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QUEUE LENGTH SIMULATIONS IN A FINITE SINGLE-LINE QUEUEING SYSTEM WITH REPEATED CALLS

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Simulated results about the queue length and the server state in a finite single server queueing system with repeated calls are presented. Formulas for the basic probability characteristics of the corresponding distributions are obtained in previous papers of the author. The numerical values computed according to these formulas are compared with the simulated results. Empirical mean values of the idle period are obtained as well.

1. Introduction

The communication systems' functioning is usually strongly affected by the presence of repeated calls (demands, orders). Many papers are devoted to the investigation of repeated calls influence on system's characteristics and technical capacity (c.f. [1]–[4]).

We consider here a finite, single line queueing system with repeated calls. The assumption of a finite number of customers is of special interest to practice, as in real situations the number of subscribers is finite. However, this assumption as well as the presence of repeated calls complicates the investigation of the system and the expressions for its basic probability characteristics. Formulas for some of these characteristics are obtained in previous works of the author [5].

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In the present paper simulation experiments about the queue length and the server state are performed in the case of exponentially distributed service time. In addition to mathematical simplicity, this case has advantage of admitting simple practical interpretations. The experimental results are compared with the corresponding ones, obtained by the explicit formulas. They confirm the theoretical predictions of the model. Furthermore, on the base of simulation experiments, empirical means of the idle channel periods, as well as some dependences of the considered probability characteristics on the system parameters, that have not been treated till now, are established. A Conclusion closes the paper and possible future investigations are proposed therein.

2. Model description and some preliminary results

We study a single-line (channel, server) queueing system with N customers (subscribers). These customers are identified as sources of primary orders (calls, demands). Each such source produces a Poisson process of primary (initial) calls with intensity λ. If the server is free (idle) at the instant of a primary call arrival, it begins service immediately. Otherwise, if the channel is busy, it forms a source of repeated calls. Such a source produces a Poisson process of repeated (secondary) calls with intensity μ. If an incoming repeated call finds a free line, it begins service and like a primary call, after service completion becomes again a source of primary calls. Otherwise, if the line is engaged at the moment of a repeated call arrival, the system state does not change.

The service time is a random variable ξ with distribution function $G(x)$ both for primary and repeated calls. The intervals between repeated trials, as well as between primary ones and the service times, are assumed to be mutually independent. Let

$$g(x) = G'(x), \overline{g}(s) = \int_0^\infty e^{-sx} dG(x) = \int_0^\infty e^{-sx}\ g(x)dx,$$

$$\nu^{-1} = -\overline{g}'(0),$$

$$\eta(x)\Delta = P\left\{\xi \in (x, x+\Delta)/\xi \geq x\right\}, \tag{2.1}$$

i.e. ν^{-1} is the mean of the service time ξ and $\eta(x)$ is the service rate at instant x after start of a service

$$\eta(x) = \frac{g(x)}{1-G(x)}.$$

For the sake of simplicity the primary (initial) orders sources will be called *primary subscribers* and the sources of repeated (secondary) calls – *secondary* *subscribers* (a sort of queue). Their number at moment t we shall denote by $I(t)$ and $R(t)$, respectively. By $C(t)$ we denote the number of busy channels at time t,

$$R(t) + I(t) = N - C(t) = \begin{cases} N, \text{ if } C(t) = 0, & \text{i.e. if the channel is free,} \\ N-1, \text{ if } C(t) = 1, & \text{i.e. if the channel is busy.} \end{cases}$$

As the process $(R(t), I(t))$ is not a Markov process, we introduce an additional random variable $z(t)$ - the time from the last service starting moment before t (in case $C(t) = 1$). The joint distribution of $C(t)$, $R(t)$ and $z(t)$, as well as the joint distribution of $C(t)$ and $R(t)$ in steady state

$$p_{1j}(x)dx = \lim_{t\to\infty} p_{1j}(x,t)dx = \tag{2.2}$$

$$= \lim_{t\to\infty} P\left\{C(t) = 1,\ R(t) = j,\ x \le z(t) < x + dx\right\},$$

$$p_{ij} = \lim_{t\to\infty} p_{ij}(t) = \tag{2.3}$$

$$= \lim_{t\to\infty} P\left\{C(t) = i, R(t) = j\right\},\ i = 0, 1, \quad j = 0, 1, \ldots, N$$

are studied in [5]. Considering the possible transitions of the system for a time interval Δt, we get a system of partial differential equations for the distribution $p_{1j}(x,t)$, $p_{ij}(t)$, $i = 0, 1$, $j = 0, 1, \ldots, N$. By means of Laplace - Stieltjes transforms of this system equations and taking limit as $t \to \infty$, we get a system of ordinary differential equations for the stationary probabilities 2.2- 2.3

$$\frac{d}{dx} p_{1j}(x) = -[(N-j-1)\lambda + \eta(x)]p_{1j}(x) + (1-\delta_{j0})(N-j)\lambda p_{1j-1}(x), \tag{2.4}$$

$$[j\mu + (N-j)\lambda]p_{0j} = \int_0^\infty p_{1j}(x)\eta(x)dx, \tag{2.5}$$

$$p_{1j}(0) = (N-j)\lambda p_{0j} + (j+1)\mu p_{0j+1}, \quad j = 0, 1, \ldots, N-1, \tag{2.6}$$

$$\sum_{j=0}^{N-1} p_{0j} + \sum_{j=0}^{N-1} \int_0^\infty p_{1j}(x)dx = 1. \tag{2.7}$$

The following theorem and corollary hold true [5].

Theorem 1. *The stationary probabilities* (2.2)–(2.3), *which solve the system* (2.4)–(2.7) *always exist and are given by the following formulas*

$$p_{1n}(x) = (-1)^{N-n-1} e^{-\int_0^x \eta(u)du} \sum_{k=0}^{n} \binom{N-k-1}{n-k} e^{-(N-k-1)\lambda x} \psi_k C,$$

$$p_{1n} = (-1)^{N-n-1} \sum_{k=0}^{n} \binom{N-k-1}{n-k} \psi_k r_k C,$$

$$p_{0n} = (-1)^{N-n-1} a_n^{-1} \sum_{k=0}^{n} \binom{N-k-1}{n-k} \psi_k g_{N-k-1} C, \quad n = 0, 1, \ldots, N-1,$$

$$p_{0N} = p_{1N}(x) = p_{1N} = 0, \quad x \geq 0,$$

where

$$r_n = \begin{cases} \frac{1-g_{N-n-1}}{(N-n-1)\lambda}, & n = 0, 1, \ldots, N-2, \\ \nu^{-1}, & n = N-1, \end{cases}$$

(2.8) $$C = N\lambda\mu \left[\mu \left(1 + N\lambda\nu^{-1}\right) \psi_{N-1} + \left(1 - g_1\right)\left(\mu - \lambda\right) \psi_{N-2}\right]^{-1},$$

(2.9) $$\psi_n = (-1)^n \binom{N-1}{n} \frac{g_{N-1}}{g_{N-n-1}} \left(1 + A_n + B_n + C_n\right), \ n = 0, 1, \ldots, N-1$$

and A_n, B_n *and* C_n *are given by the recurrent relations*

(2.10) $$A_0 = B_0 = C_0 = 0,$$

(2.11) $$A_n = \frac{1 - g_{N-n}}{g_{N-n}} \frac{a_n}{(N-n)\mu} \left(A_{n-1} + B_{n-1}\right),$$

(2.12) $$B_n = \frac{a_0}{a_{n-1}} \left(A_{n-1} + C_{n-1}\right) + B_{n-1},$$

(2.13) $$C_n = \frac{1 - g_{N-n}}{g_{N-n}} \frac{a_n}{(N-n)\mu} \left(1 + C_{n-1}\right), \ n = 1, 2, \ldots, N-1,$$

(2.14) $$a_n = (N-n)\lambda + n\mu,$$

(2.15) $$g_n = \overline{g}(n\lambda) = \int_0^\infty e^{-n\lambda x} \ g(x)dx, \ n = 0, 1, \ldots, N-1.$$

Corollary 1. *The stationary distribution of the channel state*

$$P_i = \lim_{t\to\infty} P\{C(t) = i,\}, \; i = 0, 1,$$

the mean ER and the variance DR of the secondary subscribers number (the queue length), i.e. of the distribution (2.3) *and the mean EI of the primary subscribers number in steady state have the form*

$$P_1 = \nu^{-1}\psi_{N-1}C, \quad P_0 = 1 - P_1, \tag{2.16}$$

$$ER = \left[(N-1)\nu^{-1}\psi_{N-1} + \frac{(1-g_1)(\mu-\lambda)\psi_{N-2}}{\mu\lambda}\right]C, \tag{2.17}$$

$$EI = N - ER - P_1, \tag{2.18}$$

$$DR = \left\{\psi_{N-1}(N-1)\left[\nu^{-1}(N-1) + \frac{1}{\mu-\lambda}\right] + \right. \tag{2.19}$$

$$\left. +\psi_{N-2}\left[\frac{(1-g_1)(2N-3)}{\lambda} + \frac{\mu g_1 + N\lambda(1-g_1)}{\mu(\mu-\lambda)}\right] + \psi_{N-3}\frac{(1-g_2)}{\lambda}\right\}C - (ER)^2.$$

(See (2.8)–(2.15) *for the definition of the corresponding quantities.)*

3. Simulation

Now we assume that the service time is exponentially distributed with mean ν^{-1}. Then the service rate $\eta(x)$ at instant x after service start (2.1) is equal to ν and does not depend on x, so that the system state at time t is fully determined by the number $R(t)$ of repeated subscribers and the channel state $C(t)$.

The argument t will be omitted when it is understood.

The possible transitions of the system from state $(R(t), I(t), C(t),)$ to state $(R(t+\Delta t), I(t+\Delta t), C(t+\Delta t))$, $R(t)+I(t)+C(t) = N$ for a short time Δt are as follows:

(1) If at the moment t the channel is busy, i.e. $C(t) = 1$, only one of the following events may occur:

(1a) Some primary subscriber becomes a secondary one. The probability p_{IR} of this transition is

$$p_{IR} = \lambda I(t)\Delta t, \tag{3.1}$$

and the change of the system state is

$$(R(t), I(t), C(t) = 1) \quad \rightarrow \ (R(t) + 1, I(t) - 1, C(t) = 1).$$

(1b) The subscriber under service ends its service and becomes primary again. This means that

$$(R(t), I(t), C(t) = 1) \ \rightarrow \quad (R(t), I(t) + 1, C(t) = 0).$$

The probability p_{CI} of this transition is

$$p_{CI} = \nu \Delta t. \tag{3.2}$$

(1c) With probability

$$1 - (p_{IR} + p_{CI}) \tag{3.3}$$

there will be no changes in the system state.

(2) If at the moment t the channel is free ($C(t) = 0$), the possibilities are:

(2a) Some secondary subscriber begins its service, i.e.

$$(R(t), I(t), C(t) = 0) \ \rightarrow \quad (R(t) - 1, I(t), C(t) = 1).$$

The probability p_{RC} of such a transition is

$$p_{RC} = \mu R(t) \Delta t. \tag{3.4}$$

(2b) Some primary subscriber begins its service, i.e. with probability

$$p_{IC} = \lambda I(t) \Delta t \tag{3.5}$$

we have

$$(R(t), I(t), C(t) = 0) \ \rightarrow \ (R(t), I(t) - 1, C(t) = 1).$$

(2c) With probability

$$1 - (p_{RC} + p_{IC}) \tag{3.6}$$

there are no changes in the system state.

The simulation follows the short time changes of the process $(R(t), I(t), C(t))$, defined by the transitions (1a)–(1c), (2a)–(2c) and some initial state $(R(t_0), I(t_0), C(t_0))$. Here the appropriate choice of the time interval Δt participating in formulas (3.1)–(3.6) is of importance. This interval gives the single simulation step length. As each one of the transition probabilities (3.1)–(3.6) has to be smaller than 1, in case (1) of busy channel, it is necessary to hold

$$\lambda I(t)\Delta t < 1,\ \nu\Delta t < 1,\ \ (\lambda I(t) + \nu)\,\Delta t < 1,$$

and we may choose

$$\Delta t \leq \frac{1}{m_1},\ m_1 = \lambda N + \nu,$$

and in the case (2) of free channel, in order to have

$$\mu R(t)\Delta t < 1,\ \lambda I(t)\Delta t < 1,\ \ (\mu R(t) + \lambda I(t))\,\Delta t < 1$$

we assume that

$$\Delta t \leq \frac{1}{m_2},\ m_2 = N(\lambda + \mu).$$

Finally, using generated random numbers, the direction of transitions is chosen with probability, defined by formulas (3.1)–(3.6) with

$$\Delta t = \frac{1}{m},\ m = \max(m_1, m_2)\ .$$

We accept that the initial state of the system is

$$C(t_0) = 0,\ R(t_0) = ER,\ I(t_0) = N - C(t_0) - R(t_0),$$

where ER is the stationary mean value of the secondary subscribers, calculated according to formulas (2.17), (2.8)–(2.15) in the case of exponentially distributed service time with parameter ν.

In order to evaluate the channel state probabilities, we consider the alternating sequence of idle and service periods of the channel. Denote by ξ_l and η_l the length (the number of simulation steps) of the l^{th} idle period and of the l^{th} service respectively, $l = 1, 2, \ldots$, by I and J – the number of these periods during the simulation, $I = 1, 2, \ldots$, $J = I$ or $J = I - 1$, and finally by n – the number of simulation steps. Then the quantities

$$\widehat{P}_0 = \frac{\sum_{l=1}^{I} \xi_l}{n},\ \widehat{P}_1 = \frac{\sum_{l=1}^{J} \eta_l}{n},\ \sum_{l=1}^{I} \xi_l + \sum_{l=1}^{J} \eta_l = n$$

have to be appropriate estimates of the free and busy channel probabilities, respectively.

What is more, if the obtained empirical means of the idle and service periods multiply by the single step length Δt, we should obtain values close to the real means of these distributions. As the mean of idle channel period has not been obtained till now, the comparison is out of question. In contrast to this, the service periods coincide with the service times, so we can compare the empirical mean with the service time mean value, i.e. with ν^{-1}.

4. Simulated results

Simulations are performed using the software system "MATLAB" [8].

The input parameters of the program include the number of subscribers N, the primary intensity λ, the secondary one μ, as well as the service rate ν and the duration of simulation (a number of time steps) n. Two examples are presented as illustrations: using the parameter value $\lambda = 0.012$ in the first example and $\lambda = 0.024$ in the second one. The other parameters are fixed in both examples: $N = 50$, $\nu = 1$, $\mu = 0.1$, $n = 10000$.

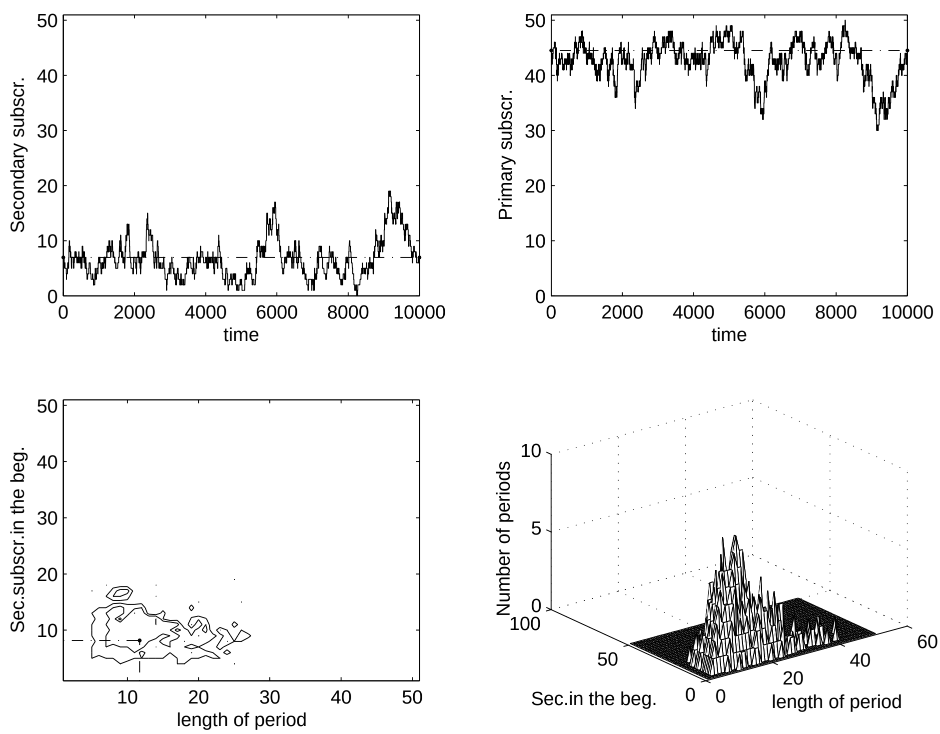

Figure 1

The empirical means and standard deviations of the secondary and of the primary subscribers number, as well as the observed probability of busy channel are given in table 1. The corresponding theoretically obtained values $ER(2.17)$, $\sqrt{DR}(2.19)$, $EI(2.18)$ and $P_1(2.16)$ are given also.

Table 1

	Busy line prob. P_1	Secondary subscr. mean ER (st.dev.)	Primary subscr. mean EI (st.dev.)
$\lambda = 0.012$			
Obs. values	0.5402	6.31 (3.58)	43.15 (3.68)
Theor.values	0.5223	5.95 (2.92)	43.53 (–)
$\lambda = 0.024$			
Obs. values	0.7278	18.93 (4.02)	30.34 (4.07)
Theor.values	0.7241	19.10 (4.15)	30.17 (–)

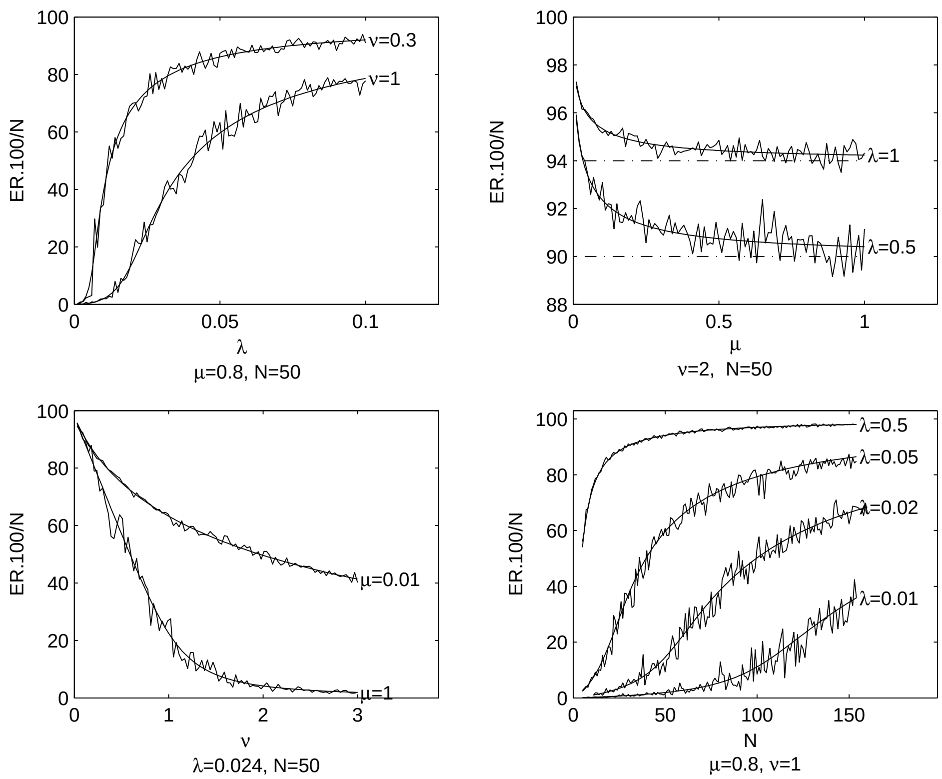

Figure 2

Table 2 shows the empirical means of the idle and the busy channel periods, the simulation step length Δt and the service mean ν^{-1}.

Table 2

	Simul.step length Δt	Mean of the idle periods	Mean of the service periods (ν^{-1})
$\lambda = 0.012$			
Obs. values	—	$5.98\times\Delta t$=0.89	$5.86\times\Delta t$=1.04
Theor. values	0.18	—	1
$\lambda = 0.024$			
Obs. values	—	$2.34\times\Delta t$=0.38	$6.26\times\Delta t$=1.01
Theor. values	0.16	—	1

In table 3, the correlation matrix for the primary subscribers number in the beginning of an idle period, the number of secondary ones in the beginning of the same period and the sum of this period length and the length of the succeeding service are given.

Table 3

$\lambda = 0.012$	Corr. matrix			$\lambda = 0.024$	Corr. matrix		
Sec. in the begin.	1			Sec. in the begin.	1		
Pr. in the begin.	-1	1		Pr. in the begin.	-1	1	
Period length	-0.07	0.07	1	Period length	0.09	-0.09	1

Fig. 1 illustrates the behaviour of the simulated system in the first example ($\lambda = 0.012$). It consists of four diagrams. Two of them represent the number of primary subscribers (the upper-right corner) and of the secondary ones (the upper-left corner) at each simulation step. The lower-right diagram is a 2-dimentional histogram of the idle period length (measured in simulation steps) and the number of secondary subscribers in the beginning of this period (this number remains the same during the whole period); on the left are the level curves of the same histogram. The joint distribution of these two variables has not been theoretically treated till now.

Comparing the obtained results, we may conclude that all characteristics under consideration are affected by the variation of the parameter λ, whereas this concerns to the least degree the correlation coefficients. The values of these coefficients in both experiments show that the channel period is not correlated considerably neither with the number of primary, nor with the number of secondary subscribers at the beginning of the period.

An attempt to determine the influence of all parameters of the system on the considered characteristics is done on Fig. 2, 3 and 4.

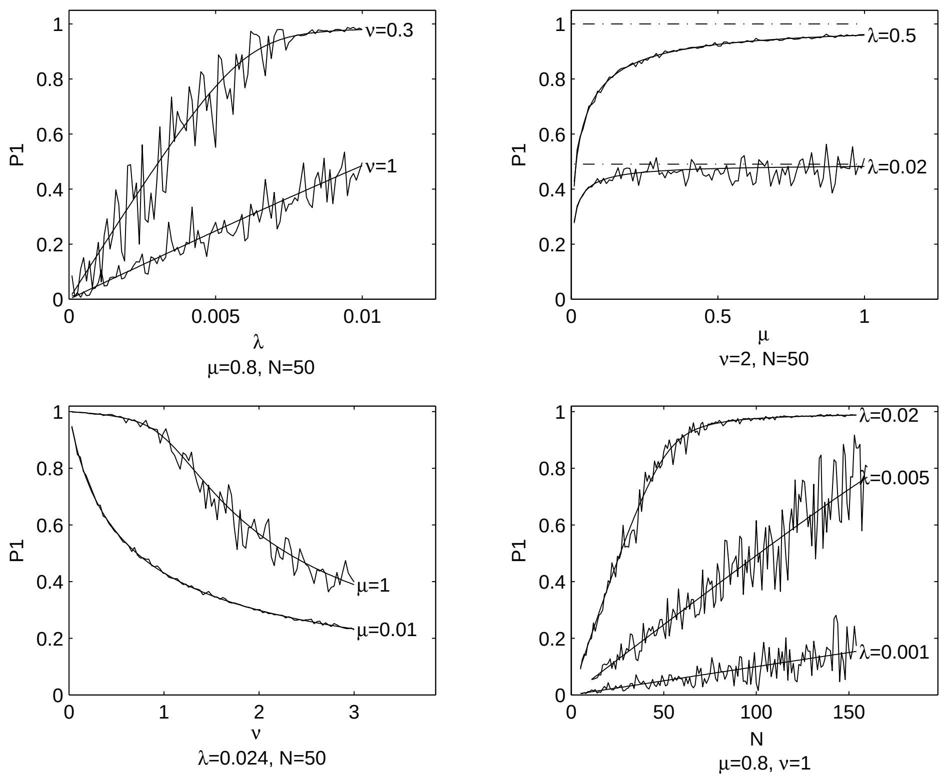

Figure 3

The graphs on Fig. 2 represent the dependence of the secondary subscribers mean percentage $ER.100/N$ on the system's parameters λ (the upper-right corner), μ (the upper-left corner), ν (the lower-left corner) and N (the lower-right corner). The mean ER is calculated according to formula 2.17 for 100 values of the varying parameter and the corresponding empirical means, obtained via simulations with 10000 steps are also presented. We can see that this percentage is an increasing function of the primary intensity λ and of the number of all customers N and a decreasing function of the service intensity υ and the secondary intensity μ. Moreover, as $\lambda \to 0$ or $N \to 1$ the mean percentage of secondary subscribers takes values close to 0, while when λ and N are large, this percentage is close to 100. The boundary values as $\nu \to 0$ and $\nu \to \infty$ are close to 100 and 0, respectively.

The behaviour of the system's characteristics as $\mu \to \infty$ is studied in [6]. The

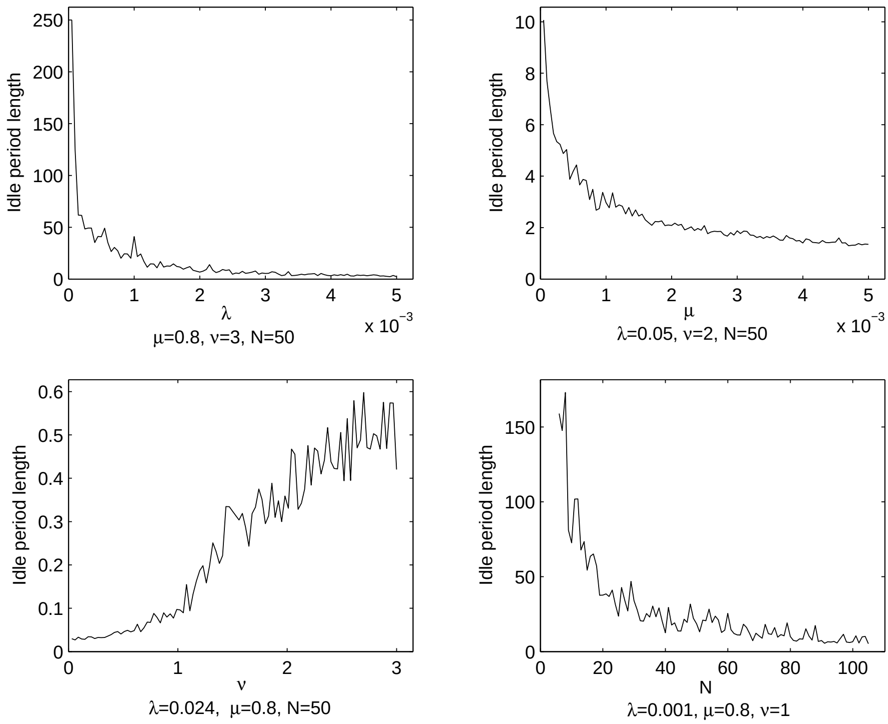

Figure 4

boundary values of $ER.100/N$ computed according to formulas, obtained in [6] are represented by dotted lines on the upper-right diagram. From both graphs (for $\lambda = 0.5$ and $\lambda = 1$) one sees, that the theoretically computed, as well as the obtained via simulation values are approaching the boundary ones yet for values of μ, close to 1.

The dependence of the busy channel probability P_1 on the system's parameters is presented on Fig. 3. The behaviour of P_1 as a function of each of the parameters λ, ν and N is similar to the behaviour of $ER.100/N$, but whereas $ER.100/N$ is a decreasing function of the parameter μ, P_1 is an increasing one. Here, as on Fig.2, dotted lines show the boundary values of P_1 as $\mu \to \infty$ [6]. Without writing down the boundary formulas, we shall note that when $\mu \to \infty$, P_1 tends to the busy channel probability in the corresponding finite single-server system with queue [7]. This is not true for the mean value ER and the variance

DR of the secondary subscribers number.

Finally, the influence of the system's parameters on the idle period length (Fig. 4) is illustrated only by simulated results, since it has not been theoretically treated till now.

5. Conclusion

The simulation experiments show satisfactory behaviour of the studied process, i.e. the observed statistics of the generated distributions are near to the theoretical ones for a wide range of parameters' values. The good accordance of the theoretical results with the observed ones shows that the values of some characteristics, like the idle channel period, not theoretically investigated yet, may be obtained by the corresponding simulated results.

Similar simulation model may be applied for investigation of the same queueing system, but for other service time distributions - normal, uniform, deterministic etc. Besides that, the dependence of the system's characteristics on its parameters, established here on the base of numerical analysis in the case of exponentially distributed service time, should be checked for other distributions of the service time or proved theoretically.

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