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BRANCHING STOCHASTIC PROCESSES: REGULATION, REGENERATION, ESTIMATION, APPLICATIONS

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This is a survey of the works of Bulgarian mathematicians in the area of Branching Stochastic Processes.

1. Introduction

Branching stochastic processes arise as models of population dynamics of particles having different nature, like photons, electrons, neutrons, protons, atoms, molecules, cells, microorganisms, plants, animals, individuals, prices, information, etc. (see e.g. Gihman and Skorohod [228], Harris [229], Assmusen and Hering [209], Sevastyanov [263], Jagers [240], Srinivasan [264], Yakovlev and N. Yanev [17], [30], [51], [81], Haccou, Jagers and Vatutin [236]). Thus, many real situations in physics, chemistry, biology, demography, ecology, economy, etc. could be modeled by different types of branching processes. Recall that the terminology "branching processes" was first introduced by Kolmogorov and his coauthors [207, 208] considering multitype branching processes in the Markov case, which have received much attention in the literature on stochastic processes.

Investigations of branching structures increase significantly during forties because of their application in studying of nuclear chain reactions. In many applications the phase space of the branching processes is the set of nonnegative integers in $\mathbf{R}$ or $\mathbf{R}^d$, but the branching processes with continuous state spaces are studied too (see [209]).

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A lot of papers deal with statistical problems for branching processes and among them at first we will mention some books and survey articles: Dion and Keiding [224], Basawa and Scott [211], Nanthi [253], Iosifescu et al. [239], Vatutin and Zubkov [268], Winnicki [269], Sankaranarayanan [259]), Badalbaev and Mukhitdinov [218], Guttorp [219], Dion [220], N. Yanev [187].

In Bulgaria some problems of the theory of branching processes were first considered in the book of Obreshkov [1]. Further a chapter in the book of Obretenov [6] gave short representation of some classical models. In fact the first Bulgarian articles on branching processes appeared in 1972. In 1985, N.Yanev gave the invited talk "Branching Stochastic Structures" [56] on the 14th Spring Conference of the Union of Bulgarian Mathematicians. The talk represented the results of Bulgarian mathematicians in the area of branching processes by that time.

During the last 25 years the Bulgarian group working in the area of branching processes attracted new members and published about 150 papers, books, chapters of books, invited talks, application works, dissertations, etc. The main goal of this article is to present the results obtained during the last two decades. Since the main directions of the studies are the same as before 1985, we will repeat partially the structure and the results presented in [56]. Among others we would like to point out the book of Yakovlev and N. Yanev [81] with applications in the field of Biology and Medicine, the book for students with classical (and some modern) models of branching processes published recently by Slavtchova-Bojkova and N. Yanev [177] and the review chapters of Mitov and N. Yanev [186], N. Yanev [187] and G. Yanev [188]. It is worth to note that twelve dissertations in the field of branching processes are defended and among them two for Dr. Math. Sci.

Finally, we would like to point out that the First World Congress on Branching Processes was organized in Varna in 1993 by the Bulgarian branching team. The chairman of the Program Committee was C.C. Heyde and the chairman of the Organizing Committee was N. Yanev. The talks are published in [238] and three of them are Bulgarian. A lot of Bulgarian papers are presented as invited talks at some international conferences and among them we would like to note the World Congresses of ISI and Bernoulli Society (Istanbul, 1996), Classical and Modern Branching Processes (Minneapolis, 1994) and Conference on Branching Processes (Oberwolfach, 1995).

2. Controlled branching processes

2.1. Bienaymé-Galton-Watson branching processes

Assume that on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is given the set of independent identically distributed (i.i.d.) integer valued random variables $\xi = \{\xi_i(t), i,$

$t = 0, 1, 2, \dots\}$ with probability generating function $F(s) = \mathbf{E}\{s^{\xi_i(t)}\}$. Then the classical Bienaymé-Galton-Watson (BGW) branching process is well defined by the recurrence:

$$(1) \qquad Z_0 = 1, \ a.s., \quad Z_{t+1} = \sum_{i=1}^{Z_t} \xi_i(t+1), \quad t = 0, 1, 2, \dots .$$

Usually $\xi_i(t+1)$ is interpreted as the number of particles in the $(t+1)-$th generation which were born by i-th particle living in the $t-$th generation. The independence of ξ's means the independence of the evolution of the particles. This restrictive assumption is satisfied in many real situations in physics, but in biology it is not always satisfied. More realistic models should assume certain kind of dependence. A possible way is the model of ϕ-controlled branching processes introduced by Sevastyanov and Zubkov [266].

2.2. General model of controlled branching processes

Assume that on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ are given two sets of i.i.d. integer valued random variables: the set of the reproduction $\xi = \{\xi_{jk}(t), j \in J, k = 1, 2, \dots, \ t = 1, 2, \dots\}$ and the set of control functions $\phi = \{\phi_{jt}(n), j \in J, t = 1, 2, \dots, \ n = 0, 1, 2, \dots\}$, where J is a certain set of indexes, either finite or infinite.

Then we define the controlled branching process as follows:

$$(2) \qquad Z_0 \geq 0, \quad Z_{t+1} = \sum_{j \in J} \sum_{k=1}^{\phi_{jt}(Z_t)} \xi_{jk}(t+1), \quad t = 0, 1, 2, \dots .$$

Evidently, if $J = \{1\}$ and $\phi_{1t}(n) \equiv n$ a.s. then we have the BGW process defined in (1).

Sevastyanov and Zubkov [266] found the conditions for extinction and non-extinction in the case $J = \{1\}$, $\phi_{1t}(n) \equiv \phi(n)$ a.s., where $\phi(.)$ is a non-random integer valued function.

The first result for ϕ-controlled branching processes with random control functions was proved by N. Yanev in [7]. He considered the case when $J = \{1\}$ and

$$\phi = \{\phi_t = \{\phi_{t1}(0), \phi_{t1}(1), \dots, \phi_{t1}(n), \dots\}, \quad t = 0, 1, 2, \dots\}$$

are independent stochastically equivalent integer valued processes. More precisely he assumed that $\phi_t(n) \sim \alpha_t n, \ \ n \to \infty$ where α_t are i.i.d. random variables. The critical parameter in this case is $\rho = \mathbf{E}\{\log \alpha_t F'(1)\}$.

Under the same assumptions for the controlled functions N. Yanev and Mitov [16] considered the case

$$\mathbf{P}\{\xi_{1k}(t) > x\} \sim Cx^{-\alpha}, \quad x \to \infty, \quad 0 < \alpha < 1,$$

i.e. when the offspring distribution has infinite mean. In this situation they obtained the asymptotic behaviour of the probability for extinction. N. Yanev and G. Yanev [78] continued these investigations.

N. Yanev [10] considered the evolution of these processes in more general situation, namely when the evolution is in random environments. Further investigations of this case were done by N. Yanev and G. Yanev [84].

Branching processes with multitype random control functions were investigated by Del Puerto and N. Yanev in the subcritical case [152], [182] and [190]. They have also shown that the most popular discrete-time branching processes can be treated as particular cases: size-dependent branching processes, branching processes in random environments, etc.

In this way the problem for extinction or non-extinction of the ϕ-controlled branching processes were studied in rather general situation. On the other hand, it became clear that some other problems which are important for a given branching process could not be solved completely in this general situation. For these reasons further investigations have been done in some particular but still enough general cases.

2.3. Random migration

The processes with random migration are particular cases of ϕ-controlled branching processes with random control functions of the following types: $J = \{1, 2\}$ and

$$\phi_{1t}(n) = \max\{0, \min\{n, n + \eta_t\}\}, \quad \phi_{2t}(n) = \max\{\eta_t, 0\}.$$

They were introduced independently by Nagaev and Han [252] and N. Yanev and Mitov [15, 20]. The random variables η_t have distributions

$$\mathbf{P}\{\eta_t = -1\} = p_t, \quad \mathbf{P}\{\eta_t = 0\} = q_t, \quad \mathbf{P}\{\eta_t = 1\} = r_t, \quad p_t + q_t + r_t = 1, t \geq 0.$$

It is not difficult to check that the above definition can be represented in the following form

$$Z_{t+1} = \begin{cases} \sum_{i=1}^{\max\{Z_t-1,0\}} \xi_{1i}(t+1), & \text{with probability} \quad p_t \\ \sum_{i=1}^{Z_t} \xi_{1i}(t+1), & \text{with probability} \quad q_t \\ \sum_{i=1}^{Z_t} \xi_{1i}(t+1) + \xi_{21}(t+1), & \text{with probability} \quad r_t, \end{cases} \tag{3}$$

where we denote the p.g.f.'s $F(s) = \mathbf{E}\{s^{\xi_{1i}(t)}\}$ and $G(s) = \mathbf{E}\{s^{\xi_{2i}(t)}\}$, for $|s| \leq 1$.

The following well known models of branching processes can be obtained as particular cases of the processes with random migration:

- Bienaymé-Galtion-Watson processes $p_t = q_t = r_t = 0$.
- Bienaymé-Galtion-Watson processes with immigration $p_t = q_t = 0, \quad q_t = 1$. (See [237], [23]).
- Bienaymé-Galtion-Watson processes with emigration $p_t = 1, \quad q_t = r_t = 0$. (See [267]).
- Bienaymé-Galtion-Watson processes with immigration stopped at zero (The process $\tilde{Z}_t$ defined by (4) with $p_t = q_t = 0, \quad r_t = 1$ (see [265]).)

The time homogeneous case when $p_t = p,\ q_t = q,\ r_t = r$ was studied together with the process $\tilde{Z}_t,\ t \geq 0$, defined by

$$\tilde{Z}_0 > 0, \quad \tilde{Z}_t = \begin{cases} Z_t, & \text{if} \quad Z_t > 0, \\ 0, & \text{if} \quad Z_t = 0. \end{cases} \tag{4}$$

The state zero is an absorbing state for the process defined by (4). Hence the time to extinction τ of the process (4) is an important characteristic. It is the time between two consecutive visits of the state zero for the process defined by (3) which is called "life period" of this process. The asymptotic of the probability $u_t = \mathbf{P}\{\tilde{Z}_t > 0 | \tilde{Z}_0 > 0\} = \mathbf{P}\{\tau > t\}$, i.e. the survival probability of the "life period" is studied in all three cases: subcritical, critical and supercritical. Then it was used to prove limit theorems under appropriate normalization.

In the subcritical case it was shown that the probability u_t decreases exponentially. It was also found that there exists a discrete limiting distribution for the process Z_t. (See [42, 35]). The supercritical case was studied by Han [252].

The most interesting is the critical case. It was firstly investigated in [15, 252], [20], and [29].

Later G. Yanev and N. Yanev [106, 109] studied a rather more general situation presented briefly below (see also [140]).

Let $\xi = \{\xi_i(t), i = 1, 2, \dots, t = 0, 1, 2, \dots\}$ be a set of non-negative integer-valued i.i.d. random variables. Denote $U_t(j) = \sum_{i=1}^{j} \xi_i(t),\ j = 1, 2, \dots$ and $U_t(0) \equiv 0$. Let additionally two independent of ξ sets $E = \{e_1(t), e_2(t)\}$ and $I = \{I_t^+, I_t^o\}$ be given. Each of these sets consists of i.i.d. non-negative random variables. Then the branching process $\{Z_t, t = 0, 1, 2, \dots\}$ is defined by the following recurrent formula

$$Z_0 = 0,\ a.s. \quad Z_t = (U_t(Z_{t-1}) + M_t)^+,\ t = 1, 2, \dots, \tag{5}$$

where $\mathbf{P}\{M_t = -(U_t(e_1(t)) + e_2(t))\} = p, \mathbf{P}\{M_t = 0\} = q, \mathbf{P}\{M_t = I_t^+ \mathbb{I}_{\{Z_{t-1}>0\}} + I_t^o \mathbb{I}_{\{Z_{t-1}=0\}}\} = r;\ p, q, r \geq 0;\ p + q + r = 1$. Here and later on $\mathbb{I}_A$ denotes the indicator of the event A.

The definition (5) shows the following three options for the further population evolution in the t-th generation due to the migration component M_t :

(i) Emigration with probability p, that is $e_1(t)$ families emigrate which means that $U_t(e_1(t))$ members leave the population (family emigration) and, additionally, $e_2(t)$ members randomly selected from different families are also eliminated from the further evolution (individual emigration);

(ii) No migration with probability q, i.e. the reproduction is as in the classical BGW branching process;

(iii) State-dependent immigration with probability r, which means that I_t^+ new members join the population in the positive states or I_t^o members appear after the state zero.

The processes defined in (5) were studied under the following conditions

$$m = \mathbf{E}\{\xi_i(t)\} = 1 \ \ and \ \ 0 < Var\{\xi_i(t)\} = 2b < \infty, \ \ critical \ \ case, \tag{6}$$

$$\begin{cases} 0 < m_I^+ = \mathbf{E}\{I_t^+\} < \infty, \quad 0 < m_E = \mathbf{E}\{e_1(t) + e_2(t)\} < \infty, \\ 0 < m_I^o = \mathbf{E}\{I_t^o\} < \infty, \\ 0 \leq e_1(t) \leq N_1 < \infty, \quad 0 \leq e_2(t) \leq N_2 < \infty, \quad a.s. \end{cases} \tag{7}$$

The authors have shown that the following critical parameter

$$\Delta = \frac{\mathbf{E}\{M_t | Z_{t-1} > 0\}}{(1/2)Var\{\xi_k(t)\}} = \frac{r m_I^+ - p m_E}{b}$$

plays an important role for the behavior of the process. Under the above conditions they proved that:

- $u_t = \mathbf{P}\{Z_t > 0\} = \mathbf{P}\{\tau > t\} \sim L_\Delta(t) t^{-(1-\Delta)^+}, \quad t \to \infty$, where $L_\Delta(t)$ is a s.v.f.
- If the conditions (6) and (7) hold then:

  (i) If $\Delta < 0$ then the process possesses a stationary distribution.

  (ii) If $\Delta = 0$ then $\lim_{t\to\infty} \mathbf{P}\left\{\frac{\log Z(t)}{\log t} \le x\right\} = x, \quad 0 < x < 1.$

  (iii) If $\Delta > 0$ then $\lim_{t\to\infty} \mathbf{P}\left\{\frac{Z(t)}{bt} \le x\right\} = \frac{1}{\Gamma(\Delta)} \int_0^x y^{\Delta-1} e^{-y} dy, \quad x \ge 0.$

The process $\tilde{Z}_t$ with immigration stopped at zero (see (4)) was studied also in all three cases: the subcritical case in [57]; the critical case in [61], [86], [111]; and the supercritical case in [63]

2.4. Nonhomogeneous Migration

The processes defined by (3) with non-homogeneous migration are better models for some real phenomena. They are also interesting from theoretical viewpoint. The investigation of these processes was initiated by N. Yanev and Mitov in [22] and continued in the papers [38, 40, 37, 53, 55] of the same authors. The most interesting results are obtained in the critical case under the assumption $q_t \to 1, (p_t, q_t \to 0)$ as $t \to \infty$. This means that the process becomes closer to the critical process without any migration as $t \to \infty$. The limiting distributions depend on the balance between decreasing emigration and immigration. It is worth representing some of the main results.

A. Predominated immigration.

If $r_t \sim \frac{C}{\log t}, p_t = o(r_t), t \to \infty$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{1 - \frac{\log Z_t}{\log t} \le x | Z_t > 0\right\} = \frac{1 - e^{-\theta x}}{1 - e^{-\theta}}$$

for $0 < x < 1$. Here $\theta = C\lambda/b, \lambda = G'(1), b = F''(1)/2$.

If $r_t \sim \frac{L(t)}{\log t}$, $p_t \sim \frac{C}{\log t}$, and $L(t)$ is a s.v.f. such that $L(t) \to \infty, t \to \infty$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{L(t)\left(1 - \frac{\log Z_t}{\log t}\right) \le x | Z_t > 0\right\} = 1 - e^{-x/b}$$

for $x \geq 0$.

B. Balanced migration.

Suppose that $r_t\lambda \equiv p_t \sim t^{-\rho}L(t) \to 0, \quad t \to 0$ for some s.v.f.

If $\rho = 0$ and $L(t) \sim \dfrac{K}{\log t}$, and $\alpha = \dfrac{K}{b} \in (0,1)$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\log Z_t}{\log t} \leq x\right\} = \frac{1}{1+\alpha} + \frac{\alpha x}{1+\alpha}, \quad 0 \leq x \leq 1.$$

The restriction $\alpha \in (0,1)$ was removed lately by Drmota, Louchard, and N. Yanev [164].

If $0 < \rho < 1$ or ($\rho = 0$ and $L(t) = o(1/\log t), \quad t \to \infty$) then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\log Z_t}{\log t} \leq x | Z_t > 0\right\} = \frac{1}{1+\alpha} + \frac{\alpha x}{1+\alpha}, \quad 0 \leq x \leq 1.$$

If $\rho = 1$, $M(t) = \displaystyle\sum_{k=0}^{t} p_k \to \infty$, and $\displaystyle\lim_{t\to\infty} \frac{L(t)\log t}{M(t)} = \gamma, \quad 0 \leq \gamma \leq \infty$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\log Z_t}{\log t} \leq x | Z_t > 0\right\} = \frac{\gamma x}{1+\gamma}, \quad 0 \leq x \leq 1$$

and

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{Z_t}{bt} \leq x | Z_t > 0\right\} = \frac{\gamma}{1+\gamma} + \frac{1-e^{-x}}{1+\gamma}, \quad x > 0.$$

If $\displaystyle\sum_{k=0}^{\infty} p_k < \infty$ then $\mathbf{P}\{Z_t > 0\} \sim \dfrac{C}{bt}, \quad t \to \infty$, and

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{Z_t}{bt} \leq x | Z_t > 0\right\} = 1 - e^{-x}, \quad x \geq 0.$$

3. Regenerative branching processes

In the seventies Foster [227] and Pakes [255] introduced BGW processes with state-dependent immigration. The processes of this type allow immigration of new particles only in state zero, i.e., the population regenerates when it becomes extinct. Thus, the state zero is not an absorbing state anymore and becomes a reflecting barrier. The process evolution consists of a sequence of cycles which are independent and stochastically equivalent.

Later the model was generalized for Markov branching processes with continuous time by Yamazato [271] and for Bellman-Harris branching processes by Mitov and N. Yanev [50], [80] and Slavchova and N. Yanev [83], [88].

In this section we represent the results, obtained by Bulgarian researchers in this direction. Evidently the regenerative property also arise in the evolution of branching processes with homogeneous migration which were represented in the previous section.

On the other hand the BGW branching processes with state-dependent immigration can be described in terms of ϕ-controlled branching processes (see definition (2)), with $J = \{1, 2\}$ and $\phi_{1t}(n) = n$, $\phi_{2t}(n) = \max\{1 - n, 0\}$, $t = 1, 2, \ldots,$ $n = 0, 1, 2, \ldots$. Then $Z_{t+1} = \sum_{i=1}^{Z_t} \xi_{1i}(t+1) + \xi_{21}(t+1)\mathbb{I}_{\{Z_t=0\}}$, where we denote the p.g.f.'s $F(s) = \mathbf{E}\{s^{\xi_{1i}(t)}\}$ and $G_t(s) = \mathbf{E}\{s^{\xi_{2i}(t)}\}$, for $|s| \le 1$.

3.1. BGW processes with state-dependent immigration

Foster [227] and Pakes [255] considered the case of time homogeneous immigration in the state zero, i.e. $G_t(s)$ does not depend on t. The BGW processes with non-homogeneous state-dependent immigration were studied in [31], [43], [32], [33], [73], [116], [118], [129] mainly in critical case.

3.2. Continuous time Markov processes

Later the Foster-Pakes model was generalized by Yamazato [271] for continuous time Markov branching processes. Further these processes were studied assuming that the immigration in the state zero is not homogeneous in time. (See [35], [39], [59].) The process was defined as the time inhomogeneous Markov chain $\{Z_t, t \ge 0\}$ with transition probabilities given by

$$P_{ij}(t, t+h) = \mathbf{P}\{Z_{t+h} = j | Z_t = i\}$$
$$= \begin{cases} \delta_{0i} + q_i(t)h + o(h), & i = 0, j \ge 0, \\ \delta_{ij} + ip_{j+1-i}h + o(h), & 1 \le i \le j+1, \\ o(h), & 0 \le j < i - 1. \end{cases} \quad \text{as } h \to 0,$$

where $q_0(t) < 0$, $q_j(t) \ge 0$, $j \ge 1$, $\sum_{j=0}^{\infty} q_j(t) = 0$, $t \ge 0$, $p_1 < 0$, $p_j \ge 0, j \ne 1$, $\sum_{j=0}^{\infty} p_j = 0$. For simplicity we assume $Z_0 = 0$, $a.s.$

Assume that

$$\sum_{j=1}^{\infty} jp_j = 0, \quad 0 < 2b = \sum_{j=1}^{\infty} j(j-1)p_j < \infty, \quad (critical \quad case), \tag{8}$$

$$0 < m_I(t) = \sum_{j=1}^{\infty} jq_j(t) \to 0, \quad c_I(t) = \sum_{j=2}^{\infty} j(j-1)q_j(t) \to 0, \tag{9}$$

$$\sup_{t\geq 0} m_I(t) < \infty, \qquad \sup_{t\geq 0} c_I(t) < \infty. \tag{10}$$

Depending on the rate of the convergence in (9) the limiting behavior of the process essentially differs. (See e.g. [39]).

Assume (8), (9), (10), and $m_I(t) \sim L(t)t^{-\theta}$, $c_I(t) = o(m_I(t)\log t)$, $t \to \infty$, where $0 \leq \theta < 1$, and $L(.)$ is a function slowly varying at infinity. Then

(a) If $0 < \theta < 1$ or $\{\theta = 0$ and $0 \leq \lim_{t\to\infty} b^{-1}L(t)\log t < 1\}$ then

$$\lim_{t\to\infty} \mathbf{P}\{\log Z_t/\log t \leq x | Z_t > 0\} = x, \quad x \in (0,1).$$

(b) If $\theta = 1$ and $\lim_{t\to\infty}(L(t)\log t)\left(\int_0^t m_I(u)du\right)^{-1} = a, \quad 0 \leq a \leq \infty$, then

(i) for $0 < x < 1$

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\log Z_t}{\log t} \leq x | Z_t > 0\right\} = \frac{ax}{1+a};$$

(ii) for $x \geq 0$

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{Z_t}{bt} \leq x | Z_t > 0\right\} = \frac{a}{1+a} + \frac{1-e^{-x}}{1+a}.$$

(c) If $\int_0^\infty m_I(t)dt < \infty$ and $c(t) = o(t^{-1})$, $t \to \infty$, then for $x > 0$

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{Z_t}{bt} \leq x | Z_t > 0\right\} = 1 - e^{-x}.$$

3.3. Age-dependent branching processes

We have already mentioned that the Foster-Pakes model was generalized to the Bellman-Harris branching processes in [50], [80] for the critical case and in [83], [88] for the noncritical cases. The general definition is as follows:

Let on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, three independent sets of random variables be given:

(i) The set $Z = \{\{Z_t(k,j), t \geq 0\}, \quad k,j = 1,2,\ldots.\}$ of i.i.d. branching processes with p.g.f. $F(t;s) = \mathbf{E}\{s^{Z_t(k,j)}\}, \quad |s| \leq 1$, and $Z_0(k,j) = 1$ a.s., i.e. $F(0;s) = s$.

(ii) The set $I = \{I_k, \ k = 1,2,\ldots\}$ of i.i.d. integer valued positive random variables with p.g.f. $g(s) = \mathbf{E}\{s^{I_k}\}, \quad |s| \leq 1$ (the number of immigrants in the state zero).

(iii) The set $X = \{X_k, \ k = 1,2,\ldots\}$ of i.i.d., positive r.v. with cumulative distribution function $A(t) = \mathbf{P}\{X_k \leq t\}, \ t \geq 0$ (stay at zero or waiting period).

Define the sequence $\{\{Z_t(k), t \geq 0\}, \ k = 1,2,\ldots\}$ of i.i.d. branching processes starting with a positive random number of ancestors I_k at time $t = 0$, as follows

$$Z_t(k) = \sum_{j=1}^{I_k} Z_t(k,j), \quad t \geq 0.$$

By the independence of the sets Z and I it follows that $\Phi(t,s) := \mathbf{E}\{s^{Z_t(k)}\} = g(F(t,s)), \quad |s| \leq 1$. Denote by T_k the life period (time to extinction) of the process $Z_t(k)$, i.e. T_k is a r.v. such that

$$Z_0(k) = I_k > 0, \ Z_t(k) > 0 \text{ for } t \in [0, T_k), \text{ and } Z_{T_k}(k) = 0.$$

It is clear that the r.v. $\{T_k, \ k = 1,2,\ldots\}$ are i.i.d. and the equivalence of the events $\{Z_t(k) = 0\}$ and $\{T_k \leq t\}$ yields $B(t) = \mathbf{P}\{T_k \leq t\} = \Phi(t,0) = g(F(t,0))$. Consider the sequence $Y_k = X_k + T_k, \quad k = 1,2,\ldots,$ of i.i.d. positive r.v. and denote their c.d.f. by

$$D(t) = \mathbf{P}\{Y_k \leq t\} = \int_0^t A(t-u)dB(u) = (A * B)(t).$$

Define the renewal epochs $S_0 = 0, \quad S_{n+1} = S_n + Y_{n+1}, \quad n = 0,1,2,\ldots$ and the ordinary renewal process $N(t) = \max\{n : S_n \leq t\}$.

The *alternating renewal sequence* $\{(S_n, \ S'_{n+1}), \ n = 0,1,2,\ldots\}$, where $S'_{n+1} = S_n + X_{n+1}, \quad n = 0,1,2,\ldots,$ allows us to define the process $\sigma(t)$, related with the moment t, as follows

$$\sigma(t) = t - S'_{N(t)+1} = t - (S_{N(t)} + X_{N(t)+1}).$$

If $\sigma(t) \geq 0$ then it can be interpreted as the "spent lifetime." If $\sigma(t) < 0$ then $-\sigma(t) > 0$ is the rest of the "waiting period" (see Figure 1).

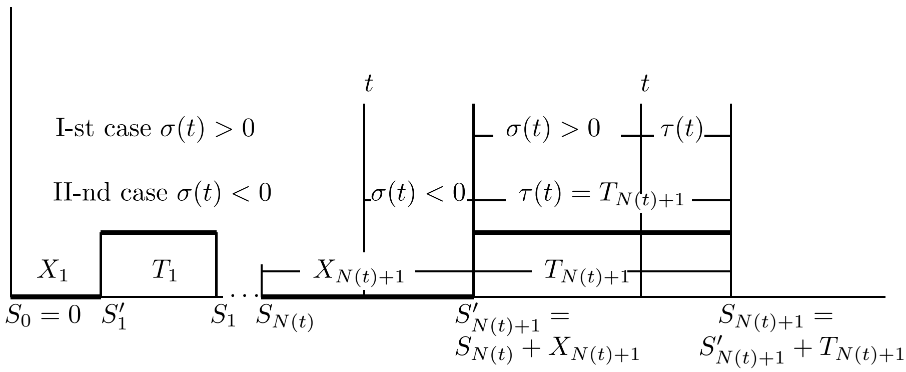

Figure 1: Spend and residual life times

Finally, we define the branching process with state-dependent immigration Z_t, $t \geq 0$, as follows

$$Z_t = Z_{\sigma(t)}(N(t)+1)\mathbb{I}_{\{\sigma(t)\geq 0\}}.$$

The process $\{Z_t, t \geq 0\}$ can be interpreted in the following way (see Figure 2).

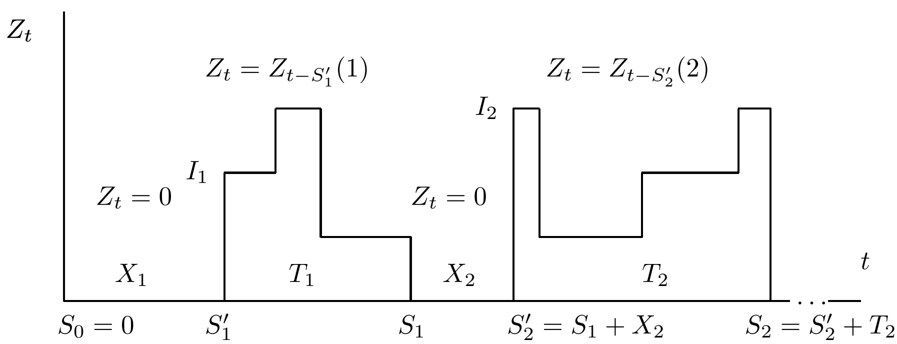

Figure 2: Branching process with state-dependent immigration

It starts at the moment $t = 0$ by $Z_0 = 0$ and stays at the state 0 during the random period $S'_1 = X_1$. Then at the moment S'_1, I_1 new particles ($I_1 > 0, a.s.$) at age 0 immigrate. They initiate the branching process $Z_{t-S'_1}(1)$ and Z_t coincides with this process up to the moment $S_1 = X_1 + T_1$ when $Z_{t-X_1}(1)$ hits the state zero. Commonly, during the waiting periods $[S_{n-1}, S'_n)$ the process Z_t is equal to zero, while in the life periods $[S'_n, S_n)$ the process Z_t coincides with the corresponding branching process starting with the positive random number I_n of new particles, which are the immigrants in the state 0.

This construction was generalized further, rejecting the assumption that the processes $\{\{Z_t(k), t \geq 0\},\ k = 1, 2, \ldots\}$ are branching, but any nonnegative excursion processes. A detailed investigation of these regenerative processes was done in [132], [130], [131]. The general results were used later for additional investigation of critical age-dependent regenerative processes in [135] and [146].

Some of the results were extended to the case of multi-type age-dependent branching processes with state dependent immigration in [108], [110].

Another class of alternating regenerative branching processes was studied by Mayster [158], [181]. The main idea in both papers is to control a branching process by another branching process. The model considered in [158] can be described starting from the Sevastyanov and Zubkov's model of ϕ-controlled branching process,

$$Z_{t+1} = \sum_{i=1}^{\phi_t(Z_t)} \xi_i(t+1), t = 0, 1, 2, \ldots .$$

The control function is defined as follows $\phi_t(Z_t) := \theta \otimes Z_t + \eta_t$, where the right hand side is the fractional thinning operator introduced by Steutel and van Harn as the "discrete multiplication". The author studied the extinction probability and proved limit theorems for reproduction by t cycles, as $t \to \infty$. In the second paper [181] the author considers the model of alternating branching process in autoregressive random environment.

Bojkova and N. Yanev [102], [105] studied processes with two types of immigration. The process of this type is a superposition of a sequence of state-dependent Bellman-Harris branching processes starting at the renewal epochs of an independent renewal process. The model was studied later in [110, 136], [144], [156].

N. Yanev [3], [4] studied the more general Sevastyanov model of age-dependent branching processes with general immigration. He obtained the asymptotic of the moments and limiting distributions in all three cases: subcritical, critical, and supercritical.

4. Branching diffusion processes

Let us consider diffusion process $X_t, t \geq 0$ in the phase-space $(\mathfrak{X}, \mathcal{A})$ with the infinitesimal operator

$$\mathcal{L}_x = \frac{1}{2} \sum_{i,j=1}^{d} a^{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x_i}.$$

Suppose that particles move on the sample paths of the diffusion process, and the intensity reproduction of a particle which is at the point x is $k(x)\Delta t + o(\Delta t)$, and the number of particles are determined by the random measure η_x. Assume that we know the initial distribution of particles in the space $\mathfrak{X}$. The processes of this type are known as diffusion branching processes and they were studied by Mayster in a series of papers [5], [11], [12], [24], [25], [26], [34], [45], [46].

In the paper [5] the author proved the asymptotic of mathematical expectation and limiting distributions for branching diffusion processes with discrete time, which evolve in a bounded region with absorbing bounds. The asymptotic of the first two factorial moments for branching diffusion processes with continuous time, which evolve in a bounded region with absorbing bounds were studied in the papers [11], [12].

The paper [24] concerns branching diffusion processes in an unbounded domain. The author found sufficient conditions for the existence of a maximal eigenvalue μ_0 in the specter of the operator $M_t f(x) = \mathbf{E}\left\{\int_{\mathfrak{X}} f(z)\mu_{xt}(dz)\right\}$, which plays role of a critical parameter. She also proved limit theorems:

- In the subcritical case ($\mu_0 < 0$) the finite dimensional distributions of the measure $\mu_{xt}(u)$ on the set $\{\mu_{xt}(u) > 0\}$ converge to the finite dimensional distributions of the given measure μ^*;

- In the critical case ($\mu_0 = 0$), under an appropriate normalization it is proved the convergence to multidimensional exponential distribution;

- In the supercritical case ($\mu_0 > 0$) it is proved that for every bounded measurable function $\nu(x)$ reduced process

$$\int_{\mathfrak{X}} \nu(z)\mu_{xt}(dz)/\omega_0^*(\nu)e^{\mu_0 t}$$

  converges in mean-square to a random variable independent of $\nu(x)$.

The results obtained in the paper [24] are used later for investigation of a harmonic oscillator in [25]. The paper [26] deals with diffusion branching processes with Poisson initial distribution. Branching random walk on a closed interval and its convergence to a diffusion branching process is investigated in the paper [45]. Some results for the mathematical expectation of the branching processes with a small diffusion are obtained in the paper [46]. It is proved that under certain conditions the mathematical expectation is close to running wave. Several particular cases are considered in details.

5. Spatial branching processes

Let us assume that the particles which were born in a given generation are distributed in the space $\mathbf{R}^d$ accordingly with the given locally-bounded random measure, independently of the position of the mother-particle, i.e. the measure is space-homogeneous (see e.g. [245]). Some results in this direction are proved in the papers of Tschobanov [18], [48], and [49]. In the paper [18], the author considered critical space-homogeneous Markov branching semi-groups with continuous time. He proved a sufficient condition for weak continuity and the necessary and sufficient condition for stability of such a semi-group. He also proved the weak convergence to the invariant distribution independently of the initial one. In the paper [48] the author generalized the space-homogeneous Bellman-Harris branching processes with finite number of types of particles, introduced in [245] to the case where the set of the types of particles constitutes a bounded complete metric space. In the third paper [49] the author solved two important problems for the space-homogeneous Bellman-Harris branching processes: the full description of their invariants is given (Theorem 4.1); the convergence of distributions of infinite population of particles to the invariant distribution is proved (Theorem 4.3).

In the paper [178] the author studied the Sevastyanov model with a motion of particles. The model is defined by:

- the motion process $X(t), t > 0$, which is a time homogeneous Markov process on $\mathbf{R}^d$;
- the life-span $L(\tau), \tau > 0$, which is a Levy process called subordinator: a right continuous increasing stochastic process on $\mathbf{R}^+$ having stationary independent and positive increments with initial point being the origin;
- the offspring number $\eta(u), u > 0$, which is an integer value measure, in general, depending on the age u of the parent-particle at the splitting time.

Now, for a fixed τ, the couple $(L(\tau), \eta(L(\tau))$ gives the Sevastyanov branching mechanism. If the offspring number η does not depend on the age u, we have the Bellman-Harris branching process. The life-span $L(.)$ and the motion process $X(.)$ are assumed to be independent.

6. Extremes in branching processes

There is a significant amount of research in the theory of branching processes devoted to extreme value problems concerning different population characteristics. The history of such studies goes back to the works of Zolotarev and Urbanik who consider the maximum generation size. Another direction of study concerns the maxima related to the offspring size.

Let M_n denote the maximum offspring size of all individuals living in the $(n-1)$-st generation of a branching process. This is a maximum of random number of independent and identically distributed (i.i.d.) integer-valued random variables, where the random index is the population size of the process. M_n has two characteristic features: (i) the i.i.d. random variables are integer-valued and (ii) the distribution of the random index is connected to the distribution of the terms involved through the branching mechanism. These two characteristics distinguish the subject matter maxima among those studied in the general extreme value theory.

The study of the sequence $\{M_n\}$ might be motivated in different ways. It provides a fertility measure characterizing the most prolific individual in one generation. It measures the maximum litter (or family) size. In the branching tree context, it is the maximum degree of a vertex. The asymptotic behavior of M_n gives us some information about the influence of the largest families on the size and survival of the entire population.

Bulgarian mathematicians also contributed to this topic. First of all we have to point out the papers of Rahimov and G.Yanev [115], [122] on the maximum family size of BGW processes. These two works initiate some further studies of Bulgarian mathematicians in this direction. The papers [116], [118], [137], [139], [117] concern with different characteristics of maxima related to the offspring size for different classes of branching processes.

The review papers [172] and [188] by G. Yanev give a comprehensive up to date overview of the results in these area of branching processes.

7. Other models

An important particular case of branching processes are the birth and death processes. In the book of Obretenov [6] it was considered non-homogeneous discrete time birth and death process with reflecting barrier at zero and absorbing barrier at $N < \infty$.

Some results for Bienaymé-Galton-Watson branching processes with increasing random number of ancestors are obtained by Dimitrov in his PhD thesis [194]. The author considered the Bienaymé-Galton-Watson branching process $Z_t, t = 0, 1, 2, \ldots$ (see (1)) with $Z_0 = \nu_n \stackrel{P}{\to} \infty,\ \ n \to \infty$, where $\lim_{n\to\infty} \mathbf{P}\{\nu_n/k_n \leq x\} = A(x),\ A(0) \neq 1$, for some increasing sequence $k_n \to \infty,\ n \to \infty$. Then in the supercritical case $m = F'(1) > 1$,

$$\frac{Z_t - m^t \nu_t}{m^t \sigma \sqrt{\nu_t (1 - 1/m)}} \xrightarrow{d} N(0, (1 - A(0))^{-1}),$$

where $\sigma^2 = F''(1) + m - m^2$ is the offspring variance. In the subcritical case $m < 1$ the author proved that under the additional condition $k_t \mathbf{P}\{Z_t > 0 | Z_0 = 1\} = \gamma, \quad 0 < \gamma < \infty$ there exists a stationary distribution on the non-degenerated sample paths. The cases when $\gamma \to 0$ and $\gamma \to \infty$ were also considered. In the critical case $m = 1$ it was found the limiting distribution with characteristic function

$$h_0(z) = \frac{\int_0^\infty \left[e^{\gamma x i z (1 - iz)} - e^{-\gamma x} \right] dA(x)}{1 - \int_0^\infty e^{-\gamma x} dA(x)}.$$

An interesting investigation of the family tree of a Bienaymé-Galton-Watson branching process is given by G. Yanev and Mutafchiev[163] and Mutafchiev [189]. Especially Mutafchiev [189] obtained the asymptotic behaviour as $t \to \infty$ of the process $V_{N,t}$ representing the count of the complete disjoint $N-$ary subtrees of height at least t, which are rooted at the ancestor.

Two interesting papers were written by Kerbashev [104], [117]. In these papers the author studied the maximum M of the critical Bienaymé-Galton-Watson branching process conditioned on its total progeny N. A limit theorem for M as $N \to \infty$ is proved, imbedding the process in a random walk. The results are transferred to the non-critical processes. A corollary for the maximal strata of a random rooted labeled tree is obtained. The results refined a result of Kolchin.

8. Statistics of branching processes

It is well known that consistent and asymptotically normal estimators for offspring characteristics of the classical Bienaymé-Galton-Watson process exist only in the supercritical case on the explosion set. This restriction can be avoided introducing an increasing random number of ancestors or an immigration component. In the first case an unified estimation theory is proposed first by N. Yanev [8] for subcritical, critical and supercritical processes. After that this idea is developed in a series of papers by N. Yanev and also Dion and N. Yanev for different sample schemes. It is shown also how to transfer the asymptotic results of the BGW processes with an increasing random number of ancestors to the BGW processes with immigration designing a suitable sampling scheme.

8.1. Classical BGW branching process

Let us begin with the classical BGW branching processes defined in Section 2.1. with the offspring distribution $p_k = \mathbf{P}\{\xi = k\}$, $\sum_{k=0}^{\infty} p_k = 1$, and the offspring mean and variance $m = \mathbf{E}\{\xi\}$, $\sigma^2 = Var\{\xi\}$.

The estimates for the offspring mean, variance, and distribution depend on the observation scheme. Further on the following **observation schemes** will be considered:

(O1) *The entire tree up to the moment t:* $\{\xi_i(k),\ i = 1, 2, \dots, Z_k,\ k = 0, 1, \dots, t\}$;

(O2) *The successive generations:* $(Z_0, Z_1, Z_2, \dots, Z_t)$;

(O3) *Two successive generations:* (Z_t, Z_{t+1});

(O4) *The initial and another generation:* (Z_0, Z_t);

(O5) *Left censored observations:* $(Z_t, Z_{t+1}, \dots, Z_{t+T})$.

Assume now the observation scheme (O1). This is the full information for the process, by which it is easy to determine the statistics $Z_j(k) = \sum_{i=1}^{Z_j} \mathbb{I}_{\{\xi_i(j)=k\}}$,which means that $Z_j(k)$ is the number of individuals in the j-th generation having exactly k offsprings in the next generation, $j = 0, 1, 2, \dots, t$; $k = 0, 1, 2, \dots$. Then using the independence of individual's evolutions we are able to obtain the maximum likelihood function (m.l.f.) $L_t(p) = \prod_{k=0}^{\infty} p_k^{\sum_{j=0}^{t-1} Z_j(k)}$,where the parameter $p = (p_0, p_1, p_2, \dots)$ is just the offspring distribution. Therefore the maximum likelihood estimators (m.l.e.) for $\{p_k\}$ are $\widehat{p}_k(t) = U_t(k)/U_t, k = 0, 1, 2, \dots,$ where $U_t(k) = \sum_{j=0}^{t-1} Z_j(k)$ is the total number of individuals in the first t generations having exactly k offsprings and $U_t = \sum_{k=0}^{\infty} U_t(k) = \sum_{i=0}^{t-1} Z_i$ is the total number of individuals up to the moment t (i.e. from 0 to $t-1$ including). Note that $\widehat{p}_k(t)$ is the relative proportion of the parents with k offsprings. Then the m.l.e. for m will be $\widehat{m}_t = \sum_{k=1}^{\infty} k\widehat{p}_k(t)$. It is not difficult to calculate that $\widehat{m}_t = (U_{t+1} - Z_0)/U_t = \dfrac{Z_1 + Z_2 + \dots + Z_t}{Z_0 + Z_1 + \dots + Z_{t-1}}$. Note first that the m.l.e. $\widehat{m}_t$ can be interpreted as a ratio of the "total number of the sons" divided by the "total number of the fathers" (which seems quite naturally). On the other hand, $\widehat{m}_t$ depends surprisingly only by the successive generations (O2). Remark that these m.l.e. were proposed by Harris [230]. It was shown in Dion [221], Feigin [226], and Keiding and Lauritzen [244] that $\widehat{m}_t$ is also a m.l.e. for m with respect only to observations (O2). Similarly it is possible to consider the m.l.e. for the individual variance $\sigma^2 : \sigma_t^2 = \sum_{k=0}^{\infty}(k-m)^2\widehat{p}_k(t)$ with known offspring mean m and $\widehat{\sigma}_t^2 = \sum_{k=0}^{\infty}(k-\widehat{m}_t)^2\widehat{p}_k(t)$ with unknown m, which cannot be presented in a closed form and depend on the entire information (O1).

Let us first consider the most important estimator $\widehat{m}_t$ for which Harris [230] proved that on the explosion set it is a consistent estimator for the critical parameter m. Later Heyde [233] improved this result showing that the Harris estimator is strongly consistent and Dion [221] obtain that it is also asymptotically normal.

Assume now the observation scheme (O3). Lotka introduced the estimator $\widetilde{m}_t = (Z_{t+1}/Z_t)\mathbb{I}_{\{Z_t>0\}} + \mathbb{I}_{\{Z_t=0\}}$. This estimator was investigated by many authors and asymptotic results on the non-extinction set are obtained. Pakes [254] introduced and investigated the corresponding estimators for the offspring probabilities: $\widetilde{p}_t(k) = (Z_t(k)/Z_t)\mathbb{I}_{\{Z_t>0\}} + \mathbb{I}_{\{Z_t=0\}}, \quad k = 0, 1, 2, \ldots$.

Suppose now that we can observe only the sample (O4). In this case Heyde [235] suggested the moment-type estimator $m_t^* = Z_t^{1/t}$, having in mind that $\mathbf{E}\{Z_t\} = m^t$. Note that the estimator m_t^* is consistent but not asymptotically normal distributed.

Another interesting problem is how to estimate the unknown age of the process if we are able to observe only the current size Z_t, where the parameter t is unknown and we have to estimate it. Stigler [258] proposed the following estimator $\widehat{t} = \{\log[Z_t(1-q)-q]\}/\log m$. Unfortunately the estimator is not consistent but in the next section we will propose a consistent and asymptotically normal estimator in the noncritical cases and an unbiased estimator in the critical case.

In the case of a sample (O5) Crump and Howe [215] introduce the following estimator for $m : \widehat{m}_{t,T} = \sum_{i=t+1}^{t+T} Z_i / \sum_{i=t}^{t+T-1} Z_i$, which can be considered as a version of the Harris estimator ($t = 0$) or the Lotka-Nagaev estimator ($T = 1$). Crump and Howe investigated the cases $t \to \infty$, $T-$ fixed and $t-$fixed, $T \to \infty$. They proved that $\widehat{m}_{t,T}$ is consistent and asymptotically unbiased estimator in the supercritical case $m > 1$ on the explosion set. In the next section we will consider also the case $t, T \to \infty$ in a more general situation and not only in the supercritical case.

Finally it is interesting to compare the different estimators. For example, the best estimator for m is the Harris estimator, which could be expected because it uses more information than the others (see [126] for more details). We will come back to this problem in the next sections. The estimation problems for σ^2 will be considered and discussed in the following section.

8.2. BGW branching processes with an increasing random number of ancestors (BGWR)

As one can see by the previous section, the statistical inference for the classical BGW processes is well developed only in the supercritical case on the explosion set. To avoid that restriction we will consider in this section the following generalization of (1):

$$Z_{t+1}(n) = \sum_{i=1}^{Z_t(n)} \xi_i(t,n), t = 0,1,2,\dots; n = 1,2,\dots, \tag{11}$$

where $\{\xi_i(t,n)\}$ are i.i.d. r.v. and $Z_0(n)$ is an independent of them r.v. (which goes to ∞ in some sense).

Note also that $Z_t(n) = \sum_{k=1}^{Z_0(n)} Z_t^{(k)}, t = 0,1,2,\dots$, where $\{Z_t^{(k)}\}$ are i.i.d. copies of the classical BGW process defined by (1).

As noted in Yakovlev and N. Yanev [81], branching processes with a large and often a random number of ancestors occur naturally in the study of cell proliferation. Such is also the case in applications to nuclear chain reactions.

In general n and t are free parameters but very often we will assume that $n = n(t) \to \infty$ as $t \to \infty$ (or *vice versa*). We will obtain consistent and asymptotically normal estimators for offspring probabilities, mean and variance in the entire range of $0 < m < \infty$.

In the case (11) we will use the same notions from the previous section adding everywhere the parameter n: $\widehat{p}_k(t,n)$, $\widehat{m}_t(n)$, $\sigma_t^2(n)$, $\widehat{\sigma}_t^2(n)$, $\widetilde{m}_t(n)$, $m_t^*(n)$, $\widehat{m}_{t,T}(n)$ and so on.

8.2.1. Estimating the mean in the case $Z_0(n) \equiv n$

The case $Z_0(n) \equiv n$ a.s. was first investigated in N. Yanev [8] and the asymptotic properties of $\widehat{m}_t(n)$ were studied. We will summarize the main results in the following theorems.

Theorem 8.1. [8] *If $m < \infty$ and $n \to \infty$ then uniformly in $0 \le t \le \infty$ the estimator $\widehat{m}_t(n)$ is strongly consistent and asymptotically unbiased, i.e. $\widehat{m}_t(n) \to m$ a.s. and $\mathbf{E}\{\widehat{m}_t(n)\} \to m$.*

Further on we will use the notation $c_t = \mathbf{E}\{U_t\} = \sum_{k=0}^{t-1} m^k$. We will put $X_k \sim N(a_k, b_k)$ for the asymptotic normality which means that $(X_k - a_k)/\sqrt{b_k} \xrightarrow{d} N(0,1)$ as $k \to \infty$.

Theorem 8.2. [8] *Let* $\sigma^2 < \infty$ *and* $t > 0$ *is fixed. Then*

$$\widehat{m}_t(n) \sim N(m, \sigma^2/nc_t), n \to \infty.$$

When n and t tend to ∞ simultaneously then the relation between them is essential for the asymptotic distribution as one can see by the following results.

Theorem 8.3. [8] *Let* $\sigma^2 < \infty$ *and* $n, t \to \infty$.
(i) If $m < 1$ *then* $\widehat{m}_t(n) \sim N(m, \sigma^2(1-m)/n)$;
(ii) If $m = 1$ *and* $n/t \to \infty$ *then* $\widehat{m}_t(n) \sim N(m, \sigma^2/nt)$;
(iii) If $1 < m < \infty$ *then* $\widehat{m}_t(n) \sim N(m, \sigma^2(m-1)/nm^t)$.

Comment 8.1. *By Theorem* 8.3 *if one has* $n/t \to \infty$ *as* $n, t \to \infty$, *then independently of the criticality of the process (i.e. only for* $m < \infty$*) the limiting distribution of* $\widehat{m}_t(n)$ *is normal (in fact the same as in Theorem* 8.2*). In the critical case the asymptotic behavior is more complicated as one can see by the following result.*

Theorem 8.4. [8] *Let* $m = 1, \sigma^2 < \infty$ *and* $n^2/t \to 0$ *as* $n, t \to \infty$. *Then* $(2n/\sigma^2)(1 - \widehat{m}_t(n)) \xrightarrow{d} 1/Y$, *where the r.v.* Y *as a stable distribution with parameter* 1/2, *i.e.* $\mathbf{E}\{e^{-\lambda Y}\} = \exp\{-\lambda^{1/2}\}$.

Remark 8.1. *For the critical processes the case* $O(n) \le t \le O(n^2), n \to \infty$, *is still an open problem.*

These investigations have been continued in N. Yanev [62] where random normalizing constants have been used.

8.2.2. Limiting distributions for BGW processes with an increasing random number of ancestors

Further on we will consider with more details the general case of random $Z_0(n)$ investigated in a series of papers of Dion and N. Yanev [85], [89], [91], [93], [97], [98], [101], [103], [107], [113]. First of of all we will show that the limiting behavior of the process is quite different from the classical BGW process. From now on we will suppose that ν is a positive r.v. with d.f. $F_\nu(x)$.

Theorem 8.5. *Let* $m < 1, \sigma^2 < \infty, Z_0(n)/n \xrightarrow{d} \nu$ *and* $n, t \to \infty$.

(*i*) *If* $nm^t \to 0$ *then* $Z_t(n) \xrightarrow{P} 0$;

(*ii*) *If* $nm^t \to C_1, 0 < C_1 < \infty$, *then* $Z_t(n) \xrightarrow{d} \eta_1$, *where* $\varphi_1(\lambda) = \mathbf{E}\{e^{-\lambda\eta_1}\} = \mathbf{E}\{\exp\{-\alpha_1\nu(1-g(e^{-\lambda}))\}\}, \quad \lambda > 0$, *and the p.g.f.* $g(s)$ *is the unique solution of the equation* $g(h(s)) = 1 - m(1-g(s))$ *and* $\alpha_1 = C_1/g'(1)$.

Theorem 8.6. *Let* $m = 1, \sigma^2 < \infty, Z_0(n)/n \xrightarrow{d} \nu$ *and* $n, t \to \infty$.

(*i*) *If* $n/t \to 0$ *then* $Z_t(n) \xrightarrow{P} 0$;

(*ii*) *If* $n/t \to C_2, 0 < C_2 < \infty$, *then* $Z_t(n)/n \xrightarrow{d} \eta_2$, *where* $\varphi_2(\lambda) = \mathbf{E}\{e^{-\lambda\eta_2}\} = \mathbf{E}\{\exp\{-\lambda\alpha_2\nu/(\alpha_2+\lambda)\}\}, \quad \lambda > 0$, *and* $\alpha_2 = 2C_2/\sigma^2$.

Remark 8.2. *If* $\nu = 1$ *a.s. then* $\varphi_2(\lambda)$ *is a Laplace transform of a compound Poisson distribution.*

Further on we will use the following condition:

$$(m > 1) \vee (m = 1, n/t \to \infty) \vee (m < 1, nm^t \to \infty), \tag{12}$$

which considers all cases not included in Theorems 8.5 and 8.6.

Theorem 8.7. *Assume condition* (12), $\sigma^2 < \infty$, $Z_0(n)/n \xrightarrow{d} \nu$ *and* $n, t \to \infty$.

(*i*) *For every* $n \geq 1$, $W(n) = \{W_t(n) = Z_t(n)/(Z_0(n)m^t), t \geq 1\}$ *is a martingale and* $W_t(n) \xrightarrow{P} 1$ *(LLN);*

(*ii*) $\sqrt{Z_0(n)A_t}(W_t(n) - 1) \xrightarrow{d} N(0,1)$ *(CLT),*

where $A_t = m^t \mathbb{I}_{\{m<1\}} + \dfrac{1}{\sigma^2 t}\mathbb{I}_{\{m=1\}} + \dfrac{m(m-1)}{\sigma^2}\mathbb{I}_{\{m>1\}}$.

8.2.3. Harris estimators

Theorem 8.8. *Let* $n, t \to \infty$.

(*i*) *If* $m < \infty$ *and* $Z_0(n) \xrightarrow{P} \infty$ *then* $\widehat{m}_t(n) \xrightarrow{P} m$ *and* $\widehat{p}_k(t,n) \xrightarrow{P} p_k$, $k = 0, 1, 2, \ldots$.

(*ii*) *If* $\sigma^2 < \infty$ *and* $Z_0(n)/n \xrightarrow{d} \nu$ *then* $\mathbf{E}\{\widehat{m}_t(n)\} \to m$.

Remark 8.3. *If* $Z_0(n) \xrightarrow{a.s.} \infty$ *then* $\widehat{m}_t(n) \xrightarrow{a.s.} m$ *and* $\widehat{p}_k(t,n) \xrightarrow{a.s.} p_k$ *for* $k = 0, 1, 2, \ldots$.

Let $\widehat{\theta}_t(n)$ denote anyone of the following statistics:
$(\widehat{p}_k(t,n) - p_k)/\sqrt{p_k(1-p_k)}, (\widehat{m}_t(n) - m)/\sigma$,
$(\sigma_t^2(n) - \sigma^2)/D$ or $(\widehat{\sigma}_t^2(n) - \sigma^2)/D$,
where $D = Var\{(\xi - m)^2\} = \mathbf{E}\{(\xi - m)^4\} - \sigma^4$.

Theorem 8.9. *Let $\sigma^2 < \infty$ and $Z_0(n)/n \xrightarrow{P} \nu$. If $n, t \to \infty$ (additionally $n/t \to \infty$ for $m = 1$) then $(\sqrt{U_t(n)}\widehat{\theta}_t(n), U_t(n)/nc_t) \xrightarrow{d} (N(0,1), \widetilde{\nu})$, where $N(0,1)$ and $\widetilde{\nu}$ are independent random variables and $\widetilde{\nu}$ is identically distributed as ν.*

Of course, in the case when $\theta_t(n)$ is an estimator for the variance σ^2 we need an additional condition $\mathbf{E}\{\xi^4\} < \infty$.

Corollary 8.1. *Under the conditions of Theorem 8.9*
(i) $\sqrt{U_t(n)}\,\widehat{\theta}_t(n) \xrightarrow{d} N(0,1)$;
(ii) $\sqrt{nc_t}\widehat{\theta}_t(n) \xrightarrow{d} N(0,1)/\sqrt{\widetilde{\nu}}$.

In the case $m > 1$ and $Z_0(n) \equiv n$ a.s. Duby and Rouault [216], [217] considered the joint asymptotic distribution of $(\widehat{m}_t(n), (\widehat{\sigma}_t^2(n))$ and obtained asymptotic normality of two independent r.v. with different norming factors. In the critical case the asymptotic behavior is more complicated and one can consider the following result as a generalization of Theorem 8.4.

Theorem 8.10. *Let $m = 1, \sigma^2 < \infty$, $Z_0(n)/n \xrightarrow{d} \nu$ and $n^2/t \to 0$ as $n, t \to \infty$. Then*
(i) $\sqrt{2U_t(n)/\sigma^2}(1 - \widehat{m}_t(n)) \xrightarrow{d} |N(0,1)|$;
(ii) $(2n/\sigma^2)(1 - \widehat{m}_t(n)) \xrightarrow{d} 1/X$, *where* $\mathbf{E}\{e^{-\lambda X}\} = \mathbf{E}\{\exp\{-\sqrt{\lambda\nu}\}\}$.

Note that if $\nu = 1$ a.s. then X has a stable distribution with parameter 1/2.

Remark 8.4. *For the critical processes ($m = 1$) the case $O(n) \leq t \leq O(n^2), n \to \infty$, is still an open problem.*

Comment 8.2. *Corollary (i) shows the asymptotic normality of $\widehat{\theta}_t(n)$ (under random norming) if one has $n/t \to \infty$ (as $n, t \to \infty$) unified for $0 < m < \infty$. This fact allows us to construct asymptotic confidence intervals (a.c.i.) and to verify hypothesis for the criticality.*

Let the level be $1-\alpha$ and $\gamma_{1-\alpha/2}$ be the upper $\alpha/2$ percentage point of the standard normal distribution: $\Phi(\gamma_{1-\alpha/2}) = 1-\alpha/2$. Then the a.c.i. for m and p_k $(k=0,1,2,\dots)$ are given as follows:

(i) $(\widehat{m}_t(n) \pm \sigma\gamma_{1-\alpha/2}/\sqrt{U_t(n)})$;

(ii) $(\{\widehat{p}_k(t,n)+\gamma^2_{1-\alpha/2}\}/\{1+\gamma^2_{1-\alpha/2}/U_t(n)\}$
$\pm\sqrt{\gamma^4_{1-\alpha/2}+4U_t(n)\gamma^2_{1-\alpha/2}\widehat{p}_k(t,n)(1-\widehat{p}_k(t,n))}/2\{U_t(n)+\gamma^2_{1-\alpha/2}\})$.

One can apply similarly Theorem 8.10 to obtain a.c.i. in the case $m=1$ and $n^2/t \to 0$.

8.2.4. Lotka-Nagaev estimator

Assume now the sample scheme (O3) and let $\widetilde{\theta}_t(n) = (\widetilde{m}_t(n)-m)/\sigma$.

Theorem 8.11. *Assume* $\sigma^2<\infty$, *condition* (12) *and* $n,t\to\infty$.

(i) If $Z_0(n)/n \xrightarrow{P} \nu$ *then* $(\sqrt{Z_t(n)}\widetilde{\theta}_t(n), Z_t(n)/nm^t) \xrightarrow{a.s.} (N(0,1),\widetilde{\nu})$ *and* $\sqrt{nm^t}\widetilde{\theta}_t(n) \xrightarrow{d} N(0,1)/\sqrt{\widetilde{\nu}}$, *where* $N(0,1)$ *and* $\widetilde{\nu}$ *are independent random variables and* $\widetilde{\nu}$ *is identically distributed as* ν.

(ii) If $Z_0(n)/n \xrightarrow{d} \nu$ *then* $\sqrt{Z_t(n)}\widetilde{\theta}_t(n) \xrightarrow{d} N(0,1)$ *and* $\sqrt{Z_0(n)m^t}\widetilde{\theta}_t(n) \xrightarrow{d} N(0,1)$.

8.2.5. Moment-type estimator of Heyde

Assume now the sample scheme (O4) and let $m^*_t(n) = [Z_t(n)/Z_0(n)]^{1/t}$.

Theorem 8.12. *Suppose* $Z_0(n)/n \xrightarrow{d} \nu$, $\sigma^2<\infty$ *and condition* (12) *with* $n,t\to\infty$. *Then*

(i) $m^*_t(n) \xrightarrow{P} m$;

(ii) $\dfrac{t}{n}\sqrt{Z_0(n)A_t}(m^*_t(n)-m) \xrightarrow{d} N(0,1)$.

Remark 8.5. *Note that in Theorem* 8.12 *we obtain consistency and asymptotic normality not only in the supercritical case but also in the critical and subcritical cases, while in the classical case the limiting distribution is not normal even it is only in the supercritical case.*

8.2.6. Estimating the unknown age of the process

The problem is how to determine the age of the process when only the size of the process is observed. In other words, we can observe the size $Z_t(n)$, where the current moment t is unknown and we would like to estimate it. Define the following estimators:

$$\widehat{t}(n) = \{\log(Z_t(n)/Z_0(n))\}/\log m \text{ if } m \neq 1; \tag{13}$$

$$\widehat{T}(n) = (Z_t(n) - Z_0(n))^2/Z_0(n)\sigma^2 \ if \ m = 1. \tag{14}$$

Theorem 8.13. *Let $\sigma^2 < \infty$, $Z_0(n)/n \xrightarrow{d} \nu$ and $n, t \to \infty$.*

(i) If $(m > 1) \vee (m < 1, nm^t \to \infty)$ then $\widehat{t}(n) - t \xrightarrow{P} 0$;

(ii) If $m > 1$ then $(\log m)\sqrt{Z_0(n)m(m-1)/\ \sigma^2}(\widehat{t}(n) - t) \xrightarrow{d} N(0,1)$;

(iii) If $m < 1$ and $nm^t \to \infty$ then $(\log m)\sqrt{Z_0(n)m^t}(\widehat{t}(n) - t) \xrightarrow{d} N(0,1)$;

(iv) If $m = 1$ then $\mathbf{E}\{\widehat{T}(n)\} = t$; if additionally $n/t \to \infty$ then $\widehat{T}(n)/t \xrightarrow{d} \chi_1^2 = [N(0,1)]^2$.

8.3. Conditional least square estimators

It is not difficult to obtain that for every $k \geq 1$

$$\begin{aligned} &\mathbf{E}\{Z_k(n)|Z_{k-1}(n)\} = mZ_{k-1}(n), \\ &\mathbf{E}\{[Z_k(n) - mZ_{k-1}(n)]^2|Z_{k-1}(n)\} = \sigma^2 Z_{k-1}(n). \end{aligned} \tag{15}$$

Therefore one can consider

$$Z_k(n) = mZ_{k-1}(n) + \alpha_k(n) \tag{16}$$

as a stochastic regression equation with the unknown coefficient m and the martingale difference error term $\alpha_k(n)$. In view of (14) it is most convenient to represent $\alpha_k(n)$ as $\epsilon_k(n)\sqrt{Z_{k-1}(n)}$. Now minimizing $S_{t,n}^2(m) = \sum_{k=1}^t \epsilon_k^2(n)$ with respect to m, one will obtain the least squares estimator (l.s.e.) for m : $\widehat{m}_t(n) = \sum_{k=1}^t Z_k(n)/\sum_{k=1}^t Z_{k-1}(n)$, which is just the m.l. Harris estimator, studied above.

Similarly by (15) one can consider the stochastic regression equation

$$[Z_k(n) - mZ_{k-1}(n)]^2 = \sigma^2 Z_{k-1}(n) + \beta_k(n) \tag{17}$$

with parameters m and σ^2 and the martingale difference error term $\beta_k(n)$.

Let

$$Y_k(n;x) = Z_{k-1}(n)[Z_k(n)/Z_{k-1}(n) - x]^2 \mathbb{I}_{\{Z_{k-1}(n)>0\}}. \tag{18}$$

In our case it will be convenient to represent $\beta_k(n)$ as $\epsilon_k(n)Z_{k-1}(n)$. Then by (17) and (18) one has $Y_k(n;m) = \sigma^2 + \epsilon_k(n)$.To obtain the l.s.e. for σ^2 one has to minimize $S^2_{t,n}(\sigma^2) = \sum_{k=1}^t \epsilon_k^2(n)$ with respect to σ^2. If m is known one obtains

(19) $$\widehat{\sigma}_t^2(n;m) = \frac{1}{t}\sum_{k=1}^{t} Y_k(n;m),$$

and similarly, if m is unknown,

(20) $$\widehat{\sigma}_t^2(n;\widehat{m}) = \frac{1}{t}\sum_{k=1}^{t} Y_k(n;\widehat{m}_t(n)).$$

As one can see by the following theorem, the estimators (19) and (20) are consistent and asymptotically normal.

Theorem 8.14. *Assume* $\mathbf{E}\{\xi^4\} < \infty$ *and condition* (12).
(i) If $Z_0(n)/n \xrightarrow{d} \nu$ *then* $\sigma_t^2(n;m) \xrightarrow{P} \sigma^2$ *and* $\sigma_t^2(n;m) \sim N(\sigma^2, 2\sigma^4/t)$;
(ii) If $Z_0(n)/n \xrightarrow{P} \nu$ *then* $\widehat{\sigma}_t^2(n;\widehat{m}) \xrightarrow{P} \sigma^2$ *and* $\widehat{\sigma}_t^2(n;\widehat{m}) \sim N(\sigma^2, 2\sigma^4/t)$.

Remark 8.6. *The case* $Z_0(n) \equiv 1$ *a.s. and* $m > 1$ *was investigated by Heyde* [234] *and Dion* [222]. *Dion proved that on the explosion set* $\sigma_t^2(\widehat{m}_t)$ *is consistent and asymptotically normal estimator for* σ^2. *Heyde obtained similar results for* $\sigma_t^2(\widetilde{m}_t)$, *where* $\widetilde{m}_t$ *is the Lotka-Nagaev estimator.*

8.4. Censored estimators

Consider $(Z_t(n), Z_{t+1}(n), \ldots, Z_{t+T}(n))$ for some t and T, which can be interpreted as a left censored sample. For m we will use the estimator:

(21) $$\widehat{m}_{t,T}(n) = \sum_{i=t+1}^{t+T} Z_i(n) \Big/ \sum_{i=t}^{t+T-1} Z_i(n).$$

For σ^2 we will define the estimators

(22) $$\sigma^2_{t,T}(n;m) = \frac{1}{T}\sum_{k=t+1}^{t+T} Y_k(n;m),$$

if m is known and

(23) $$\widehat{\sigma}^2_{t,T}(n;\widehat{m}) = \frac{1}{T}\sum_{k=t+1}^{t+T} Y_k(n;\widehat{m}_t(n)),$$

if m is unknown.

Note that for $t = 0$ the estimator(22) is equivalent to (19) and (23) to (20).

Theorem 8.15. *Assume condition* (12), $\sigma^2 < \infty$ *and* $Z_0(n)/n \xrightarrow{d} \nu$. *If* $n, t \to \infty$ *then uniformly in* $1 \le T \le \infty$ *it follows:*

(*i*) $\widehat{m}_{t,T}(n) \xrightarrow{P} m$ *and* $\mathbf{E}\{\widehat{m}_{t,T}(n)\} \to m$;

(*ii*) $\sqrt{Z_t(n) + \cdots + Z_{t+T-1}(n)}(\widehat{m}_{t,T}(n) - m)/\sigma \xrightarrow{d} N(0,1)$.

Theorem 8.16. *Assume condition* (12), $\mathbf{E}\{\xi^4\} < \infty$ *and* $Z_0(n)/n \xrightarrow{P} \nu$. *If* $n, T \to \infty$ *then uniformly in* $1 \le t \le \infty$ *it follows:*

(*i*) $\sigma^2_{t,T}(n; m) \xrightarrow{P} \sigma^2$, $\mathbf{E}\{\sigma^2_{t,T}(n; m)\} \to \sigma^2$ *and* $\sqrt{2T/\sigma^2}(\sigma^2_{t,T}(n; m) - \sigma^2) \xrightarrow{d} N(0,1)$.

(*ii*) $\widehat{\sigma}^2_{t,T}(n; \widehat{m}) \xrightarrow{P} \sigma^2$, $\mathbf{E}\{\widehat{\sigma}^2_{t,T}(n; \widehat{m})\} \to \sigma^2$ *and* $\sqrt{2T/\sigma^2}(\widehat{\sigma}^2_{t,T}(n; \widehat{m}) - \sigma^2) \xrightarrow{d} N(0,1)$.

Theorem 8.17. *Let* $m > 1$ *and* $\sigma^2 < \infty$. *Then under the condition of non-extinction one has:*

(*i*) *If* $t \to \infty$ *then uniformly by,* $1 \le T \le \infty$, *the estimator* $\widehat{m}_{t,T}$ *is consistent, asymptotically unbiased and asymptotically normal:* $\widehat{m}_{t,T} \xrightarrow{P} m$, $E\ \widehat{m}_{t,T} \to m$ *and* $\sqrt{Z_t + \cdots + Z_{t+T-1}}(\widehat{m}_{t,T} - m)/\sigma \xrightarrow{d} N(0,1)$;

(*ii*) *If* $\mathbf{E}\{\xi^4\} < \infty$ *and* $t, T \to \infty$ *then* $\sigma^2_{t,T}(m)$ *and* $\sigma^2_{t,T}(\widehat{m})$ *are consistent and asymptotically normal estimators for* σ^2 : $\sigma^2_{t,T}(m) \xrightarrow{P} \sigma^2$, $\widehat{\sigma}^2_{t,T}(\widehat{m}) \xrightarrow{P} \sigma^2$ *and* $\sigma^2_{t,T}(m) \sim N(0, 2\sigma^4/T), \widehat{\sigma}^2_{t,T}(\widehat{m}) \sim N(0, 2\sigma^4/T)$.

Remark 8.7. *The proofs of Theorems* 8.5, 8.6, 8.8–8.11 *are given in Dion and N. Yanev* [113]. *In general they use random summation methods and a generalization of the Gnedenko-Fahim's transfer limit theorem. Theorems* 8.7,8.12 *and* 8.13 *are proved in Dion and N. Yanev* [107], *and Theorems* 8.14–8.17 — *in Dion and N. Yanev* [101].

8.5. BGW branching processes with immigration (BGWI)

Traditionally, the branching processes with or without immigration have been treated separately. In this section we will show that the estimation theory developed in the previous section can be transferred to processes with immigration designing a suitable sampling scheme.

It is well known that the BGW process with immigration can be defined as follows:

$$Y_0 = 0, \quad Y_{t+1} = \sum_{i=1}^{Y_t} \xi_i(t+1) + I_{t+1}, \quad t = 0, 1, 2, \ldots, \tag{24}$$

where the offsprings $\{\xi_i(t)\}$ and the immigration component $\{I_t\}$ are independent. We will suppose that r.v. $\{\xi_i(t)\}$ are i.i.d. and have the same individual characteristics as in Section 8.1. One assumes usually that $\{I_t\}$ are also non-negative integer valued i.i.d. r.v. with $\lambda = \mathbf{E}\{I_t\}$, $b^2 = Var\{I_t\}$.

Remark 8.8. *The first branching process with immigration was investigated by Sevastyanov* [262] *in the continuous time Markov case. The discrete time model* (24) *was introduced by Heathcote* [237].

Similarly as in (16), it is not difficult to obtain from (24) that for every $k \geq 1$

$$\mathbf{E}\{Y_k|Y_{k-1}\} = mY_{k-1} + \lambda. \tag{25}$$

From(25) it follows that one can consider $Y_k = mY_{k-1} + \lambda + \alpha_k$ as a stochastic regression equation with the unknown parameters m and λ and a martingale difference error term α_k. Now minimizing $S_t^2(m,\lambda) = \sum_{k=1}^t \alpha_k^2$ with respect to m and λ, one can obtain the conditional l.s.e.

$$\widehat{m}_t = \{\sum_{k=1}^t Y_k \sum_{k=0}^{t-1} Y_k - t\sum_{k=1}^t Y_kY_{k-1}\}/\{(\sum_{k=0}^{t-1} Y_k)^2 - t\sum_{k=0}^{t-1} Y_k^2\}, \tag{26}$$

$$\widehat{\lambda}_t = \{\sum_{k=0}^{t-1} Y_k \sum_{k=1}^t Y_kY_{k-1} - \sum_{k=1}^t Y_k \sum_{k=0}^{t-1} Y_k^2\}/\{(\sum_{k=0}^{t-1} Y_k)^2 - t\sum_{k=0}^{t-1} Y_k^2\}. \tag{27}$$

The estimators (26) and (27) were introduced and investigated in the subcritical case by Heyde and Seneta [232], [231]. Note that the subcritical case was studied also by Quine [257].

Theorem 8.18. *Let $\sigma^2 < \infty$ and $b^2 < \infty$. If $m < 1$ then $\widehat{m}_t$ and $\widehat{\lambda}_t$ are strongly consistent and asymptotically normal estimators for m and λ.*

The estimation of the variances σ^2 and b^2 was first discussed and investigated by N. Yanev and Tchoukova-Dantcheva [14], [60]. Since $\mathbf{E}\{U_k^2|Y_{k-1}\} = \sigma^2 Y_{k-1} + b^2$, where $U_k = Y_k - mY_{k-1} + \lambda$, then one can consider $U_k^2 = \sigma^2 Y_{k-1} + b^2 + \beta_k$ as a stochastic regression equation with unknown parameters m, λ, σ^2 and b^2 and a martingale difference error term β_k.

Now assuming that m and λ are known and minimizing $S_t^2(\sigma^2, b^2) = \sum_{k=1}^t \beta_k^2$ with respect to σ^2 and b^2, one can obtain the conditional l.s.e.

$$\overline{\sigma}_t^2 = \sigma_t^2(m,\lambda) = \sum_{k=1}^t U_k^2(Y_k - \overline{Y}_t)/\sum_{k=1}^t (Y_k - \overline{Y}_t)^2, \tag{28}$$

$$\overline{b}_t^2 = b_t^2(m,\lambda) = \sum_{k=1}^{t} U_k^2(Y_k^2 - Y_k\overline{Y}_t)/t\sum_{k=1}^{t}(Y_k - \overline{Y}_t)^2, \tag{29}$$

where $\overline{Y}_t = \dfrac{1}{t}\sum\limits_{k=1}^{t} Y_k$.

If the means m and λ are unknown then one has to consider the estimators

$$\widehat{\sigma}_t^2 = \sigma_t^2(\widehat{m}_t,\ \widehat{\lambda}_t),\ \widehat{b}_t^2 = b_t^2(\widehat{m}_t,\ \widehat{\lambda}_t), \tag{30}$$

which are obtained by (28) and (29) replacing U_k with $\widehat{U}_k = Y_k - \widehat{m}_t Y_{k-1} +\ \widehat{\lambda}_t$.

Theorem 8.19. *Let* $\mathbf{E}\{\xi^4\} < \infty$ *and* $\mathbf{E}\{I_t^4\} < \infty$. *If* $m < 1$ *then the estimators* (28)–(30) *are consistent and asymptotically normal for* σ^2 *and* b^2.

Now we will follow the papers of Dion and N. Yanev [107], [113] where it is shown how to transfer the asymptotic results of the BGW process with a random number of ancestors (BGWR) to the BGW process with immigration (BGWI).

Consider the branching tree underlining the process $\{Y_t\}$ and let $Z_t(n)$ be the number of individuals, among generations $t+1, \ldots, t+n$, whose ancestors immigrated exactly t generations ago. For example, this type of data may be available in the genealogy or in the cell biology. Obviously, $Z_0(n) = \sum_{k=1}^{n} I_k$ is the total number of immigrants in the first n generations, $Z_1(n)$ is the total number of their offsprings and so on. Therefore $\{Z_t(n)\}$ is a BGWR process investigated in Section 8.2. In fact, $Z_t(n)$ is the cardinality of a particular "diagonal stopping line" over the sampling tree of the BGWI process, while Y_t is the cardinality of a "horizontal stopping line" (one can see in Chauvin [214] for the notion of stopping line). This duality is very important and can be exploited to transfer some results and properties from branching processes of one type to the other.

Actually, to estimate the offspring characteristics in a branching process with immigration one has to consider the proposed "diagonal" sampling scheme. It means to have the observations $(Z_0(n), Z_1(n), Z_2(n), \ldots, Z_t(n))$ (or even part of them). Then one has to apply the results of Section 8.2. and to estimate m and σ^2.

Assume that it is also possible to observe $Z_j(n,k) = \{$*the number of individuals in the first* $(j+n)$ *generations of the process* (1) *having exactly* k *offsprings in the next generation and whose ancestors immigrated exactly* j *generations ago*$\}$. Then $U_t(n,k) = \sum_{j=0}^{t-1} Z_j(n,k)$ is the total number of the individuals in the first $(n+t)$ generations of the process (1) having k offsprings in the next generation and whose ancestors immigrated up to t generations ago. Now we can calculate

$U_t(n)=\sum_{k=0}^{\infty} U_t(n,k)=\sum_{i=0}^{t-1} Z_i(n)$, which is the total number of the individuals of the first $(n+t)$ generations of process (1) whose ancestors immigrated up to t generations ago. Therefore one can obtain the estimator $p_k(t,n) = U_t(n,k)/U_t(n)$ of the offspring probability $p_k, k = 0, 1, 2, \ldots$, and one can apply Theorems 8.8 and 8.9.

Note that in the case (24) the r.v. $\{I_t\}$ are i.i.d. Therefore by the LLN $Z_0(n)/n=\frac{1}{n}\sum_{k=1}^{n} I_k \stackrel{a.s.}{\to} \lambda$. In this way applying the results of Section 8.2. we obtain strongly consistent and asymptotically normal estimators for offspring characteristics.

Remark that for the estimation theory of Section 8.2. we need only that $Z_0(n)/n = \frac{1}{n}\sum_{k=1}^{n} I_k \to \nu$ (in probability on even in distribution for some cases), where ν is a positive random variable. Note that this is a very serious generalization of the model (24).

Suppose now that $\{I_t\}$ are i.i.d. in (24). In this case the following estimators for λ are proposed:

$$\widetilde{\lambda}_t(n) = Z_t(n)/nm^t, \quad \widehat{\lambda}_t(n) = Z_t(n)/n[\widehat{m}_t(n)]^t, \tag{31}$$

where $\widehat{m}_t(n)$ is the Harris estimator studied in Section 8.2.

Theorem 8.20. *Assume that $\sigma^2 < \infty$ and the condition* (12) *holds. Then as $n, t \to \infty$: $\widetilde{\lambda}_t(n) \stackrel{P}{\to} \lambda, \quad \widehat{\lambda}_t(n) \stackrel{P}{\to} \lambda$.*

Remark 8.9. *The theorem is still valid if $\{I_t\}$ is a stationary ergodic process for which $\frac{1}{n}\sum_{k=1}^{n} I_k$ converges in probability to some positive constant λ.*

Theorem 8.21. *Assume $\sigma^2 < \infty, b^2 < \infty$ and $n, t \to \infty$.*

(i) If $m < 1$ and $nm^t \to \infty$ then $\sqrt{nm^t}(\widetilde{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, \lambda)$, $\sqrt{nm^t}(\widehat{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, \lambda)$;

(ii) If $m = 1$ and $n/t \to \infty$ then $\sqrt{n/t}(\widetilde{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, \lambda\sigma^2)$, $\sqrt{n/t}(\widehat{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, b^2)$;

(iii) If $m > 1$ then $\sqrt{n}(\widetilde{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, \lambda\sigma^2/m(m-1) + b^2)$, $\sqrt{n}(\widehat{\lambda}_t(n) - \lambda) \stackrel{d}{\to} N(0, \lambda\sigma^2/m(m-1) + b^2)$.

Robust modifications of the estimators (31) are investigated and compared also with $\widehat{\lambda}_t$ and $\widetilde{\lambda}_t$ in Atanasov et al. [174].

Note that Wei and Winnicki [270] showed that in the supercritical case the immigration mean λ does not admit a consistent estimator on the basis of $\{Y_t\}$ alone (see Theorem 8.20, (ii)).

8.6. Conclusion remarks and comments

Assuming that offspring probabilities in (1) depend on an unknown parameter, the parametric estimation theory is developed in many papers. Some of the results for the classical BGW process are given in the book of Guttorp [219] (see also the references therein). Parametric estimation for BGWR processes is developed in Stoimenova and N. Yanev [154].

The Bayes statistical methods are considered in the book of Guttorp [219] (see also the references therein) and results for the classical BGW process are presented. Tsokos and G. Yanev [121] obtained some new results in this direction. Johnson et al. [241] developed a nonparametric Bayesian estimation in the case of age-dependent branching processes (note that the process is non-Markov with a continuous time).

Sequential estimation in BGWI processes is proposed by Sriram et al. [261] and Qi and Reeves [256].

Jacob and Lalam [242], [243] and Lalam et al. [247] developed an estimation theory for size-dependent branching processes. Jacob et al. [160] proposed a statistical inference for some classes of regenerative branching processes.

Robust estimation for BGW process is considered in Sriram and Vidyashankar [260]. Robust methods in BGWR processes are developed by Stoimenova et al. [149], [150], Stoimenova [155] and Atanasov et al. [174]. A software system for simulation and estimation of branching processes with random migration is considered and discussed in Nitcheva and N. Yanev [126]. A new version and further development of this system in MATLAB environment is developed by Atanasov and Stoimenova.

Finally we would like to point out that a lot of branching models are studied by the probabilistic point of view, where the statistical results are only a few (or even absent), which means that there are a lot of possibilities for further statistical investigations. For more details see N. Yanev [187].

9. Applications of branching processes

9.1. Biological applications

In the papers [17], [28], [30], [51] the authors investigated biological systems with initiated cell proliferation. They apply the classical branching processes and they also define and study some new models. So, in the paper of Yakovlev and N. Yanev [17] it is assumed that the evolution of the cell follows the classical Bellman-Harris branching process. Together with the total number of cells in the given generation the following characteristics are of interest for biologists

(i) $\mu_n(t)$ = the number of cells in the n generation alive at time t;

(ii) $u_n(t)$ = the number of cells in the first $n-1$ generations, which are still alive at time t;

(iii) $N_k(t,x)$ = the number of cells in the kth generation which are younger than x at moment t;

(iv) $\mu^i(t) = (\mu^i_1(t), \dots, \mu^i_m(t))$, where $\mu^i_j(t)$ is the number of cells of type T_j born up to the moment t by an initial cell of type T_i.

These results are used in the paper [30] in the investigations of some characteristics of the mitotic cycle of the cells. First of all, the authors studied the number of cells which synthesize DNA (S-phase) under certain proliferating agent. The processes of main interest are

$N^C_S(t)$ = the number of cells which went into the $S-$phase up to time t;

$N_S(t)$ = the number of cells which are in $S-$phase at time t.

The last two processes correspond to the continuous and impulsive labels with radioactive ancestors.

The problem for the distribution of the labels was solved completely in the paper of Yakovlev and N. Yanev [51] in the case of impulsive labels at $t = 0$ and initial Poisson distribution $Po(\theta)$ of the labels among cells. The authors proved that the states of the system can be described by a Bellman-Harris branching process with infinitely many types of particles

$$\tilde{Z} = (Z_0(t), Z_1(t), \dots, Z_j(t), \dots),$$

where $Z_j(t)$ is the number of cells with label j at time t. For the distribution

$$\Pi_j(t) = \frac{\mathbf{E}\{Z_j(t)\}}{\mathbf{E}\left\{\left(\sum\limits_{k=0}^{\infty} Z_k(t)\right)\right\}}$$

they obtained (in case of synchronized population) that

$$\Pi_j(t) = \left[\frac{\theta^j}{j!}\sum_{k=0}^{\infty}(p2^{1-j})^k e^{-\theta/2^k}(\bar{G} * G^{*k})(t)\right] \Big/ \left[\sum_{k=0}^{\infty}(2p)^k(\bar{G} * G^{*k})(t)\right],$$

where $G(t)$ is the c.d.f. of the mitotic cycle.

These investigations are continued in the papers [64] and [72]. A comprehensive presentation of biological applications of branching processes by that time can be found in the book of Yakovlev and N. Yanev [81].

Recently, in a series of publications [161], [166], [167], [175], [176], [183], [184] and [185], some new applications of branching processes for modeling of cell proliferation kinetics of normal and cancer cells were developed.

First of all, some generalizations in the case of continuous labels are given in [166] and [175]. Note that processes with continuous labels are first considered by Kolmogorov.

On the other hand, some new models of renewing cell populations (*in vivo*) using age-dependent branching processes with non-homogeneous Poisson immigration are proposed in [167]. A model of leukemia cell kinetics with a stem cell immigration component is studied in [161].

Age-dependent branching processes with randomly chosen paths of evolution are proposed in [183] as models of progenitor cell populations (*in vitro*) with estimating of the offspring distributions using real data as well as bootstrap methods.

An interesting and important problem arising from cell proliferation kinetics is the definition and the limiting behavior of age and residual lifetime distributions for branching processes considered in [176].

The relative frequencies of distinct types of cells in multitype branching processes with a large number of ancestors are investigated in [184] and [185]. The reported limiting results are of advantage in cell kinetics studies where the relative frequencies but not the absolute cell counts are accessible to measurement. In [184] some relevant statistical applications are discussed in the context of asymptotic maximum likelihood inference for multitype branching processes. In [185] the asymptotic behavior of multitype Markov branching processes with discrete or continuous time is investigated in the positive regular and nonsingular case when both the initial number of ancestors and the time tend to infinity. Some limiting distributions are obtained as well as multivariate asymptotic normality is proved. The results from [184] and [185] have a specific applications in cell proliferation kinetics.

Finally it is worth to point out that new problems in the theory of branching processes appeared as a result of cell proliferation modeling.

9.2. Applications in economy

The "call" option on the given *underlying stock* is the right to buy a share of the stock at a certain fixed price K (the "strike price") at a certain fixed time T in the future (the "maturity date"). Let us denote by $S(t),\ t \geq 0$ the price of a share of the underlying stock at time t. The buyer is paying today (at $t < T$) some money (present value of the option $C(T;t)$) in return for the *right* to force the seller to sell him a share of the stock, if the buyer wants it, at the strike price K on the maturity date T. Clearly, if $S(T) > K$, then the buyer of the option will exercise his right at time T, buying the stock for K and selling it for $S(T)$, gaining a net profit $S(T) - K$. If $S(T) \leq K$, then it is not profitable to buy the stock at price K, so the option is not exercised, and the gain at time T is 0. In summary, for the call option, the gain at maturity date is $(S(T) - K)^+ = \max(S(T) - K, 0)$.

The main problem in the option pricing is to determine the "fair" present value of the option. Apparently, it should depend at least on the value $S(t)$ of the underlying stock, the time $T - t$ to maturity, and the strike price K.

Thus, the problem for finding security prices processes which agree well with the market data is the central problem for both practitioners and scientists. Since 1973 the most widely used model is the Black-Scholes [210] model. It assumes that $S(t)$ is a geometric Brownian motion, which implies that

(i) trading takes place continuously in time;

(ii) the price dynamics of the stock have a continuous sample path with probability one;

(iii) the distribution of log-returns is normal with constant volatility.

Many empirical investigations during the last thirty years do not agree with these assumptions (see e.g. [248]).

A possible way for relaxing the assumption that stock prices follow a geometric Brownian motion is to specify an alternative stochastic process for the price (pure jump models; jump diffusion models; models with stable distributions of the returns) see e.g. [213]).

At 1996 T. Epps [225] introduced a randomly indexed branching process for modeling of daily stock prices. The model is constructed by Galton-Watson branching processes subordinated with the Poisson process. It has the following attractive features (see [225]):

(i) The extra randomness introduced by the subordination produces in the increments and in the returns the same high proportion of outliers observed in high-frequency stock data; but, unlike the other tick-tailed models of stock returns this one capture the discreteness of prices.

(ii) The model predicts an inverse relation between variance of returns and the initial price which is well documented empirically.

(iii) The possibility of extinction of the stock price process and the distribution of the extinction time have natural interpretation as bankruptcy of the corresponding firm and can be applied in investigations of bankruptcy risk.

Further investigations on the model are done under the following assumptions:

(i) The offspring of a particle in the branching process has the *two parameter geometric distribution.*

(ii) The subordinator is a *Poisson process.*

A formula for pricing of European vanilla call option is derived in [173]. In the paper [192] a formula for price of European barrier call option is proved.

The process introduced by Epps was generalized from theoretical point of view in the papers [165] and [193].

Here we have to mention an earlier paper of Stojkov and Dimitrov [27] where the authors apply the Galton-Watson branching processes to the so called multiplicative effect in the economy.

Some applications of branching processes in economy and demography are proposed by Gavazki [96], [124], [134], [142], [148], [169], [180].

10. Concluding remarks

Note that the book of Obreshkov on Probability Theory (1963) is the first Bulgarian publication where Markov branching processes are considered in Ch. 11 (pp. 226-234). Starting from the first two articles published by Yanev in 1972, the publications of the Bulgarian mathematicians in the field of Branching Processes and their applications increased up to 190 during the last 37 years (given year by year in the References). After that the Bulgarian dissertations on Branching Processes and Applications are presented – among them 11 for the first scientific degree "PhD" and 2 for the second degree "Doctor of Mathematical Sciences". It is interesting to point out that the main contributions of the papers are in the following areas: Controlled Branching Processes (Regulation) – 35, Regenerative Branching Processes – 39, Statistical Inference for Branching Processes – 31, Applications of Branching Processes – 29, Diffusion Branching Processes – 10. A lot of these papers are published in the most prestigious international journals and in four books.

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