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STABILITY OF THE INVENTORY-BACKORDER PROCESS IN THE (R, S) INVENTORY/PRODUCTION MODEL

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The aim of this paper is to obtain the sufficient conditions for the uniform ergodicity and the strong stability of the inventory-backorder process in a single-item, single location, (R, S) inventory/production model with limited capacity of production per period and uncertain demands. In this order some intermediate results are established and an overview about the main stability methods for stochastic processes and the performance measure in the inventory models are also considered.

1. Introduction

A stochastic inventory models are generally a complex system which includes many parameters which can be deterministic or stochastic. The first contribution to analyze the inventory model under the deterministic demand is published in 1913 by F.W. Harris [4]. The research concerning the inventory model under assumption that the demand is a stochastic process were considered earlier by Arrow, Harris and Marsschak [17]. It is well known that the general inventory model is usually depend on a large class of parameters and so the real system is often generally considered as a complex system which depends in complicated way on its parameters. It is out put to precise that the inventory models are the first stochastic models for which the qualitative property of monotonicity is

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established [9, 19]. Thus, Boylan have proved, in 1969, that the solution of the optimal inventory equation, which determines the dynamic of the model, depends continually on its parameters including the demand distribution which is the main parameter [2].

Usually, the performance measures of inventory systems are generally calculated in terms of cost, which depends on the parameter of the model, and recently Chen and Zheng (1997) have showed that the inventory cost in an (R, s, S) model is relatively insensitive to the changes in $D = S - s$. However, in certain industries, customer waiting times, backorder or shortfall level have become more important as performance measure [12, 11]. Unfortunately, the parameter's system are not often known exactly because they are obtained by statistical methods from empirical data. Therefore, the analysis of the performance measure (depending on the parameter of the model) of this type of complex systems don't allow us to obtain explicitly analytical formula. Also, if we are able to establish analytical results they are not generally useful in practice.

For this, we try usually to replace the complex system, by an other which is more simpler in structure and/or component, and close to it in some particular sense. So the real complex system can be considered as a perturbed of some parameters of the simpler one which may be considered that is the ideal system. However, in order to justify these approximations and estimate the resultant error, it is of interest to precise the kind and the type of this perturbation and so the stability problem arises. In particular, the stability problem in inventory model theory arises to establish the domain within which an ideal inventory system may be considered as a good approximation, in some sense, to the real complex inventory system under consideration.

The previous approximation must be precise in what sense that is means. For this, various methods to investigate the stability problem are developed. The first type of methods which are considered concerns the quantitative stability of stochastic sequence as the metric method [21], test function method [8], method of the proximity points [7], the renevating events or renewal method [1], the weak convergence method [16] applied for queuing system, stability method of nonhomogeneous discret Markov chain with continuous time [20], the method of the uniform stability [6] applied for the finite irreductible Markov chain and the strong stability method [10] for homogeneous Markov chain in general state space. The second type of existing methods concerns the qualitative stability as the stochastic stability [13], the stability and continuity of denumerable Markov chains method [3] and the method of the perturbation of a special form [5, 18].

In this paper, we established the sufficient conditions for the qualitative property of the strong stability of the inventory backorder process considering under an (R, S) inventory/production model with uncertain demands in which a production capacity is limited per period and generally random leadtime of replenishment. The treatments's cause of this type of model is suggested by Maasaaki Kijima and Tetsu Takimoto [12].

Therefore, our strong stability investigation, and in contrast to other methods such that are cited previously, allow us to obtain the error of approximation of the stationary and non stationary characteristics for two inventory-production backorder processes considered in (R, S) inventory-production model (Σ) (ideal model) and $(\widetilde{\Sigma})$ (real model) under different demand distribution. Namely, we are interesting about the effect of perturbation of the total demand distribution on the behavior of the inventory-production backorder process.

This paper is organized as follows: in section 2., we give a briefly description of the (R, S) inventory/production model with uncertain demands in which a capacity production is limited per period. In particular, we will interesting about the inventory-backorder process. In the third section we introduce the strong stability approach and all some notations which we need in this paper. Sections 4. and 5. are avoided to compute the transition kernel and to establish ergodicity condition respectively of the inventory backorder process. The strong stability and the uniform ergodicity of the backorder process is established in section 6.. Moreover, the rate of convergence of the t-fold power to the stationary projector of the process is also obtained. Finally, we give a concluding remarks where we specify some research perspectives.

2. Description of the model

We consider the single-item, single location, (R, S) inventory/production model (Σ) with limited capacity of production per period and uncertain demands. Let us fix $R = 1$ in order to define the time unit appropriately and the time interval $]n-1, n]$ is called the n-th period. Let D_n represent the total demand during period n. When sufficient stock is available for the demand, it is served at the end period. Otherwise the available stock is supplied, the remaining demand is backordered and the customer wait until the demand is fully satisfied. We assume there are an infinite waiting room for the waiting customers and they are served on an FCFS basis. Moreover, the inventory position IP is reviewed at the end of each period (at the beginning of the next period) and a replenishment order is placed to the production system in order to raise the IP to the fixed level S. Then, we introduce the following basic notations, which have needed in the sequel of this section. First S, I_n, B_n and J_n are the order-up-to-level which is strictly positive, physical inventory level, backorder and inventory/backorder level which is equal to $I_n - B_n$, respectively, at the end of period n. Moreover, we denote c the positive fixed capacity production. Replenishment orders are produced in the production system with a finite capacity c and are delivred at the end each period, if any, to satisfy the customer demand. For more details about the structure and the policy of this model, we have to consult [12, 11].

Assume that $D_n, n = 0, 1, \ldots$ are an independent identically distributed nonnegative, integer-valued random variables with common probabilities

$$d_k = \mathbb{P}(D_1 = k), \;\; k = 0, 1, \ldots$$

Let us consider another (R, S) inventory-production model $(\widetilde{\Sigma})$ with the same structure (limited capacity c of production and random leadtime), but with different demands $\widetilde{D}_n, n = 0, 1, \ldots$ having the common probability

$$\widetilde{d}_k = \mathbb{P}(\widetilde{D}_1 = k) \;\; k = \ldots, 0, 1, \ldots$$

Let us consider $J = \{J_n : n = 0, 1, \ldots\}$ and $\widetilde{J} = \{\widetilde{J}_n : n = 0, 1, \ldots\}$ the embedded Markov chains corresponding to the on-hand backorder level at the end of periods and we denote by P and Q their respective transition operators and by E their common state space.

3. Notations and definitions

We notice that all notations used in this paper are introduced in many references [10, 14, 15] and adapted for the homgeneous discrete non finite Markov chain.

Let us consider $J = (J_n, n \in \mathbb{N})$ an homogeneous Markov chain taking values in a measurable state space $E = \{\ldots, -2, -1, 0, 1, \ldots, S\}$ and defined on the phase space $(E, \mathcal{E})$ where $\mathcal{E}$ is the countably algebra generated by the singletons of the set E. Moreover, we assume that the chain X have a regular transition kernel $P(x, A), \;\; x \in E, \;\; A \in \mathcal{E}$ with a unique and finite stationary distribution π.

Furthermore, we provide the space $m\mathcal{E} = \{\mu_j\}_{j \in E}$ of finite measures on the σ-algebra $\mathcal{E}$ with some weight norm $\|.\|_v$ which have the following form

$$\|\mu\|_v = \sum_{j \in E}^{S} v(j)|\mu_j| = \sum_{j=-\infty}^{S} v(j)|\mu_j| \tag{1}$$

where v is a $\mathcal{E}$-measurable function bounded below by a positive constant (not necessary finite), i.e, $\inf_{i \in E} v(i) = k > 0$. So it is obvious that the subspace

$\mathcal{M} = \{\mu \in m\mathcal{E} : \|\mu\|_\beta < \infty\}$ has a Banach space structure with respect to the norm $\|.\|_v$.

For all measure $\mu = (\mu_j)_{j\in E}$ belonged to $m\mathcal{E}$, then the induced corresponding norms in the space $\mathcal{J}$ of measurable $\mathcal{E}$-function on E and the space $\mathcal{B}$ of linear operators on $\mathcal{M}$ to $\mathcal{M}$ have, respectively, the following form

$$\|f\|_v = \sup_{j\in E} \frac{|f(j)|}{v(j)} \ \forall f \in \mathcal{J}$$

and

$$\|Q\|_v = \sup_{j\in E} \frac{1}{v(j)} \sum_{i=0}^{+\infty} |Q|_{ij} v(i) \ \ \forall Q \in \mathcal{B}$$

The action of each transition kernel $Q \in \mathcal{B}$ on $\mu = (\mu_j)_{j\in E} \in m\mathcal{E}$ and $f \in \mathcal{J}$, is defined as follows

$$(\mu Q)_i = \sum_{j\in E} Q_{ij}\mu_j \,\forall i \in \mathcal{E}$$

and

$$Qf(i) = \sum_{j\in E} Q_{ij} f(j) \,\forall i \in E$$

We introduce also the operator $f \circ \mu$ defined on $E \times \mathcal{E}$ as follows: for all $(i, \{j\}) \in E \times \mathcal{E}$, we have

$$f \circ \mu(i, \{j\}) = f(i)\mu(\{j\}) = f(i)\mu_j$$

Moreover, we denote the product PQ of two transition kernels P and Q, the kernel defined by

$$(P.Q)_{ij} = \sum_{k\in\mathbb{N}} P_{ik}Q_{kj} \ \forall (i,j) \in E \times E.$$

Remark 1. *See for example N.V.Kartashov in [10] and D. Revuz for the construction of the trial (test or Lyapounov) function for different classes of Markov chains.*

We denote the stationary projector of the chain J by $\Pi = \mathbf{1} \circ \pi$ where $\mathbf{1}$ est the function identically equal to the unit.

Definition 1. *The chain J is said to be uniformly ergodic with respect to the norm $\|\cdot\|_v$ if it has a unique invariant measure π and*

$$\lim_{t\to+\infty} \|t^{-1} \sum_{n=1}^{t} P^t - \Pi\|_v = 0$$

Let us denote the stationary projector Π of the chain X by $\Pi = \mathbf{1} \circ \pi$, where π is the invariant probability measure of the transition kernel P of the chain X.

Definition 2. *The chain X is said to be uniformly ergodic with respect to the norm $\|.\|_v$ if it has a unique invariant measure π and*

$$\lim_{t \to +\infty} \|t^{-1} \sum_{n=1}^{t} P^t - \Pi\|_v = 0$$

Definition 3. *The chain X is said to be strongly stable with respect to the norm $\|.\|_v$ if*

1) $\|P\|_v < \infty$

2) Each transition kernel Q in some neighborhood $\{Q : \|Q - P\|_v < \epsilon\}$, has a unique invariant measure $\nu = \nu(Q)$ and we have $\|\pi - \nu\|_v \longrightarrow 0$ as $\|Q - P\|_v \longrightarrow 0$ uniformly in this neighborhood.

The main purpose of this paper is to investigation the sufficient conditions of the strong stability of our backorder process. In this order, we will apply the strong stability criterion given by the following theorem 1.

Theorem 1 ([10], Theorem 2.3, p. 29) *Assume that a Markov chain X with the regular transition operator P has a unique invariant probability measure π that satisfies the following conditions*

A) $\|P\|_v < \infty$

B) There exist a natural n, a measure $\alpha \in m\mathcal{E}^+$, a function $h \in \mathcal{J}^+$ such that $\pi h > 0$, $\alpha \mathbf{1} = \alpha(E) = 1$, $\alpha h > 0$ and the residual transition kernel

$$T = P^n - h \circ \alpha$$

is nonnegative.

Then, the Markov chain X is strongly stable and aperiodic with respect to the norm $\|.\|_v$ if and only if we have

(T) $\|T^m\|_v \leq \rho$ for some $m \geq 1$, $\rho < 1$, where the kernel T is defined according to the condition (B) for the same n, α and h from condition (B).

Furthermore, the uniform ergodicity and aperiodicity of the chain X under condition (A) imply that condition (T) is fulfilled for all n, α and h satisfying condition (B).

Remark 2. *Under condition $\|P\|_v < \infty$, the concept of the strong stability and uniform ergodicity are equivalent. For more details, we may consult the monograph of N.V. Kartashov [10].*

4. Transition kernel

We start our investigation by computing the regular transition kernel $P(x, A)$ of the backorder Markovian process J. In this order, from Kijima and Takimoto [12] we have the following recursive equation

$$J_{n+1} = J_n + \min\{c, S - J_n\} - D_{n+1} \tag{2}$$

Since, J_{n+1} depends only on J_n and D_{n+1} where D_{n+1} is independent of n, then the inventory-backorder process $\{J_n\}$ is an homogeneous Markovian process with a phase space $(E, \mathcal{E})$ where the state space $E = \{\dots, -2, -1, 0, 1, \dots, S\}$ and $\mathcal{E}$ is the σ-algebra generated by the singletons of E.

Let us compute the one step transition probabilities. For this, suppose that at $t_n = nR$ we have $J_n = i$, then we get

$$P_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i) = \mathbb{P}(D_{n+1} = i - j + \min(c, S - i)), \ \forall (i, j) \in E \times E.$$

So if we denote by $d_i = \mathbb{P}(D_1 = i)$ for all $i \in E$, thus we have to consider some cases:

a) If $i \geq S - c$, then we have

$$P_{ij} = \mathbb{P}(D_{n+1} = S - j) = d_{S-j}.$$

b) If $i < S - c$, then we have

$$P_{ij} = \mathbb{P}(D_{n+1} = i - j + c).$$

Thus, we distinguish the following two cases:

for $j \leq i + c$, we obtain

$$P_{ij} = \mathbb{P}(D_{n+1} = i - j + c) = d_{i-j+c}.$$

For $j > i + c$, we get

$$P_{ij} = \mathbb{P}(D_{n+1} = i - j + c) = 0.$$

In other way, we have for $i < S - c$

$$P_{ij} = \mathbb{P}(D_{n+1} = i - j + c) = \begin{cases} d_{i-j+c} & j \leq i + c \\ 0 & j > i + c. \end{cases}$$

Hence, we get

$$P_{ij} = \begin{cases} d_{S-j} & i \geq S-c \quad j \in E \\ d_{i-j+c} & i < S-c \quad j \leq i+c \\ 0 & i < S-c \quad j > i+c. \end{cases}$$

Finally, the transition kernel P of the homogeneous inventory backorder process J is represented by the following infinite transition matrix which has the following form:

$(P_{ij}) =$

	$\dots$	o	$\dots$	$c-1$	c	$\dots$	$S-2$	$S-1$	S
$\vdots$	$\vdots$	$\dots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$
0	$\dots$	d_c	$\dots$	d_1	d_0	$\dots$	0	0	0
1	$\dots$	d_{c+1}	$\dots$	d_2	d_1	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$S-c-1$	$\dots$	d_{S-1}	$\dots$	d_{S-c}	d_{S-c-1}	$\dots$	d_1	d_0	0
$S-c$	$\dots$	d_S	$\dots$	d_{S-c+1}	d_{S-c}	d_{S-c-1}	$\dots$	d_1	d_0
$S-c+1$	$\dots$	$\dots$	d_S	d_{S-c+1}	d_{S-c}	d_{S-c-1}	$\dots$	d_1	d_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S	$\vdots$	$\dots$	d_S	d_{S-c+1}	d_{S-c}	d_{S-c-1}	$\dots$	d_1	d_0

5. Ergodicity conditions

In order to establish conditions for the ergodicity of the embedded Markov chain J, we need to use the following change of variables

(3) $$X_n = S - J_n - D_n \text{ for all } n = 0, 1, \dots$$

and

$$\xi_n = D_n - c \text{ for all } n = 0, 1, \dots.$$

Then, we get the following well known random walk recursive equation

(4) $$X_{n+1} = \max(0, X_n + \xi_n) \text{ for all } n = 0, 1, \dots.$$

Then, $X = \{X_n, n = 0, 1, \ldots\}$ is a stochastic process with the infinite space state $\{0, 1, \ldots\}$. Since $(D_n)_n$ is the independent identically distributed random variables, so the $(\xi_n)_n$ is also an iid random variables. In this case, the process X is called the Lindley process and it is easily showed extensively in the literature [1], and especially in queuing theory, that under the assumption $\mathbb{E}(\xi_1) < \infty$ the process X is ergodic if and only if the following condition $\mathbb{E}(\xi_1) < 0$ hold. Consequently, the process X is ergodic under the following condition

$$c > \mathbb{E}(D_1). \tag{5}$$

From equation (3) and by the fact that $D_n : n = 0, 1, \ldots$ is the independent identically distributed random variables, so we deduce that X admits a stationary distribution if and only if J admits also one. Therefore, we obtain that the homogeneous Markov process J is ergodic under the same condition (5).

Remark 3. *This condition is meaningful because the stability of the model is guaranteed if the capacity production is greater than the mean of the demand* D_1.

In other words, the homogeneous Markov chain J admits a unique stationary distribution $\pi = \{\pi\}_{j\in E} \in \mathcal{E}$ corresponding to the unique solution of the following linear infinite equation's system

$$\begin{cases} \pi P = \pi \\ \sum\limits_{j\in E} \pi_j = \sum\limits_{j=-\infty}^{S} \pi_j = 1. \end{cases} \tag{6}$$

6. Strong stability analysis

We consider the test function

$$\begin{aligned} v: \; E &\longrightarrow \mathbb{R}_+^* \\ j &\longrightarrow v(j) = \beta^{-j}, \end{aligned}$$

where $\beta > 1$ is a parameter and $E = \{\ldots, -2, -1, 0, 1, \ldots\}$ is the measurable state space of the Markov chain J. We denote by $\mathcal{E}$ the countably algebra generated by the singletons of the set E. Hence, the special weight variation norm given by relation (7) in the space $m\mathcal{E}$ of the finite measures $\mu = \{\mu_j\}_{j\in E}$ has the following form

$$\|\mu\|_v = \sum_{j\in E} v(j)|\mu|(\{j\})| = \sum_{j\in E} \beta^{-j}|\mu_j| = \sum_{j=-\infty}^{S} \beta^{-j}|\mu_j|. \tag{7}$$

We also assume the following cramér condition

$$\exists a > 0 : \mathbb{E}[x^D] < \infty. \tag{8}$$

In order to establish the strong stability of the embedded homogeneous Markov chain J, under the same condition (5), with respect to the norm $\|.\|_v$ where $v(j) = \beta^{-j}$ for all $j \in E$ and β belonging to a convenient interval, we need some intermediate results.

Lemma 1. *Assume that the following conditions (5) and (8) hold. Then it exists $\omega \in]1, +\infty[$ such that for all $x \in]1, \omega[$ we have $\mathbb{E}(\omega^{D_1-c}) < 1$, i.e.,*

$$\frac{\sum\limits_{l=0}^{+\infty} \omega^l d_j}{\omega^c} < 1.$$

P r o o f. Let us consider the $\mathcal{E}$-measurable function g defined on $[0, +\infty[$ by $g(x) = \mathbb{E}(x^{D_1-c}) = \sum\limits_{l=0}^{+\infty} x^{l-c} d_j$. Then, according to condition (8), the function g is continous and diffentiable in $[0, a[$. Furthemore, according to condition (5) we have $g'(1)) = \mathbb{E}(D_1) - c < 0$. Moreover, it is not hard to see that g is a convex function and since $g(1) = 1$, then it exists $\omega \in]1, +\infty[$ such that $g(\omega) = 1$ and for all $x \in]1, \omega[$ we have $g(x) < 1$. □ Now let us consider the measure α defined on the countably generated σ-algebra $\mathcal{E}$ by

$$\forall\, j \in E \,:\, \alpha(\{j\}) = \alpha_j = d_{S-j}.$$

Moreover, we introduce the $\mathcal{E}$-measurable function h defined as follows $h(i) = h_i = \mathbb{I}_{\{i\geq S-c\}}$, explicitly given as below

$$h(i) = h_i = \begin{cases} 1 & i \geq S-c \\ 0 & i < S-c. \end{cases}$$

Then, we have the following decomposition of the transition kernel P of the Markovian chain J.

Lemma 2. *The transition kernel P of the Markovian process J admits the following canonical decomposition*

$$P = T + h \circ \alpha \tag{9}$$

where the residual kernel T is a nonnegative operator and further $\alpha \boldsymbol{1} = 1$.

Proof. Setting $T = (T_{ij})$, then we have

$$T_{ij} = T(i, \{j\}) = P_{ij} - h_i \alpha_j = \begin{cases} P_{ij} - d_{S-j} & i \geq S - c \\ P_{ij} & i < S - c. \end{cases}$$

Hence, we obtain

$$T_{ij} = P_{ij} - h_i \alpha_j = \begin{cases} 0 & i \geq S - c \quad j \in E \\ d_{i-j+c} & i < S - c \quad j \leq i + c \\ 0 & i < S - c \quad j > i + c \end{cases}$$

Consequently, T is a nonnegative operator and $P = T + h \circ \alpha$.

Furthermore, we have $\alpha \mathbf{1} = \sum_{j=-\infty}^{S} \alpha_j = \sum_{l=0}^{+\infty} d_l = 1$. □

Lemma 3. *Under conditions of lemma 1, we get $\alpha h > 0$ and $\pi h = d_0^{-1} \pi_S > 0$.*

Proof. We have

$$\alpha h = \sum_{j=S-c}^{S} \alpha(\{j\}) h(j) = \sum_{j=S-c}^{S} d_{S-j} = \sum_{j=0}^{c} d_l > 0.$$

Moreover, the system of linear equations (6), leads to the following identity

$$\pi h = \sum_{j=-\infty}^{S} h_j \pi_j = d_0^{-1} \pi_S. \tag{10}$$

Moreover, if $\pi h = 0$ then we obtain $\pi_S = 0$ and by induction we obatin $\pi_i = 0$ for all $i \in E$ which is impossible and so $\pi h > 0$. Hence, the proof is finished. □

Lemma 4. *Suppose that conditions of lemme 1 hold. Then, it exists* $\rho = \rho(\beta) \in [0,1[$ *such that*

$$Tv(k) \leq \rho v(k) \ \textit{for all } k \in E \textit{ and } \beta \in]1,\omega[,$$

where

$$\rho(\beta) = \mathbb{E}(\beta^{D_1 - c}) = \frac{\mathbb{E}(\beta^{D_1})}{\beta^c}. \tag{11}$$

Proof. We have to compute $Tv(k)$ for all $k \in E$. Indeed, we have for all $k \in E$, $Tv(k) = \sum_{j=-\infty}^{S} v(j) T_{kj}$ and some cases must be considered.

1) If $k < S - c$, then we get

$$\begin{aligned} Tv(k) &= \sum_{j=-\infty}^{k+c} \beta^{-j} d_{k-j+c} = \sum_{l=0}^{+\infty} \beta^{l-k-c} d_l \\ &= \beta^{-k} \sum_{l=0}^{\infty} \beta^{l-c} d_l = v(k) \sum_{l=0}^{\infty} \beta^{l-c} d_l \end{aligned}$$

2) If $k \geq S - c$, then $Tv(k) = 0$ because $T_{kj} = 0$ for all $k \geq S - c$.

Putting $\rho\beta) = \mathbb{E}(\beta^{D_1 - c})$, we obtain $Tv(k) \leq \rho(\beta) v(k)$ for all $k \in E$.

Moreover, according to lemma 1, we have $\rho(\beta) < 1$ for all $\beta \in]1, \omega[$. Finally, we have established that $Tv(k) \leq \rho(\beta) v(k)$ for all $k \in E$ where $0 < \rho(\beta) < 1$ and

$$\rho(\beta) = \mathbb{E}(\beta^{D_1 - c}) = \frac{\mathbb{E}(\beta^{D_1})}{\beta^c}.$$

So the result is established. □

Proposition 1. *Under conditions of lemma 1, we have* $\|T\|_v \leq \rho = \rho(\beta)$.

Proof. Using the inequality $Tv(k) \leq \rho v(k)$ for all $k \in E$, then we have

$$\begin{aligned} \|T\|_v &= \sup_{k \in E} \frac{1}{v(k)} \sum_{j=-\infty}^{S} T(k, \{j\}) v(j) = \sup_{k \in E} \frac{1}{v(k)} \sum_{j=-\infty}^{S} T_{kj} v(j) \\ &= \sup_{k \in E} \frac{1}{v(k)} Tv(k) \leq \sup_{k \in E} \frac{1}{v(k)} \rho v(k) = \rho. \end{aligned}$$

□

The following result establishes that the transition operator P of the chain J is bounded with respect to the weight variation norm $\|.\|_v$.

Lemma 5. *Under conditions of lemma 1, the transition operator P is bounded with respect to the norm $\|.\|_v$ and we have*

$$\|P\|_v = \beta^c \rho(\beta) < \infty,$$

where $\rho = \rho(\beta)$ is given by the identity (11).

Proof. Using simple computation, the result is easy to establish. Indeed,

$$\|P\|_v = \sup_{k \in E} \beta^k \sum_{j=-\infty}^{S} P_{kj} \beta^{-j} = \max(A, B),$$

where

$$\begin{aligned} A &= \sup_{k \geq S-c} \beta^k \sum_{j=-\infty}^{S} P_{kj} \beta^{-j} = \sup_{k \geq S-c} \beta^k \sum_{j=-\infty}^{S} \beta^{-j} d_{S-j} \\ &= \sup_{k \geq S-c} \beta^k \sum_{l=0}^{+\infty} \beta^{l-S} d_l = \sum_{l=0}^{+\infty} \beta^l d_l \\ &= \beta^c \rho \end{aligned}$$

and

$$\begin{aligned} B &= \sup_{k < S-c} \beta^k \sum_{j=-\infty}^{S} P_{kj} \beta^{-j} = \sup_{k < S-c} \beta^k \sum_{j=-\infty}^{k+c} \beta^{-j} d_{i-j+c} \\ &= \sup_{k < S-c} \beta^k \sum_{l=0}^{+\infty} \beta^{l-k-c} d_l = \sum_{l=0}^{+\infty} \beta^{l-c} d_l = \rho \end{aligned}$$

Therefore, the result is established. □

The next result establishes the strong stability of the embedded Markov chain J with respect to the norm $\| \cdot \|_v$ in the (R, S) model with uncertain demand, limited production capacity per period and stochastic leadtime order.

Theorem 2. *Suppose that conditions of lemme 1 are satisfied. Then, the discrete inventory bachorder process J in (R, S) inventory-production model with uncertainty demand, limited capacity production and random leadtime is uniformly ergodic, strongly stable and aperiodic, with respect to the norm $\| \cdot \|_v$ for all $\beta \in]1, \omega[$ where ω is defined in lemma 1 and $v(k) = \beta^{-k}$ for all $k \in E$.*

P r o o f. Using lemma 1, 2, 3, 4 and 5, then the strong stability and the aperiodicity of the chain J are directly deduced from theorem 1. Moreover, from the remark 2 the uniform ergodicity of the Markovian chain J is immediately established. □

Remark 4. *The rate of convergence $\rho(\beta)$ decreases to zero when $c \longrightarrow +\infty$. Then, we will expect that the bound of the uniform ergodicity and the strong stability estimates will be kind when the capacity production c is sufficiently greater. In this case, the inventory/production model can be approximated by the classical inventory model (unlimited capacity). The later will be the subject of an other paper.*

Concluding remarks

In this paper, we have investigated the uniform ergodicity and the strong stability of the backorder process in the (R, S) model with randomness demand, limited production capacity per period and with stochastic leadtime order. However, the convergence of the transition operator of the chain to the stationary projector, the estimate of the rate of convergence and some stability estimates of the stationary and non stationary characteristics of the perturbed process under the perturbation of the demand distribution will be presented in a separate paper.

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