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NONPARAMETRIC ESTIMATION IN THE CLASS OF BISEXUAL BRANCHING PROCESSES WITH POPULATION-SIZE DEPENDENT MATING

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In this paper the class of bisexual branching processes with population-size dependent mating is considered. Nonparametric estimators and confidence intervals for the main parameters involved in such a class of stochastic models are provided. For the proposed estimators, the main conditional to non-extinction and unconditional moments are established and some asymptotic properties are investigated. As illustration, a simulated example is given.

1. Introduction.

Recently, in order to describe the probabilistic evolution of populations where females and males coexist and form couples (mating units) which reproduce independently, some branching models have been investigated. In particular, the bisexual process with population-size dependent mating, introduced by Molina et al. (2002), allows that the function governing the mating changes in each generation depending on the total number of mating units in the previous one. Some results concerning its extinction probability and limiting behaviour has been developed in Molina et al. (2002, 2004a, 2004b). This paper aims to continue this research, studying several inferential questions about it. In Section 2, the

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probability model description and some working assumptions are given. Sections 3 is devoted to determining, from a nonparametric setting, maximum likelihood estimators for the offspring probability distribution, the offspring mean vector and covariance matrix, and the growth rate. Their main conditional, assuming non-extinction, and unconditional moments are established and some asymptotic properties are investigated. Confidence intervals are also obtained and an illustrative simulated example is provided. In order to allow a more comprehensible reading, the proofs are relegated to Section 4.

2. The Probabilistic Model.

The bisexual process with population-size dependent mating (BPSDM) is a two–type sequence $\{(F_n, M_n)\}_{n\geq 1}$ defined in the form:

$$Z_0 = N, (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}), \quad Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1}),\ n = 0, 1, \ldots$$

where the empty sum is considered to be $(0, 0)$, N is a positive integer, $\{(f_{ni}, m_{ni}),\ n = 0, 1, \ldots;\ i = 1, 2, \ldots\}$ is a sequence of independent and identically distributed, non-negative, integer-valued random variables and $\{L_k\}_{k\geq 0}$, with $L_k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, is a sequence of mating functions assumed to be monotonic non-decreasing in each component, integer-valued on the integers, and such that $L_k(f, 0) = L_k(0, m) = 0, \quad k, f, m = 0, 1, \ldots$ Intuitively, (f_{ni}, m_{ni}) represents the number of females and males produced by the i-th mating unit in the generation n, their common probability law is called offspring probability distribution. It follows that (F_{n+1}, M_{n+1}) is the total number of females and males in the $(n+1)$-th generation, which form Z_{n+1} mating units according to the mating function L_{Z_n}. It is easy to prove that $\{(F_n, M_n)\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 0}$ are homogeneous Markov chains. When the mating functions L_k are the same for every k, it is obtained as particular case the bisexual branching process introduced by Daley (1968).

Remark. In two-sex populations it is reasonable to allow an individual's mating behaviour to depend on the population size, e.g. it might seem conceivable that by environmental or social changes, or by another factors, the same number of females an males gives rise to different number of mating units in different generations. In this context, the BPSDM could be an appropriate model to describe such a behaviour.

Definition 1. *Given a BPSDM we define the mean growth rates per mating unit as*

$$r_k := E[Z_{n+1}Z_n^{-1} \mid Z_n = k] = k^{-1}E[\ L_k(\ \sum_{i=1}^{k}(f_{ni}, m_{ni})\ )\ ], \quad k = 1, 2, \ldots$$

Note that r_k represents the expected growth rate per mating unit when, in certain generation, there are k mating units.

In what follows we shall consider a BPSDM verifying the working assumptions:

(A1): $L : \mathbb{Z}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, defined by $L(k, x, y) := L_k(x, y)$, is a superadditive function, namely for $k_i \in \mathbb{Z}^+,\ x_i, y_i \in \mathbb{R}^+,\ i = 1, 2$, it verifies

$$L(k_1 + k_2, x_1 + x_2, y_1 + y_2) \geq L(k_1, x_1, y_1) + L(k_2, x_2, y_2).$$

(A2): $r := \lim_{k\to\infty} r_k > 1$, N is such that $P(Z_n \to \infty \mid Z_0 = N) > 0$, and $\{Z_n \to \infty\} = \{W > 0\}$ where W denotes the almost sure limit of $\{r^{-n}Z_n\}_{n\geq 0}$.

Remark. Assumption $A1$ extends the classical superadditivity condition usually imposed to the mating function in the Daley's bisexual branching process literature (see for example Hull (1982)) and expresses the intuitive fact that if the females and males originated by $k_1 + k_2$ mating units coexist together then, the number of matings produced will be greater than that one obtained from the females and males originated by k_1 and by k_2 mating units living separately. Under $A1$, it was proved in Molina et al. (2002) the existence of the growth rate r. Assumption $A2$ is considered in order to investigate asymptotic properties for the proposed estimators. Under $A1$ and $A2$, some necessary and sufficient conditions which guarantee the almost sure and L^1 convergence of $\{r^{-n}Z_n\}_{n\geq 0}$, $\{r^{-n}F_n\}_{n\geq 1}$ and $\{r^{-n}M_n\}_{n\geq 1}$ to non-degenerate random variables have been established in Molina et al.(2004b).

Given an event A, we shall consider, when necessary, the simplified notation $P_A(\cdot) := P(\cdot|A)$, $E_A[\cdot] := E[\cdot \mid A]$, $Var_A[\cdot] := Var[\cdot \mid A]$, $Cov_A[\cdot] := Cov[\cdot \mid A]$.

3. Maximum likelihood Estimation.

3.1. Estimation of the offspring probability distribution

Let us denote by $\mathbf{p} := (p_{k,l},\ (k, l) \in S)$, $\quad S \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$, the offspring probability distribution. We will assume the observation of the entire family tree up to the current n-th generation, namely

$$\{(f_{ij}, m_{ij})\ ,\ i = 0, \ldots, n\ ;\ j = 1, \ldots, Z_i\}.$$

For $(k,l) \in S$, let $Z_{i;k,l} := \sum_{j=1}^{Z_i} \mathbf{1}_{\{(f_{ij},m_{ij})=(k,l)\}}$ and $Y_{n;k,l} := \sum_{i=0}^{n} Z_{i;k,l}$. It is clear that

$$Z_i = \sum_{(k,l)\in S} Z_{i;k,l} \quad \text{and} \quad (F_{i+1}, M_{i+1}) = \sum_{(k,l)\in S} (k,l) Z_{i;k,l}$$

and it is deduced the log-likelihood function

$$\ell_n(\mathbf{p}) = \sum_{i=0}^{n} \log(\ Z_i! \ (\prod_{(k,l)\in S} Z_{i;k,l}! \ )^{-1} \ ) + \sum_{(k,l)\in S} Y_{n;k,l} \log(p_{k,l}).$$

Hence, it is derived the maximum likelihood estimator for $\mathbf{p}$:

$\widehat{\mathbf{p}}_n = Y_n^{*-1} \mathbf{Y}_n$ where $Y_n^* := \sum_{(k,l)\in S} Y_{n;k,l} = \sum_{i=0}^{n} Z_i$ and $\mathbf{Y}_n := (Y_{n;k,l}, \ (k,l) \in S)$.

Consequently,

$$\widehat{p}_{n;k,l} = Y_n^{*-1} Y_{n;k,l}, \quad (k,l) \in S. \tag{1}$$

Next results provide some moments and asymptotic properties for $\widehat{\mathbf{p}}_n$.

Theorem 1. *Let us denote by $Q_n := \{Z_n > 0\}$, then:*

(i) $E_{Q_n}[\ \widehat{\mathbf{p}}_n \ ] = E_{Q_n}[\ Y_n^{*-1} Z_n \ ]\mathbf{p} + E_{Q_n}[\ Y_n^{*-1} \mathbf{Y}_{n-1} \ ]$,

(ii) $Cov_{Q_n}[\ \widehat{\mathbf{p}}_n \ ] = E_{Q_n}[\ Y_n^{*-2} Z_n \ ](\mathbf{J} - \mathbf{p}^t\mathbf{p}) + Cov_{Q_n}[\ Y_n^{*-1}(\ \mathbf{Y}_{n-1} + Z_n\mathbf{p} \ ) \ ]$,

(iii) $E[\ \widehat{\mathbf{p}}_n \ ] = E[\ Y_n^{*-1} Z_n \ ]\mathbf{p} + E[\ Y_n^{*-1} \mathbf{Y}_{n-1} \ ]$,

(iv) $Cov[\ \widehat{\mathbf{p}}_n \ ] = E[\ Y_n^{*-2} Z_n \ ](\mathbf{J} - \mathbf{p}^t\mathbf{p}) + Cov[\ Y_n^{*-1}(\ \mathbf{Y}_{n-1} + Z_n\mathbf{p} \ ) \ ]$,

where $\mathbf{J} := (\ J_{(k,l),(u,v)} \ )_{(k,l),(u,v)\in S}$, $J_{(k,l),(u,v)} := p_{k,l}\delta_{(k,l),(u,v)}$, with $\delta_{a,b}$ being the kronecker delta, and $\mathbf{p}^t$ denotes the transpose vector of $\mathbf{p}$.

Theorem 2. *Let $Q := \{Z_n \to \infty\}$, then for $(k,l) \in S$:*

(i) *On Q, $\hat{p}_{n;k,l}$ is a strongly consistent estimator for $p_{k,l}$.*

(ii) *If P^* is a probability absolutely continuous with respect to P_Q (i.e. $P^* << P_Q$) then, for $x \in \mathbb{R}$:*

(a) $\lim_{n\to\infty} P^* \Big(\ (p_{k,l}(1-p_{k,l}) \ )^{-\frac{1}{2}} Y_n^{*\frac{1}{2}} (\widehat{p}_{n;k,l} - p_{k,l}) \leq x \Big) = \phi(x)$,

(b) $\lim_{n\to\infty} P^* \Big((p_{k,l}(1-p_{k,l})(r-1))^{-\frac{1}{2}} (r^{n+1}-1)^{\frac{1}{2}} (\widehat{p}_{n;k,l} - p_{k,l}) \leq x \Big) = \phi^*(x)$,

where:

$$\phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}dt, \ \phi^*(x) = \int_{0}^{\infty} \phi(xw^{1/2})dF_Q(w), \ F_Q(\omega){=}P_Q(W \leq \omega),$$

and we recall that W *is the almost sure limit of* $\{r^{-n}Z_n\}_{n\geq 0}$.

Notice that, by Lemma 2.3 in Guttorp (1991), we can replace P^* by P_{Q_n} in Theorem 2 and we deduce that $(\ \hat{p}_{n;k,l} \pm \lambda_\alpha(\ \hat{p}_{n;k,l}(1-\hat{p}_{n;k,l}\ )Y_n^{*-1}\ )^{1/2}\ )$ is a $(1-\alpha)$-level asymptotic confidence interval for $p_{k,l}$, where λ_α is such that $\phi(\lambda_\alpha) = 1-\alpha/2,\ \ \alpha \in (0,1)$. Its length depends on the order of magnitude of Y_n^* (see Molina et al. (2004a) for details about the rate of growth of $\{Y_n^*\}_{n\geq 0}$).

3.2. Estimation of the offspring mean vector.

We now consider the estimation of $\boldsymbol{\mu} = (\mu_1, \mu_2) := E[(f_{01}, m_{01})]$. Taking into account (1) and Zehna theorem, it is deduced that the maximum likelihood estimator of $\boldsymbol{\mu}$ based on the entire family tree up to the current n-th generation is:

$$\widehat{\boldsymbol{\mu}}_n = (\widehat{\mu}_{n;1}, \widehat{\mu}_{n;2}) := \sum_{(k,l)\in S} (k,l)\widehat{p}_{n;k,l} = Y_n^{*-1}\sum_{i=0}^{n}(F_{i+1}, M_{i+1}). \tag{2}$$

Remark. It can be proved (see Jagers (1975), p. 24) that (2) is also the maximum likelihood estimator for $\boldsymbol{\mu}$ based on the sample $\{Z_0,\ (F_k, M_k),\ k = 1,\dots,n+1\}$.

Next result provides some conditional and unconditional moments of $\widehat{\boldsymbol{\mu}}_n$. Let us denote by $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,2} := Cov[(f_{01}, m_{01})]$, i.e. the offspring covariance matrix, and by simplicity, let $\boldsymbol{\Psi}_n = (\Psi_{n;1}, \Psi_{n;2}) := \sum_{i=0}^{n}(F_{i+1}, M_{i+1})$.

Theorem 3. *Let* $Q_n := \{Z_n > 0\}$*, then:*

(i) $E_{Q_n}[\ \widehat{\boldsymbol{\mu}}_n\ ] = E_{Q_n}[\ Y_n^{*-1}\boldsymbol{\Psi}_{n-1}\ ]\ +\ E_{Q_n}[\ Y_n^{*-1}Z_n\ ]\boldsymbol{\mu}$,

(ii) $Cov_{Q_n}[\ \widehat{\boldsymbol{\mu}}_n\ ] = E_{Q_n}[\ Y_n^{*-2}Z_n\ ]\boldsymbol{\Sigma} + Cov_{Q_n}[\ Y_n^{*-1}(\ \boldsymbol{\Psi}_{n-1} + Z_n\boldsymbol{\mu}\ )\ ]$,

(iii) $E[\ \widehat{\boldsymbol{\mu}}_n\ ] = E[\ Y_n^{*-1}\boldsymbol{\Psi}_{n-1}\ ]\ +\ E[\ Y_n^{*-1}Z_n\ ]\boldsymbol{\mu}$,

(iv) $Cov[\ \widehat{\boldsymbol{\mu}}_n\ ] = E[\ Y_n^{*-2}Z_n\ ]\boldsymbol{\Sigma} + Cov[\ Y_n^{*-1}(\ \boldsymbol{\Psi}_{n-1} + Z_n\boldsymbol{\mu}\ )\ ]$.

The next theorem establishes some asymptotic properties.

Theorem 4. *Let* $Q := \{Z_n \to \infty\}$*, then for* $i = 1, 2$*, if is verified:*

(i) *On* Q*,* $\widehat{\mu}_{n;i}$ *is a strongly consistent estimator of* μ_i*,*

(ii) *If* $P^* << P_Q$ *then, for* $x \in \mathbb{R}$*:*

(a) $\lim_{n\to\infty} P^* \big(\ (\ \sigma_{ii}^{-1} Y_n^* \ )^{1/2} (\widehat{\mu}_{n;i} - \mu_i) \leq x \ \big) = \phi(x)$*,*

(b) $\lim_{n\to\infty} P^* \big(\ (\ \sigma_{ii}^{-1} (r^{n+1} - 1)(r-1)^{-1} \ )^{1/2} (\widehat{\mu}_{n;i} - \mu_i) \leq x \ \big) = \phi^*(x)$*,*

where ϕ *and* ϕ^* *are the distribution functions introduced in Theorem 2.*

3.3. Estimation of the offspring covariance matrix.

¿From $\widehat{\mathbf{p}}_n$ and $\widehat{\boldsymbol{\mu}}_n$ we derive the maximum likelihood estimator for $\boldsymbol{\Sigma}$:

$$\widehat{\boldsymbol{\Sigma}}_n = \boldsymbol{\Delta}_n - \widehat{\boldsymbol{\mu}}_n^t \widehat{\boldsymbol{\mu}}_n$$

where

$$\boldsymbol{\Delta}_n = (\Delta_{n;i,j})_{i,j=1,2}, \quad \Delta_{n;i,j} = \sum_{(k,l)\in S} a_{ij}(k,l) \widehat{p}_{n;k,l},$$

being $a_{11}(k,l) = k^2, \quad a_{22}(k,l) = l^2, \quad a_{12}(k,l) = a_{21} = kl.$

Thus,

$$\widehat{\sigma}_{n;ij} = \sum_{(k,l)\in S} a_{ij}(k,l) \widehat{p}_{n;k,l} - \widehat{\mu}_{n;i} \widehat{\mu}_{n;j}, \quad i,j = 1,2. \tag{3}$$

Note that, from Theorems 1 and 3, it can be derived the corresponding conditional to non-extinction and unconditional moments of $\widehat{\sigma}_{n;ij}$. On the other hand, applying a similar reasoning to that one used in Theorems 2 and 4, we can establish the following asymptotic properties:

Theorem 5. *For* $i, j = 1, 2$*,*

(i) *On* Q*,* $\widehat{\sigma}_{n;ij}$ *is a strongly consistent estimator for* σ_{ij}*,*

(ii) *If* $P^* << P_Q$*, then for* $x \in \mathbb{R}$*:*

(a) $\lim_{n\to\infty} P^* \left(\ \tau_{ij}^{-\frac{1}{2}} Y_n^{*\frac{1}{2}} (\ \widehat{\sigma}_{n;ij} - \sigma_{ij} \ ) \leq x \ \right) = \phi(x)$*,*

(b) $\lim_{n\to\infty} P^* \Big(\ (\ \tau_{ij}(r-1) \ )^{-\frac{1}{2}} (\ r^{n+1} - 1 \ )^{\frac{1}{2}} (\ \widehat{\sigma}_{n;ij} - \sigma_{ij} \ ) \leq x \ \Big) = \phi^*(x)$*,*

where $\tau_{ij} := E[(\xi^i_{01} - \mu_i)^2(\xi^j_{01} - \mu_j)^2] - \sigma^2_{ij}$, $i, j = 1, 2$, *with* $\xi^{(1)}_{01} := f_{01}$ *and* $\xi^{(2)}_{01} := m_{01}$, *being* ϕ *and* ϕ^* *the functions introduced in Theorem 2.*

Using again Lemma 2.3 in Guttorp (1991), P^* can be replaced by P_{Q_n} in Theorems 4 and 5, and we deduce the $(1-\alpha)$-level asymptotic confidence intervals for μ_i and σ_{ij}, respectively:

$$\left(\widehat{\mu}_{n,i} \pm \lambda_\alpha(\widehat{\sigma}_{n;ii} Y_n^{*-1})^{1/2} \right), \quad i = 1, 2,$$

$$\left(\widehat{\sigma}_{n,ij} \pm \lambda_\alpha(\widehat{\tau}_{n;ij} Y_n^{*-1})^{1/2} \right), \quad i, j = 1, 2,$$

where

$$\widehat{\tau}_{n;ij} = \sum_{(k,l)\in S} (b_i - \widehat{\mu}_{n;i})^2 (b_j - \widehat{\mu}_{n;j})^2 \widehat{p}_{n;k,l} - \widehat{\sigma}^2_{n;ij},$$

being $b_1 = k$, $b_2 = l$, and λ_α such that $\phi(\lambda_\alpha) = 1 - \alpha/2$, $\alpha \in (0, 1)$, with ϕ being the standard Normal distribution function.

3.4. Estimation of the growth rate.

Using the fact that the growth rate r can be obtained as $r = \varphi(\boldsymbol{\mu})$ where $\varphi(x, y) := \lim_{k\to\infty} L_k(kx, ky)$, (see Molina et al. (2002)), from $\widehat{\boldsymbol{\mu}}_n$ we derive the following estimator for r based in the observation of the entire family tree up to the current n-th generation:

$$\widehat{r}_n = \varphi(\widehat{\boldsymbol{\mu}}_n). \tag{4}$$

Taking into account that φ is a continue function and considering that, on Q, $\widehat{\boldsymbol{\mu}}_n$ is a strongly consistent estimator for $\boldsymbol{\mu}$, we can establish the following result:

Theorem 6. *On* Q, $\widehat{r}_n$ *is a strongly consistent estimator for* r.

3.5. Illustrative example.

As illustration, we have considered a bisexual model with population-size dependent mating with the offspring trinomial probability distribution:

$$P(f_{01}{=}k, m_{01}{=}l) {=} \frac{2}{k!\ l!\ (2-k-l)!} 0.55^k 0.20^l 0.25^{2-k-l}, \ k, l = 0, 1, 2;\ k + l \leq 2$$

and we have assumed that the mating between females and males is governed through the sequence of mating functions $\{L_k\}_{k\geq 0}$ where

$$L_k(x,y) = \min\{x, \lfloor 3ky(1+k)^{-1} \rfloor\}, \quad k = 0, 1, \ldots$$

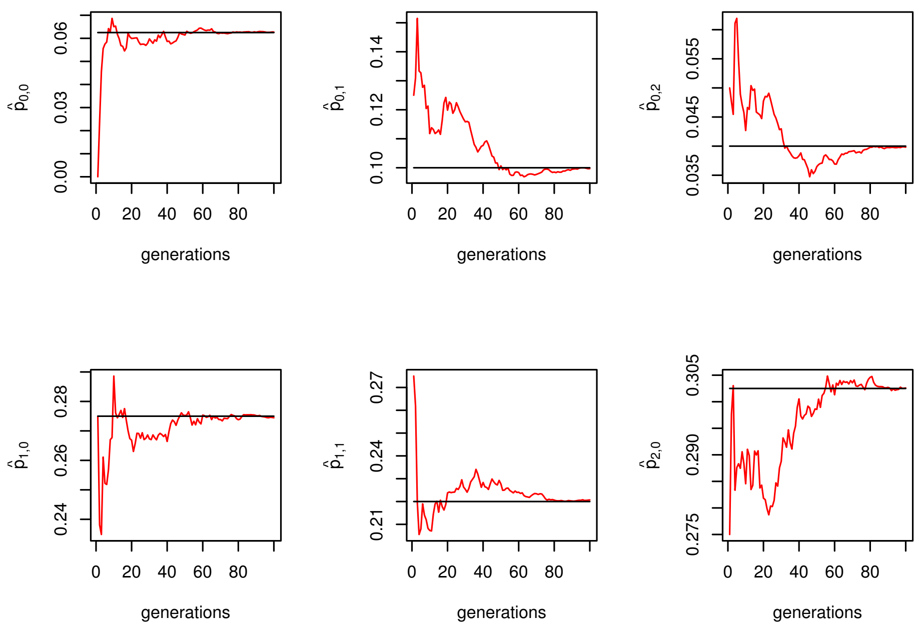
with $\lfloor z \rfloor$ denoting the integer part of z.

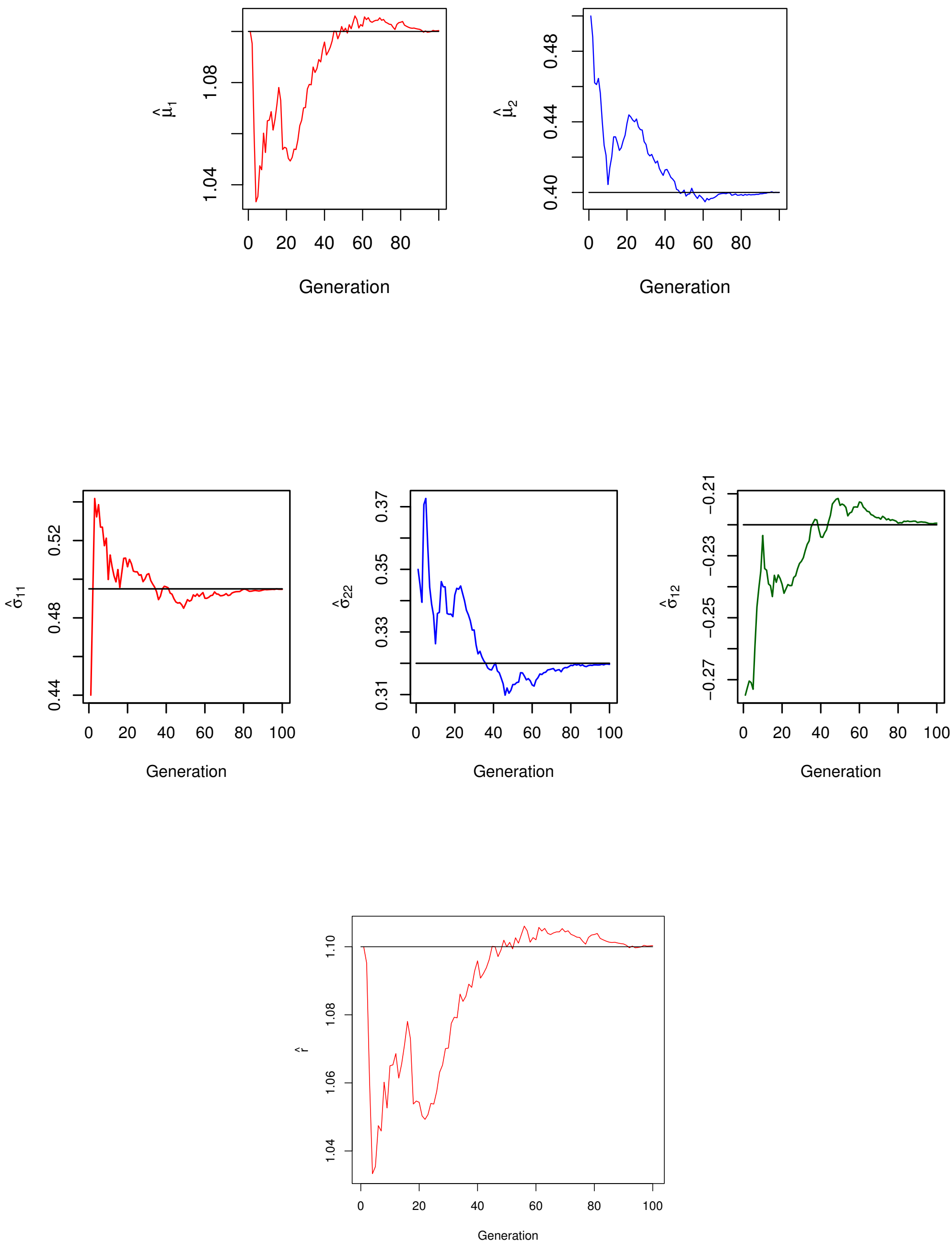
Under these conditions, $S = \{\ (0,0),\ (0,1),\ (0,2),\ (1,1),\ (1,0),\ (2,0)\ \}$ and it is matter of some straightforward calculations to obtain that

$$p_{0,0} = 0.0625,\ p_{0,1} = 0.1,\ \ p_{0,2} = 0.04,\ \ p_{1,0} = 0.275,\ \ p_{1,1} = 0.22,\ p_{2,0} = 0.3025,$$

$$\mu_1 = 1.1, \quad \mu_2 = 0.4, \quad \sigma_{11} = 0.495, \quad \sigma_{12} = -0.22, \quad \sigma_{22} = 0.32, \quad r = 1.1.$$

Starting with $N = 40$ mating units we have simulated a total of 100 generations for such a BPSDM and from (1), (2), (3) and (4), we have calculated the corresponding estimates for $p_{k,l}$, $(k,l) \in S$, μ_i, σ_{ij}, $i,j = 1,2$ and r, respectively. The following graphics, where the horizontal line represents the true value of the parameter, show the evolution of such estimates:





$\hat{\mu}_1$
1.04
1.08
0
20
40
60
80
Generation
$\hat{\mu}_2$
0.40
0.44
0.48
0
20
40
60
80
Generation
$\hat{\sigma}_{11}$
0.44
0.48
0.52
0
20
40
60
80
100
Generation
$\hat{\sigma}_{22}$
0.31
0.33
0.35
0.37
0
20
40
60
80
100
Generation
$\hat{\sigma}_{12}$
−0.27
−0.25
−0.23
−0.21
0
20
40
60
80
100
Generation
$\hat{r}$
1.04
1.06
1.08
1.10
0
20
40
60
80
100
Generation

4. Proofs.

Proof of Theorem 1.

Taking into account that

$$Y_{n;k,l} = Y_{n-1;k,l} + \sum_{j=1}^{Z_n} \mathbf{1}_{\{(f_{nj},m_{nj})=(k,l)\}}, \quad (k,l) \in S,$$

we deduce

$$E[\ \widehat{\mathbf{p}}_n \ \mid\ \mathbf{Y}_{n-1},\ Z_n\ ] = (\ \mathbf{Y}_{n-1} + Z_n\mathbf{p}\ )Y_n^{*-1} \quad a.s.$$

and

$$E[\ \widehat{\mathbf{p}}_n^t\widehat{\mathbf{p}}_n \ \mid\ \mathbf{Y}_{n-1},\ Z_n\ ] = \mathbf{\Phi}_n Y_n^{*-2} \quad a.s,$$

where

$$\mathbf{\Phi}_n := \mathbf{Y}_{n-1}^t\mathbf{Y}_{n-1} + Z_n(Z_n-1)\mathbf{p}^t\mathbf{p} + Z_n(\ \mathbf{p}^t\mathbf{Y}_{n-1} + \mathbf{Y}_{n-1}^t\mathbf{p} + \mathbf{J}\ ),$$

with $\mathbf{J} := (J_{(k,l),(u,v)})_{(k,l),(u,v)\in S}, \quad J_{(k,l),(u,v)} := p_{k,l}\delta_{(k,l),(u,v)}$.
Hence, (i) and (ii) are obtained.
On the other hand,

$$E[\ \widehat{\mathbf{p}}_n\ ] = E[\ E[\ \widehat{\mathbf{p}}_n \ \mid\ \mathbf{Y}_{n-1},\ Z_n\ ]\ ] = E[\ (\ \mathbf{Y}_{n-1} + Z_n\mathbf{p}\ )Y_n^{*-1}\ ],$$

so (iii) holds. Finally, using the fact that

$$Cov[\ \widehat{\mathbf{p}}_n\ ] = E[\ Cov[\ \widehat{\mathbf{p}}_n \ \mid\ \mathbf{Y}_{n-1},\ Z_n\ ]\ ] + Cov[\ E[\ \widehat{\mathbf{p}}_n \mid \mathbf{Y}_{n-1}, Z_n\ ]\ ]$$

and taking into account that

$$Cov[\ \widehat{\mathbf{p}}_n \ \mid\ \mathbf{Y}_{n-1},\ Z_n\ ] = Z_nY_n^{*-2}(\ \mathbf{J} - \mathbf{p}^t\mathbf{p}\ ) \quad a.s,$$

we get,

$$Cov[\ \widehat{\mathbf{p}}_n\ ] = E[\ Z_nY_n^{*-2}\ ](\ \mathbf{J} - \mathbf{p}^t\mathbf{p}\ ) + Cov[\ Y_n^{*-1}(\ \mathbf{Y}_{n-1} + Z_n\mathbf{p}\ )\ ].$$

Therefore, (iv) is proved. ∎

Proof of Theorem 2

(i) It is clear that

$$\hat{p}_{n;k,l} = Y_n^{*-1} \sum_{i=0}^{n} \sum_{j=1}^{Z_i} \mathbf{1}_{\{(f_{ij},m_{ij})=(k,l)\}}, \quad (k,l) \in S.$$

Now, given $\epsilon > 0$, applying Chebyshev conditioned inequality we deduce,

$$P\left(\left| Z_n^{-1} \sum_{j=1}^{Z_n} \mathbf{1}_{\{(f_{nj},m_{nj})=(k,l)\}} - p_{k,l} \right| > \epsilon | \mathcal{F}_n \right) \leq \epsilon^{-2} p_{k,l}(1-p_{k,l}) Z_n^{-1} \quad a.s,$$

where $\mathcal{F}_n := \sigma(Z_0, \ldots, Z_n)$, $n = 0, 1, \ldots$ Thus, on $Q = \{Z_n \to \infty\}$, it is derived that

$$P(\lim_{n\to\infty} Z_n^{-1} \sum_{j=1}^{Z_n} \mathbf{1}_{\{(f_{nj},m_{nj})=(k,l)\}} = p_{k,l}) = 1.$$

By Toeplitz's lemma, the result is obtained.

(ii) Note that, for each $(k,l) \in S$, the variables $\widehat{p}_{n;k,l}$ and $Y_n^{*-1} \sum_{j=1}^{Y_n^*} \mathbf{1}_{\{(f_j,m_j)=(k,l)\}}$ have the same probability distribution, where (f_j, m_j) are independent and identically distributed random vectors with the same probability distribution than (f_{01}, m_{01}). Consequently, for $x \in \mathbb{R}$,

$$P^* \Big((p_{k,l}(1-p_{k,l}))^{-1/2} Y_n^{*1/2} (\hat{p}_{n;k,l} - p_{k,l}) \leq x \Big) =$$

$$P^*((p_{k,l}(1-p_{k,l}) Y_n^*)^{-1/2} \sum_{j=1}^{Y_n^*} \Big(\mathbf{1}_{\{(f_j,m_j)=(k,l)\}} - p_{k,l} \Big) \leq x).$$

Applying Theorem A.1(i) (see Appendix) with

$$a_n = (r^{n+1}-1)(r-1)^{-1}, \quad \nu_n = Y_n^*, \quad \theta = W$$

and

$$X_n^*(t) = (\ p_{k,l}(1-p_{k,l})Y_n^*\ )^{-1/2} \sum_{j=1}^{\lfloor Y_n^* t \rfloor} (\ \mathbf{1}_{\{(f_j, m_j\ )=(k,l)\}} - p_{k,l}\ ),$$

and taking into account that $\{Z_n \to \infty\} = \{W > 0\}$, we deduce (ii)(a). Finally, by Theorem A.1(ii) with

$$X'_n(t) = (\ Y_n^{*-1}(r-1)^{-1}(r^{n+1}-1)\ )^{1/2} X_n^*(t)$$

and using the fact that $\{(r^{n+1}-1)^{-1}(r-1)Y_n^*\}_{n\geq 1}$ converges almost surely, as $n \to \infty$ to W (see Molina et al.(2004a)) we obtain (ii)(b). ∎

Proof of Theorem 3. It can be verified that

$$E[\ \widehat{\boldsymbol{\mu}}_n \mid Y_{n-1}^*,\ \boldsymbol{\Psi}_{n-1},\ Z_n\ ] = (Y_{n-1}^* + Z_n)^{-1}(\boldsymbol{\Psi}_{n-1} + Z_n\boldsymbol{\mu}) \quad a.s.$$

and

$$E[\ \widehat{\boldsymbol{\mu}}_n^t \widehat{\boldsymbol{\mu}}_n \mid Y_{n-1}^*,\ \widehat{\boldsymbol{\Psi}}_{n-1},\ Z_n\ ] = (Y_{n-1}^* + Z_n)^{-2}\boldsymbol{\Omega}_n \quad a.s.$$

where $\boldsymbol{\Omega}_n := \boldsymbol{\Psi}_{n-1}^t\boldsymbol{\Psi}_{n-1} + Z_n(\boldsymbol{\mu}^t\boldsymbol{\Psi}_{n-1} + \boldsymbol{\Psi}_{n-1}^t\boldsymbol{\mu} + \boldsymbol{\Sigma} + Z_n\boldsymbol{\mu}^t\boldsymbol{\mu})$ and therefore, since $Y_n^* = Y_{n-1}^* + Z_n$, (i) and (ii) are deduced. Expressions (iii) and (iv) are derived using the fact that

$$E[\ \widehat{\boldsymbol{\mu}}_n\ ] = E\,[\ E[\ \widehat{\boldsymbol{\mu}}_n \mid \mathcal{A}_n\ ]\ ] = E[\ Y_n^{*-1}(\boldsymbol{\Psi}_{n-1} + Z_n\boldsymbol{\mu})\ ]$$

and

$$E\left[\ \widehat{\boldsymbol{\mu}}_n^t\widehat{\boldsymbol{\mu}}_n\ \right] = E\left[\ Y_n^{*-2}\boldsymbol{\Omega}_n\ \right],$$

where $\mathcal{A}_n := \sigma\{Z_0, (F_1, M_1), \ldots, (F_n, M_n)\},\ n = 1, 2, \ldots$ ∎

Proof of Theorem 4.

(i) On Q, it has been proved (see Molina et al.(2004a)), that $\{r^{-n}Y_n^*\}_{n\geq 0}$ and $\{r^{-(n+1)}\Psi_n\}_{n\geq 1}$ are almost surely convergent to $r(r-1)^{-1}W$ and $(r-1)^{-1}W\boldsymbol{\mu}$, respectively. Consequently, considering that

$$\widehat{\mu}_n = r^{-(n+1)}\Psi_n\ (\ r^{-n}Y_n^*\ )^{-1} r$$

we deduce the result.

(ii) For $i = 1, 2$ we have

$$P^*\left((\sigma_{ii}^{-1}Y_n^*)^{1/2}(\widehat{\mu}_{n;i} - \mu_i) \leq x\right) = P^*(\ (\ \sigma_{ii}Y_n^*\ )^{-1/2}\sum_{l=1}^{Y_n^*}(\xi_{0l}^i - \mu_i) \leq x\ ).$$

So, by Theorem A.1(i) with $a_n = (r^{n+1} - 1)(r - 1)^{-1}$, $\nu_n = Y_n^*$, $\theta = W$ and

$$X_n^*(t) = (\sigma_{ii}Y_n^*)^{-1/2}\sum_{l=1}^{\lfloor Y_n^* t\rfloor}(\xi_{0l}^i - \mu_i),$$

we derive (ii)(a). In a similar way, considering

$$X_n'(t) = (\ Y_n^{*-1}(r-1)^{-1}(r^{n+1} - 1)\ )^{1/2}X_n^*(t),$$

by Theorem A1(ii) we deduce (ii)(b). ∎

Appendix.

On the probability space $(\Omega, \mathcal{F}, P)$ we consider:

(i) A sequence $\{\xi_n\}_{n\geq 1}$ of independent and identically distributed random variables such that $E[\xi_1] = 0$ and $\sigma^2 := E[\xi_1^2] < \infty$.

(ii) For $t \in [0, 1]$, the random variables

$$X_n^*(t) := X_{\nu_n}(t) \quad \text{if} \quad \nu_n > 0 \quad \text{or} \quad 0 \quad \text{otherwise},$$

and

$$X_n'(t) := (a_n\nu_n^{-1})^{1/2}X_n^*(t),$$

where $X_n(t) := \sigma^{-1}n^{-1/2}S_{\lfloor nt\rfloor}$ being $S_n := \sum_{i=1}^n \xi_i$, $n = 1, 2, \ldots$ and $\{\nu_n\}_{n\geq 1}$ are non-negative random variables.

Theorem A.1 If there exists a sequence of constants $\{a_n\}_{n\geq 1}$ such that

(a) $\lim_{n\to\infty} a_n = \infty$,

(b) $\{\nu_n a_n^{-1}\}_{n\geq 1}$ converges in probability to a non-negative random variable θ with $P(\theta > 0) > 0$,

then, for any $P^* << P_{\mathcal{D}}$ with $\mathcal{D} := \{\theta > 0\}$ and $t \in [0,1]$

(i) $\lim_{n\to\infty} P^*(X_n^*(t) \leq x) = P^*(W^*(t) \leq x), \quad x \in \mathbb{R},$

(ii) $\lim_{n\to\infty} P^*(X_n'(t) \leq x) = P^*(W^*(t)\theta_0^{-1/2} \leq x), \quad x \in \mathbb{R},$

where W^* denotes the Wiener process, θ_0 is P^*-independent of W^* and $P^*(\theta_0 \leq x) = P^*(\theta \leq x),\ x \in \mathbb{R}$.

For the proof of this theorem, we refer the reader to Dion (1978).

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