p_\mu}{(1 - q_\mu s)} \times \frac{\left(1 - q_\mu^{[y]+1} \mathbb{E}s^{\xi_y}\right)}{1 - q_\mu^{[y]+1}}.$$

Because X_1 is geometrically distributed with parameter p_μ, we have

$$\mathbb{E}s^{X_1} = \frac{p_\mu}{1 - q_\mu s},$$

and therefore

$$\Phi(s,y) = \mathbb{E}s^{X_1} \times \frac{\left(1 - q_\mu^{[y]+1}\mathbb{E}s^{\xi_y}\right)}{1 - q_\mu^{[y]+1}}.$$

2. The probability $\mathrm{P}(X_2 = k | X_1 \le y)$ will be found from the equation

$$\mathrm{P}(X_2 = k|X_1 \le y) = \frac{\Phi^{(k)}(0,y)}{k!} \tag{6}$$

where $\Phi(s,y) = \mathbb{E}(s^{X_2}|X_1 \le y)$ and $\Phi^{(k)}$ is the k-th derivative of Φ with respect to s.

Now 1. entails

$$\mathrm{P}(X_2 = k|X_1 \le L_n(x)) = \frac{\Phi^{(k)}(0,y)}{k!} =$$

$$= \frac{\mathrm{P}(X_1 = k) - q_\mu^{[L_n(x)]+1}\mathrm{P}(X_1 + \xi_{[L_n(x)]+1} = k)}{1 - q_\mu^{[L_n(x)]+1}}.$$

3. By 2. we have

$$\mathrm{P}(X_2 \le y|X_1 \le y) = \sum_{k=0}^{[y]} \mathrm{P}(X_2 = k|X_1 \le y) =$$

$$= \sum_{k=0}^{[y]} \frac{\mathrm{P}(X_1 = k) - q_\mu^{[y]+1}\mathrm{P}(X_1 + \xi_y = k)}{1 - q_\mu^{[y]+1}} =$$

$$= 1 - \frac{q_\mu^{[y]+1}}{1 - q_\mu^{[y]+1}} \sum_{k=0}^{[y]} \mathrm{P}(X_1 + \xi_y = k) =$$

$$= 1 - \frac{q_\mu^{[y]+1}}{1 - q_\mu^{[y]+1}} \mathrm{P}(X_1 + \xi_y \le [y]). \qquad \square$$

Denote by $M_n = \max(X_1, X_2, \ldots, X_n)$. As discussed in the introduction we cannot obtain non-degenerate weak limit under linear normalizations. We are looking for upper and lower bounds of $\mathrm{P}(L_n^{\leftarrow}(M_n) < x)$, where L_n are strictly increasing and continuous mappings, i.e. they preserve the operation maxima. Further we are going to prove that

$$|\mathrm{P}(M_n \le L_n(x)) - \mathrm{P}^n(X_1 \le L_n(x))| \to 0, \text{ as } n \to \infty.$$

In this way we could investigate the asymptotic behavior of M_n by investigating the probabilities $\mathrm{P}^n(X_1 \leq L_n(x))$.

Theorem 2. *Let $\{X_n : n = 0, 1, \dots\}$ be NGINAR(1) process, $M_n = \max(X_1, X_2, \dots, X_n)$ and $L_n(x) = -\log_{q_\mu}((n+1)f(x)) - 1$, where $f(x)$ is continuous, positive and strictly increasing function.*

1. $|\mathrm{P}^n\left(X_1 \leq L_n(x)\right) - \left(1 - q_\mu^{[-\log_{q_\mu}((n+1)f(x))]}\right)^n| \to 0,$ *as* $n \to \infty$.

2. $|\mathrm{P}(M_n \leq L_n(x)) - \mathrm{P}^n(X_1 \leq L_n(x))| \to 0,$ *as* $n \to \infty$.

3. *For all* $n \in \mathbb{N}$,

$$\exp\left\{-(q_\mu f(x))^{-1}\right\} < \liminf_{n\to\infty} \mathrm{P}(M_n \leq L_n(x)) \leq \mathrm{P}(M_n \leq L_n(x)) \leq$$

$$\leq \limsup_{n\to\infty} \mathrm{P}(M_n \leq L_n(x)) \leq \exp\left\{-f^{-1}(x)\right\}.$$

P r o o f. 1. This limit relation follows from the particular form the geometric distribution's d.f. and the d.f. of maxima.

2. Since $L_n(x)$ are non-decreasing and $\{M_n : n = 1, 2, \dots\}$ is a stationary Markov chain we have

$$\mathrm{P}(M_n \leq L_n(x)) = \mathrm{P}(\bigcap_{i=1}^{n} X_i \leq L_n(x)) =$$

$$= \mathrm{P}(X_1 \leq L_n(x))\mathrm{P}(X_2 \leq L_n(x)|X_1 \leq L_n(x)) \dots$$
$$\dots \mathrm{P}(X_n \leq L_n(x)|X_{n-1} \leq L_n(x)) =$$
$$= \mathrm{P}(X_1 \leq L_n(x))\mathrm{P}^{n-1}(X_2 \leq L_n(x)|X_1 \leq L_n(x)).$$

Then

$$|\mathrm{P}(M_n \leq L_n(x)) - \mathrm{P}^n(X_1 \leq L_n(x))| \leq$$

$$\leq (n-1)\mathrm{P}(X_1 \leq L_n(x))|\mathrm{P}(X_2 \leq L_n(x)|X_1 \leq L_n(x)) - \mathrm{P}(X_1 \leq L_n(x))|.$$

Now using 3. from Theorem 1 we obtain

$$|\mathrm{P}(M_n \leq L_n(x)) - \mathrm{P}^n(X_1 \leq L_n(x))| \leq$$

$$\leq (n-1)\mathrm{P}(X_1 \leq L_n(x)).$$

$$\cdot\left|1-\frac{q_\mu^{[L_n(x)]+1}}{1-q_\mu^{[L_n(x)]+1}}\mathrm{P}(X_1+\xi_{[L_n(x)]+1}\le [L_n(x)])-\mathrm{P}(X_1\le L_n(x))\right| =$$

$$= (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}(X_1\le [L_n(x)], X_1+\xi_{[L_n(x)]+1}\ge [L_n(x)])\le$$

$$\le (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}(X_1+\xi_{[L_n(x)]+1}\ge [L_n(x)]) =$$

$$= (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}(X_1+1+\xi_{[L_n(x)]+1}\ge [L_n(x)]+1) =$$

$$= (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}\left(\frac{X_1+1}{[L_n(x)]+1}+\frac{\xi_{[L_n(x)]+1}}{[L_n(x)]+1}\ge 1\right).$$

From the Kolmogorov's SLLN we have that

$$\frac{\xi_{[L_n(x)]+1}}{[L_n(x)]+1} = \mathbb{E}(W)+o(1) = \frac{1-p_\alpha}{p_\alpha}+o(1).$$

Then

$$|\mathrm{P}\{M_n\le L_n(x)\}-\mathrm{P}^n\{X_1\le L_n(x)\}|\le$$

$$\le (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}\left(\frac{X_1}{[L_n(x)]+1}+\frac{1-p_\alpha}{p_\alpha}+o(1)\ge 1\right)$$

$$= (n-1)q_\mu^{[L_n(x)]+1}\times \mathrm{P}\left(X_1\ge ([L_n(x)]+1)\left(1-\alpha+o(1)\right)\right).$$

For the probability in the right hand side of the last expression we have that

$$\mathrm{P}\left(X_1\ge ([L_n(x)]+1)\left(1-\alpha+o(1)\right)\right) =$$

$$= 1-\mathrm{P}\left(X_1< ([L_n(x)]+1)\left(1-\alpha+o(1)\right)\right)\to 1-\mathrm{P}\left\{X_1<\infty\right\} = 1-1 = 0,$$

because $\alpha\in[0,1)$ and $X_1<\infty$ a.s.

We substitute $L_n(x)$ and obtain

$$(n-1)q_\mu^{[L_n]+1} = (n-1)q_\mu^{[-\log_{q_\mu}((n+1)f(x))]}\le$$

$$\le (n-1)q_\mu^{-\log_{q_\mu}((n+1)f(x))-1} = \frac{n-1}{n+1}\frac{1}{q_\mu f(x)} < \frac{1}{q_\mu f(x)} < \infty.$$

In this way, we proved that

$$|\mathrm{P}(M_n\le L_n(x))-\mathrm{P}^n(X_1\le L_n(x))|\to 0,\ \text{as } n\to\infty.$$

3. It is clear that

$$\left(1-q_\mu^{-\log_{q_\mu}((n+1)f(x))}\right)^n \geq \left(1-q_\mu^{[-\log_{q_\mu}((n+1)f(x))]}\right)^n >$$

$$\left(1-q_\mu^{-\log_{q_\mu}((n+1)f(x))-1}\right)^n,$$

$$\liminf_{n\to\infty}\left(1-q_\mu^{[-\log_{q_\mu}((n+1)f(x))]}\right)^n \geq \exp\left\{-\frac{1}{q_\mu f(x)}\right\}$$

and

$$\limsup_{n\to\infty}\left(1-q_\mu^{[-\log_{q_\mu}((n+1)f(x))]}\right)^n \leq \exp\left\{-\frac{1}{f(x)}\right\}.$$

Then for all $n \in \mathbb{N}$,

$$\exp\left\{-(q_\mu f(x))^{-1}\right\} < \liminf_{n\to\infty} F^n(L_n(x)) =$$

$$= \liminf_{n\to\infty} \mathrm{P}(M_n \leq L_n(x)) \leq \mathrm{P}(M_n \leq L_n(x)) \leq$$

$$\leq \limsup_{n\to\infty} \mathrm{P}(M_n \leq L_n(x)) = \limsup_{n\to\infty} F^n(L_n(x)) \leq \exp\left\{-f^{-1}(x)\right\}. \qquad \square$$

Notes: 1. We do not use Anderson's (1970) result in our proof, because of difficulties in determination of the centering constants c_n.

2. If one takes $p_\mu = p_\mu(n) \to 0$, as $n \to \infty$, i.e. $q_\mu(n) \to 1$, then

$$\lim_{n\to\infty} \mathrm{P}(M_n \leq L_n(x))$$

exists and $\mathrm{P}(M_n \leq L_n(x)) \to \exp\left\{-\frac{1}{f(x)}\right\}$, as $n \to \infty$. Similar results for normalized maxima of i.i.d. discrete random variables with changing parameters, including geometrically distributed, were proved in Nadarajah and Mitov (2002). They use linear transformations.

3. Finally, denote $h(x) = \log f(x)$, then $f(x) = e^{h(x)}$ and $\frac{1}{f(x)} = e^{-h(x)}$. In this way if we take $p_\mu = p_\mu(n) \to 0$, as $n \to \infty$ we obtain

$$\mathrm{P}\{M_n \leq L_n(x)\} \to \exp\left\{-e^{-h(x)}\right\} =: H(x), \quad h(x) \in \mathbf{R}.$$

We could choose, for example, the following three special cases for the function $h(x)$:

- if $h(x) = x, x \in \mathbf{R}$, then $H(x) = e^{-e^{-x}}$;

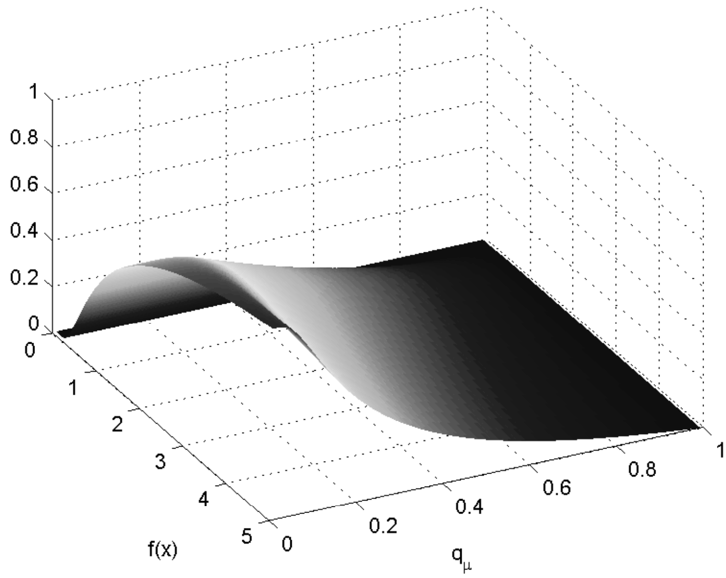

Fig. 1. The distance between the upper and lower bound as a function of q_μ and x.

- if $h(x) = \beta \log x, x > 0, \beta > 0$, then $H(x) = e^{-x^{-\beta}}$;
- if $h(x) = -\beta \log(-x), x < 0, \beta > 0$, then $H(x) = e^{-(-x)^{\beta}}$.

The latter means, that under nonlinear normalization when the parameter $p_\mu = p_\mu(n) \to 0$, as $n \to \infty$, $X_1, X_2, \dots$ are neither independent nor identically distributed, but we could obtain any of the three max-stable distributions as a limit.

Figure 1. shows the distance between the upper and lower bound as a function of $q_\mu = 1 - p_\mu$ and x. It is clear that the lower the value of q_μ the greater will be the distance between the upper and lower bounds.

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