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CONTROLLED MULTITYPE BRANCHING MODELS: GEOMETRIC GROWTH

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In this work we deal with a multitype branching process that puts together control in the number of reproductive units of each type and population-size-dependent reproduction. Moreover, unlike other branching models, it is possible interaction between individuals at reproduction time. We investigate sufficient conditions for such a model to have asymptotically a geometric growth, considering almost sure and L^α, $1 \leq \alpha \leq 2$, convergences. We pay special attention to L^2 convergence, taking advantage of the Hilbertian properties of this space.

1. Introduction

The multitype Galton-Watson process is a well-known branching model which has received considerable attention in the scientific literature (e.g. see Mode (1971)). From this model, it is possible to obtain other homogeneous branching models for a more suitable description of some real situations. One can think of different modifications by adding new features to the original multitype Galton-Watson process:

– To consider population-size-dependent reproduction.

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– To establish a control of the number of each type of progenitor according to the population size.

– To allow interaction between individuals of the same generation at reproduction time, i.e. "dependent offspring".

Control in the population was proposed by Sevast'yanov and Zubkov (1974) in a deterministic way and population-size-dependent reproduction was considered by Klebaner (1989).

González et al. (2005a) have introduced a multitype model that puts together control and size dependent reproduction, generalization of the one-dimensional model with control and reproduction dependence on the population size considered by Küster (1985). In this model, called *Controlled Multitype Branching Process with Random Control and Population-Size-Dependent Reproduction (CMPD)*, the reproduction phase in a generation is conditioned by the size of the previous generation, the number of progenitors of each type is controlled by means of a random mechanism and possible dependence among individuals of the same generation at reproduction time is allowed. The introduction of dependence is a major novel feature with respect to the classical branching models, since their implicit assumption of independence can be only considered to be a mere theoretical simplification of the more complex types of reproductive behaviour in nature.

In the next section we define mathematically the CMPD and also present some results in the literature about this process. Specifically we present results about the indefinite growth of the process, and the almost sure and L^α, $1 \leq \alpha \leq 2$, convergence of the CMPD, normed by a geometric progression, to a non-degenerate and finite random vector. In Section 3 we pay special attention to the convergence in L^2, showing a new, and in some situations weaker, set of conditions that guarantee also the geometric growth of the CMPD.

2. The probability model

Let $\{X^{i,n,j}(z) : i = 1, \dots, m; n = 0, 1, \dots; j = 1, 2, \dots; z \in \mathbb{N}_0^m\}$ and $\{\phi^n(z) : n = 0, 1, \dots; z \in \mathbb{N}_0^m\}$ be two independent sequences of m-dimensional, non-negative, integer-valued random vectors, where

(i) The stochastic processes $\{\phi^n(z) : z \in \mathbb{N}_0^m\}$, $n \geq 0$ are independent and for each $z \in \mathbb{N}_0^m$, the vectors $\{\phi^n(z) : n = 0, 1, \dots\}$ are identically distributed.

(ii) The stochastic processes $\{X^{i,n,j}(z) : i = 1, \dots, m; j = 1, 2, \dots; z \in \mathbb{N}_0^m\}$, $n \geq 0$ are independent and identically distributed. Moreover, for each $i \in$

$\{1,\ldots,m\}$ and $z\in\mathbb{N}_0^m$ the vectors $\{X^{i,n,j}(z) : n=0,1,\ldots; j=1,2,\ldots\}$ are identically distributed.

The sequence of m-dimensional random vectors $\{Z(n)\}_{n\geq 0}$ defined recursively as

$$Z(0)=z\in\mathbb{N}_0^m;\quad Z(n+1)=\sum_{i=1}^{m}\sum_{j=1}^{\phi_i^n(Z(n))} X^{i,n,j}(Z(n)),\quad n\geq 0$$

will be denominated Controlled Multitype Branching Process with Random Control and Population-Size-Dependent Reproduction (CMPD).

Intuitively, m represents the number of different types in the population. Each random vector $X^{i,n,j}(z)$ governs the reproduction phase and, intuitively, represents the vector of descendants of the jth i-type individual in the nth generation given that the number of individuals of the different types in the $(n-1)$st generation was given by the vector z. Each random vector $\phi^n(z)$ describes the control made on the population and, intuitively, represents the number of individuals of the different types allowed to be parents in the nth generation, provided that the population size in this generation is given by the vector z. Consequently the vector $Z(n+1)$ represents the total number of individuals of the different types in the $(n+1)$st generation.

The CMPD is a homogeneous multitype Markov chain that includes as particular cases not only the controlled multitype branching process proposed by Sevast'yanov and Zubkov (1974) and the population-size-dependent multitype branching process introduced by Klebaner (1989), but also other homogeneous multitype branching models.

Along this paper we will assume that the distributions of reproduction and control variables satisfy the following regularity properties:

(A1) For each $z\in\mathbb{N}$, and $i\in\{1,\ldots,m\}$ the random vectors $X^{i,0,k}(z)$, $k\geq 0$ are identically distributed. Moreover there exist $r_{ij}\geq 0$, $i,j\in\{1,\ldots,m\}$, such that

$$E[X_j^{i,0,1}(z)]=r_{ij}+h_{ij}^r(z)\qquad\text{and}\qquad h_{ij}^r(z)=o(1).$$

(A2) For each $z\in\mathbb{N}$, there exist $\lambda_i\geq 0$, $i\in\{1,\ldots,m\}$, such that

$$E[\phi_i^0(z)]=\lambda_i z_i+h_i^c(z)\qquad\text{and}\qquad h_i^c(z)=o(\|z\|),$$

where $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^m$.

(A1) is an usual hypothesis in population-size-dependent models and a possible intuitive interpretation is the stabilization of the average number of descendants per individual as the population size grows indefinitely.

On the other hand, (A2) means that the average number of progenitors of a type is proportional to the number of individuals of this type plus/minus certain quantity of progenitors which is negligible with respect to the total amount of population. Notice that under (A2) immigration/emigration of progenitors of each type is allowed. For each type, immigration is possible even if there are not individuals of that type. This could not happen if $h_i^c(z) = z_i o(1)$. However, in this case we could determine λ_i explicitly as $\lambda_i = \lim_{\|z\|\to\infty : z_i \neq 0} z_i^{-1} E[\phi_i^0(z)]$.

The matrix $M = (r_{ij}\lambda_i)_{1\leq i,j\leq m}$ is a basic parameter in order to describe the behaviour of a CMPD. In fact it is not hard to show that

$$E[Z(n+1)|Z(n) = z] = zM + h(z) \tag{1}$$

with $h(z) = o(\|z\|)$. Along this paper we will assume that M is a positively regular matrix with Perron-Frobenius eigenvalue ρ and an associated right eigenvector $\mu \in \mathbb{R}_+^m$.

As we indicated in the previous section, the CMPD was introduced in González et al. (2005a). In that paper, under more restrictive conditions than (A1) and (A2), the extinction and the explosion of such a process was investigated in the framework of a more general theory about homogeneous multitype Markov chains. This theory can be also applied here in order to obtain the following result relative to the indefinite growth of a CMPD under (A1) and (A2):

Proposition 1. *Let $\{Z(n)\}_{n\geq 0}$ be a CMPD satisfying (A1) and (A2). If $\rho > 1$, $z^{(0)}$ large enough, and for some $\delta \geq 0$ and every $i \in \{1,\ldots,m\}$*

$$E[\phi_i^n(z)^{1+\delta}]E\left[|X^{i,n,1}(z)\mu - E[X^{i,n,1}(z)\mu]|^{1+\delta}\right] = O(\|z\|^\delta)$$

and

$$E[|\phi_i^n(z) - E[\phi_i^n(z)]|^{1+\delta}](E[X^{i,n,1}(z)\mu])^{1+\delta} = O(\|z\|^\delta)$$

then $P[\|Z(n)\| \to \infty | Z(0) = z^{(0)}] > 0$.

Now the question that arises is to determine the rate of growth of a CMPD provided the conditions of Proposition 1 are fulfilled. González et al. (2005b) show that the CMPD grows geometrically and ρ itself is the rate of growth of the process. More specifically these authors prove the following result:

Theorem 1. *Let $\{Z(n)\}_{n\geq 0}$ be a CMPD satisfying (A1) and (A2) with Perron - Frobenius eigenvalue $\rho > 1$. Given α, $1 \leq \alpha \leq 2$, suppose that there exist sequences $\{\overline{h}^r(n)\}_{n\geq 1}$, $\{\overline{h}^c(n)\}_{n\geq 1}$ and $\{\overline{g}_\alpha(n)\}_{n\geq 1}$ such that*

i) For each non-null vector z

$$\max_{1\leq i,j\leq m}\{|h^r_{ij}(z)|\} \leq \overline{h}^r(z\mathbf{1}), \quad \max_{1\leq i\leq m}\{|h^c_i(z)|\} \leq (z\mathbf{1})\overline{h}^c(z\mathbf{1}),$$

and

$$\max_{1\leq i,j\leq m}\{\|\phi^0_i(z) - E[\phi^0_i(z)]\|_\alpha, \|\phi^0_i(z)\|_\alpha \|X^{i,0,1}_j(z) - E[X^{i,0,1}_j(z)]\|_\alpha\} \leq \overline{g}_\alpha(z\mathbf{1}),$$

with $\|\cdot\|_\alpha$ denoting the norm in L^α.

ii) $\{\overline{h}^r(n)\}_{n\geq 1}$, $\{\overline{h}^c(n)\}_{n\geq 1}$ and $\{n^{-1}\overline{g}_\alpha(n)\}_{n\geq 1}$ are non-increasing and

$$\sum_{n=1}^{\infty} n^{-1}\overline{h}^r(n) < \infty, \ \sum_{n=1}^{\infty} n^{-1}\overline{h}^c(n) < \infty \ \text{ and } \ \sum_{n=1}^{\infty} n^{-2}\overline{g}_\alpha(n) < \infty.$$

Then the sequence $\{\rho^{-n}Z(n)\}_{n\geq 0}$ converges almost surely and in L^α to a finite and non-degenerate at $\mathbf{0}$ random vector W.

This result is a generalization of the Kesten-Stigum Theorem for the classical multitype Galton-Watson process (see Kesten and Stigum (1966)) and, intuitively, states that $Z(n) \sim W\rho^n$, as n goes to infinity, almost surely and in L^α, $1 \leq \alpha \leq 2$, so the rhythms of growth of the coordinates of the process $\{Z(n)\}_{n\geq 0}$ and the sequence $\{\rho^n\}_{n\geq 0}$ are similar.

3. Geometric growth: L^2-convergence

The question about the geometric growth of a CMPD seems to be answered in the previous section. However, the techniques used to obtain the convergence in L^α, $1 \leq \alpha \leq 2$, do not consider the Hilbertian structure of L^2. In the following result we provide a new set of conditions for the L^2-convergence of $\{\rho^{-n}Z(n)\}_{n\geq 0}$ that, in some situations, can improve the hypothesis of Theorem 1 for $\alpha = 2$.

Theorem 2. *Let $\{Z(n)\}_{n\geq 0}$ be a CMPD satisfying (A1) and (A2) with Perron - Frobenius eigenvalue $\rho > 1$. Suppose that there exist sequences $\{\overline{h}^r(n)\}_{n\geq 1}$, $\{\overline{h}^c(n)\}_{n\geq 1}$ and $\{\overline{d}(n)\}_{n\geq 1}$ such that* *i) For each non-null vector* z

$$\max_{1\le i,j\le m}\{|h^r_{ij}(z)|\}\le \overline{h}^r(z\mathbf{1}),\quad \max_{1\le i\le m}\{|h^c_i(z)|\}\le (z\mathbf{1})\overline{h}^c(z\mathbf{1}),$$

$$\max_{1\le i_1,i_2\le m}\{|\operatorname{Cov}[\phi^0_{i_1}(z),\phi^0_{i_2}(z)]|\}\le \overline{d}(z\mathbf{1})$$

and

$$\max_{1\le i_1,i_2,j_1,j_2\le m}\{E[\phi^0_{i_1}(z)]E[\phi^0_{i_2}(z)]\,|\operatorname{Cov}[X^{i_1,0,1}_{j_1}(z),X^{i_2,0,1}_{j_2}(z)]|\}\le \overline{d}(z\mathbf{1}).$$

ii) $\{\overline{h}^r(n)\}_{n\ge1}$, $\{\overline{h}^c(n)\}_{n\ge1}$ *and* $\{n^{-2}\overline{d}(n)\}_{n\ge1}$ *are non-increasing and*

$$\sum_{n=1}^{\infty}n^{-1}\overline{h}^r(n)<\infty,\ \sum_{n=1}^{\infty}n^{-1}\overline{h}^c(n)<\infty\ \text{ and }\ \sum_{n=1}^{\infty}n^{-3}\overline{d}(n)<\infty.$$

Then the sequence $\{\rho^{-n}Z(n)\}_{n\ge0}$ *converges almost surely and in* L^2 *to a finite and non-degenerate at* $\mathbf{0}$ *random vector* W*.*

Proof. Consider the CMPD as a homogeneous multytipe Markov chain and let us show that Theorem 6 in González et al. (2006) applies. In order to check the hypotheses of such a theorem, let us introduce some notation. Denote by $\{\mu^{(1)},\dots,\mu^{(m)}\}$ a basis of complex right generalized eigenvectors of M such that $\mu^{(1)}=\mu$. For each $z\in\mathbb{N}_0^m$ and $i,j\in\{1,\dots,m\}$, we define $G^{(i,j)}(z)$ by

$$G^{(i,j)}(z)=\operatorname{Cov}[Z(n+1)\widetilde{\mu}^{(i)},Z(n+1)\widetilde{\mu}^{(j)}|Z(n)=z].$$

Notice that $G^{(i,i)}(z)=\operatorname{Var}[Z(n+1)\widetilde{\mu}^{(i)}|Z(n)=z]$ for all $i\in\{1,\dots,m\}$.

Since $h^r_{ij}(z)$ and $h^c_i(z)$ are conveniently bounded so is $h(z)\mu^{(i)}$, $i\in\{1,\dots,m\}$, with $h(z)$ as defined in (1). Now we only have to find suitable bounds for $G^{(l_1,l_2)}(z)$, $l_1,l_2\in\{1,\dots,m\}$. But it is a matter of straightforward computations to show that

$$|G^{(l_1,l_2)}(z)|\le\sum_{\substack{j_1,j_2=1\\ i_1,i_2=1}}^{m}|\mu^{(l_1)}_{j_1}||\mu^{(l_2)}_{j_2}|\Big(\,E[\phi^0_{i_1}(z)]E[\phi^0_{i_2}(z)]\,|\operatorname{Cov}[X^{i_1,0,1}_{j_1}(z),X^{i_2,0,1}_{j_2}(z)]|$$
$$+\,E[X^{i_1,0,1}_{j_1}]E[X^{i_2,0,1}_{j_2}]\,|\operatorname{Cov}[\phi^0_{i_1}(z),\phi^0_{i_2}(z)]|\,\Big)$$

and each of the summands in the brackets is conveniently bound by hypothesis, and therefore the proof is finished. □

Remark 1. The bounds given for both, the reproduction and the control variances in Theorem 2 are weaker than those given in Theorem 1 for $\alpha = 2$. Indeed, in Theorem 1,

$$\max_{1\leq i_1,i_2\leq m} \mathrm{Var}[\phi^0_{i_1}(z)] \leq \overline{g}^2_2(z\mathbf{1})$$

and

$$\max_{1\leq i_1,j_1\leq m}\{E[\phi^0_{i_1}(z)]^2\ \mathrm{Var}[X^{i_1,0,1}_{j_1}(z)]\}\leq \overline{g}^2_2(z\mathbf{1}).$$

Taking $\overline{d}(n) = \overline{g}^2_2(n)$, the sequence $\{n^{-2}\overline{d}(n)\}$ is non-increasing and

$$\sum_{n=1}^{\infty}\frac{\overline{d}(n)}{n^3} = \sum_{n=1}^{\infty}\frac{\overline{g}(n)}{n}\frac{\overline{g}(n)}{n^2} < +\infty,$$

so the conditions of Theorem 2 hold for the variances.

On the other hand, in Theorem 1 we do not assume conditions for the covariances. In this sense Theorem 2 can be viewed as an improvement of Theorem 1 only in some particular situations. In general, Theorem 2 just provides a new set of conditions for the L^2-convergence of the CMPD suitably normed.

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