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Pliska Stud. Math. Bulgar. **17** (2005), 171–186

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SOME INEQUALITIES OF THE UNIFORM ERGODICITY AND STRONG STABILITY OF HOMOGENEOUS MARKOV CHAINS

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In this paper we have established some uniform ergodicity and strong stability estimates for homogeneous Markov chains under mixing conditions. As a general rule, the initial parameters values of the most complex systems has approximately known (they are defined on basis statistics methods), which involve errors for the calculus of research characteristics for each studied system. For this, the stability inequalities obtained in this paper allow us to use them in order to estimate numerically the error of definition for concerned characteristics, for a small perturbations of system's parameters. As an example of application, we are interesting about the well known waiting process where we consider the perturbation for the characteristics of the system when we apply a small perturbation for the control sequence.

1. Introduction

In this paper, we have investigated some uniform ergodicity and strong stability estimate. In moreover of continuity qualitative affirmation, we obtain quantitative uniform ergodicity and strong stability estimates for homogeneous general Markov chains.

We must be precise that in the difference of the method proposed by Kalashnikov in [9] at the chapter 5 and Zolotarev in [14], we suppose that the perturbation of the corresponding transition kernel of the Markov chain is small with

2000 *Mathematics Subject Classification:* 60J45, 60K25

Key words: Quantitative estimates, Uniform ergodicity, Stability, Strong Stability, Perturbation.

respect to some operators norm. This condition, more stringent than other usual conditions, enable us to obtain more better approximation for the perturbed stationary distributions. Furthermore, the strong stability method give us an exactly calculus of constants which we allowed us to test the power of the results.

As a general rule, the initial parameters values of systems has approximately known (they are defined on basis statistics methods), which drives about errors for the calculus of research characteristics for each studied system. For this, the uniform ergodicity and stability inequalities obtained in this paper give us the possibilities to use them in order to estimate numerically the error of definition for concerned characteristics, for a small perturbations of system's parameters. As an example of application we can study the $M/G/1$ system (see. D. Aïssani and V.V. Kartashov [2]).

In the second section, we give some definitions and recall some results obtained by Aissani and Kartashov [1]. In the third section, we expose some intermediate results. The main results of this paper are exposed in sections 4 and 5.

The last section concerns an application of those results for a Markov chain X which taking values in $\mathbb{R}^+$, generated by the recursive equality $X_{n+1} = (X_n + \xi_{n+1})^+$, $n \geq 0$ where $(\xi_n)_{n\geq 0}$ is a sequence of independent random variables identically distributed with a common distribution function. This process can be represent the time waiting $G/G/1$ queuing model.

2. Preliminaries

Let $X = (X_n, n \in \mathbb{N})$ be a homogenous Markov chain taking values in a measurable space $(E, \mathcal{E})$ with a countably generated σ-algebra $\mathcal{E}$ and having a regular transition kernel $P(x, A)$, $x \in E$, $A \in \mathcal{E}$. The unique invariant probability measure π of the kernel P is finite, $\pi(E) = 1$.

We consider $m\mathcal{E}$ $(m\mathcal{E}^+)$, a space of finite measures on E (nonnegatives) and $\mathcal{J}$ $(\mathcal{J}^+)$ a space of the measurable bounded functions (nonnegatives). The corresponding transition kernel P of the chain X acting on $\mu \in m\mathcal{E}$ and $f \in \mathcal{J}$ as follows:

$$\mu P(A) = \int_E P(x, A)\mu(dx)\, \forall A \in \mathcal{E} \quad and \quad Pf(x) = \int_E P(x, dy) f(y)\, \forall x \in E$$

The symbol μf denotes the integral $\mu f = \int_E \mu(dx) f(x)$. The product of two transition kernels P and Q is the kernel defined by

$$P.Q(x,A) = \int_E P(x,dy)Q(y,A)\,\forall x \in E,\, \forall A \in \mathcal{E}$$

We provide the space $m\mathcal{E}$ with some norm $\|.\| = \|.\|_v$, defined by

$$\|\mu\|_v = \int_E v(x)|\mu|(dx) \tag{1}$$

where $|\mu|$ is a variation of the measure μ and v is an arbitrary bounded positive measurable function $v : E \longrightarrow \mathbb{R}_+^*$ which satisfies the following assertions

1) $\sup[v(x)^{-1}, x \in E] = \varrho > 0$.

2) v is $\mathcal{E}$-measurable.

The endowed norm on the space $\mathcal{J}$ is

$$\|f\|_v = \sup\left(\frac{|f(x)|}{v(x)},\quad x \in E\right)$$

It bring to the fore in class of endomorphism on $m\mathcal{E}$ a space $\mathcal{B}$ of bounded linear operators, with norm

$$\|P\|_v = \sup\left(\frac{\int_E v(y)|P(x,dy)|}{v(x)},\quad x \in E\right)$$

It is easy to verify that those endowed norms satisfy the following inequalities $\forall\, \mu \in m\mathcal{E}$, $\forall\, f \in \mathcal{J}$, $\forall\, P \in \mathcal{B}$, we get

a) $\|\mu P\|_v \leq \|\mu\|_v\ \|P\|_v$.

b) $|\mu f| \leq \|\mu\|_v\ \|f\|_v$.

c) $\|f \circ \mu\|_v \leq \|f\|_v\ \|\mu\|_v$, where "$\circ$" is the tensoriel product of the measure μ and a function f.

d) $\|Pf\|_v \leq \|P\|_v\ \|f\|_v$.

e) $\|\mathbf{1}\|_v = \sup[v(x)^{-1}, x \in E] = \varrho$, where $\mathbf{1}$ is the function equals to the unit, $\mathbf{1} \in \mathcal{J}$.

f) $\|PQ\|_v \leq \|P\|_v \ \|Q\|_v$.

g) $|\mu|(A) \leq \varrho \ \|\mu\|_v$, where $A \in \mathcal{E}$.

Remark 1. *See for example Kartashov [6] and Revuz [14] for the construction of the test function for different class of Markov chains.*

We denote the stationary projector of the transition kernel P by $\Pi = \mathbf{1} \circ \pi$.

Definition 1. *The chain X is said to be uniformly ergodic with respect to the norm $\|.\|$ if it has a unique invariant measure π and*

$$\lim_{t \to +\infty} \|t^{-1} \sum_{n=1}^{t} P^n - \Pi\| = 0$$

Definition 2. *The chain X is said to be strongly stable with respect to the norm $\|.\|$ if*

1) $\|P\| < \infty$

2) Each transition kernel Q in some neighborhood $\{Q : \|Q - P\| < \epsilon\}$, has a unique invariant measure $\nu = \nu(Q)$.

3) There is a constant $C = C(P)$, such that

$$\|\nu - \pi\| \leq C\|P - Q\|$$

Definition 3. *The Markov chain X strongly stable with respect to the norm $\|.\|_v$ is said strongly v-stable.*

The aim of this paper is to obtain the quantitative estimates of the rate of convergence and of the strong v-stability of the Markov chain X. Let us suppose that X is strongly v-stable. From the theorem of Kartashov [1], this is equivalent to impose the following conditions for the kernel P:

A) $\|P\|_v < \infty$.

B) $P = T + h \circ \alpha$, where T is a nonnegative kernel, $h \in \mathcal{J}^+$, $\alpha \in m\mathcal{E}^+$ such that $\|\alpha\|_v < \infty$ and $\|h\|_v < \infty$.

C) $\exists\, \rho \in]0, 1]$, $\exists\, c > 0$ such that $\|T^n\| \leq c\rho^n$ for any $n \in \mathbb{N}$.

In this paper, we suppose a condition A) holds, implicitly, in ever results which presented and we omit the index v in the writing of the norm.

Remark 2. *For $v \equiv 1$, the conditons of the mentioned theorem are equivalent to the Doblin conditions (quasi-compacity of the kernel P (see Neveu [11] at chapter 5).*

3. Preliminary results

In order to obtain the limit theorems which established for ergodicity and stability estimates, presented in section 4 and 5, we need some intermediate results.

Lemma 1. *The function* $f = \sum\limits_{i=0}^{+\infty} T^i h$ *is constant and verifies:*

$$f \equiv \frac{1}{\alpha \mathbf{1}} \quad \textit{and} \quad \alpha f = 1$$

Proof. The series $f = \sum\limits_{i=0}^{\infty} T^i h$ converges in norm, therefore the function f is well defined. But, $P\mathbf{1} = \mathbf{1}$ which implies

$$\mathbf{1} - T\mathbf{1} = \mathbf{1} - (P - h \circ \alpha)\mathbf{1} = \alpha \mathbf{1} h.$$

This implies that $f \equiv 1/\alpha\mathbf{1}$ and $\alpha f = 1$. The proof is achieved. $\square$

Lemma 2. *The measure* $\mu = \sum\limits_{i=0}^{+\infty} \alpha T^i$ *is constant and verifies*

$$\mu = \frac{\pi}{\pi h}$$

Proof. The series $\sum\limits_{i=0}^{+\infty} \alpha T^i$ converges in norm, therefore the measure is well defined. But, from the previous lemma 1, we have:

$$\mu P = \sum_{i=0}^{+\infty} \alpha T^i (T + h \circ \alpha) = \sum_{i=0}^{+\infty} \alpha T^{i+1} + \alpha \sum_{i=0}^{+\infty} \alpha T^i h = \mu.$$

Moreover,

$$\pi - \pi T = \pi - \pi(P - h \circ \alpha) = \pi h \alpha$$

hence the result is established. $\square$

Theorem 1. *Setting* $p_n = \alpha T^{n-1} h$, $n \geq 1$ *and we consider the sequence* $\lambda(n)$ *which satisfies the renewal equation below*

$$\begin{cases} \lambda(n) = \sum\limits_{k=1}^{n-1} \lambda(k) p_{n-k} & \textit{if } n \geq 2, \\ \lambda(1) = 1,\ \lambda(n) = 0 & \textit{if } n \leq 0 \end{cases} \tag{2}$$

Then, we have

1) For all $n \geq 1$, p_n is a probability distribution and $|p_n| \leq H \,.\, \rho^{n-1}$, where $H = \|h\| \,.\, \|\alpha\|\, c$.

2) For all $n \geq 1$,

$$P^n = T^n + \sum_{\substack{i,j \geq 0 \\ i+j \leq n-1}} \lambda_{ij}^n T^i h \circ \alpha T^j$$

Where $\lambda_{ij}^n = \lambda(n-i-j)$ and a sequence $\lambda(n)$ satisfies (2).

Proof.

1) Since, T, h and α are positives, consequently

$$p_n = \alpha T^{n-1} h \geq 0 \;\text{ For all }\; n \geq 1$$

From the two lemma 1 and 2, we have,

$$\sum_{n \geq 1} p_n = \sum_{n \geq 1} \alpha T^{n-1} h = 1$$

Moreover, $|p_n| = |\alpha T^{n-1} h| \leq \|\alpha\| \; \|h\| \, c \, \rho^{n-1}$

2) Noting that $(T^i h \circ \alpha T^j)\;(h \circ \alpha) = p_{j+1} T^i h \circ \alpha$, the result is obvious when we use the recurrence.

Finally, the proof is achieved. □

Let $d = GCD\{n \geq 1, p_n > 0\} \geq 1$. For $d = 1$, the behaviour of the sequence $\lambda(n)$, when $n \to \infty$, was studied by Karlin [5] at chapter 3.

Lemma 3. $\lambda(n) = 0$, *for* $n \not\equiv 1[d]$ *and it exist* $\beta \in [0, 1[$, $\bigwedge \geq 0$ *such that*

$$|\lambda(kd+1) - d\lambda| \leq \bigwedge \beta^{kd} \quad if \quad n \equiv 1[d]$$

where

$$\lim_{k \to +\infty} \lambda(kd+1) = \lambda = \frac{1}{\sum\limits_{n=1}^{+\infty} n p_n}$$

Proof. We consider the generating functions:

$$f(z) = \sum_{n \geq 1} p_n z^n \quad and \quad \psi(z) = \sum_{n \geq 1} \lambda(n) z^n$$

From the renewal equation, we get $\psi(z) = f(z)\psi(z) + z$. Hence,

$$\psi(z) = \frac{z}{1 - f(z)}$$

The condition $d \geq 1$ minds that $f(z) = f_1(z^d) = \sum\limits_{s \geq 1} p_{sd} z^{sd}$. The function $f_1(z_1)$ is analytic for $|z_1| < \rho^{-d}$, we get

$$|f_1(z_1)| < 1, \, \forall \, |z_1| < 1$$

Therefore

$$\psi_1(z_1) = \frac{\psi(z)}{z} = \frac{1}{1 - f_1(z_1)}$$

is analytic for all $|z_1| < 1$. It admits a simple pole on $z_1 = 1$.

From the definition of d, $\psi_1(z_1)$ has not other singular points on the circumference S^1, $|z_1| = 1$. Since the circle disk is compact, it exists necessary a disk Γ such that:

$$|z_1| < \beta^{-d} < \rho^{-d}, \beta \in [0, 1[$$

where, the function $\psi_1(z_1)$ have only one singular point $z_1 = 1$. Therefore, from the residus theorem, we have

$$\mathcal{R}es_{z_1=1}[\psi_1(z_1)] = -d\,\lambda$$

where $\mathcal{R}es(z)$ is the real part of the a complex number z. Consequently,

$$\psi_1(z_1) = \frac{d\lambda}{1 - z_1} + \Phi(z_1)$$

where, $\Phi(z_1)$ is analytic on Γ. Hence,

$$\psi(z) = \frac{d\lambda z}{1 - z^d} + z\Phi(z^d)$$

which achieves the proof. $\square$

4. Estimates of the uniform ergodicity

The results established in this section have the advantage that the constants appeared in the inequality depend only on the norm of the kernel T, the measure α and the function h which implicitly known.

Theorem 2. *Let X be a Markov chain satisfies the conditions A), B), C) of section 1 (i.e X is strongly v-stable). Then,*

$$\|P^{nd+s} - \Pi_s\| \leq \kappa n^2 \beta^{nd} \tag{3}$$

where κ is, independent of n, finite positive constant and can be computed exactly, and

$$\Pi_s = d\lambda \sum_{\substack{i,j\geq 0 \\ i+j\equiv s-1[d]}} T^i h \circ \alpha T^j$$

Proof. From the lemma (3),

$$P^{nd+s} = T^{nd+s} + \sum_{\substack{i,j\geq 0 \\ i+j\leq nd+s-1 \\ i+j\equiv s-1[d]}} \lambda(nd+s-i-j) T^i h \circ \alpha T^j$$

and

$$P^{nd+s} - \Pi_s =$$

$$T^{nd+s} + \sum_{\substack{i,j\geq 0 \\ i+j\leq nd+s-1 \\ i+j\equiv s-1[d]}} (\lambda(nd+s-i-j) - d\lambda) T^i h \circ \alpha T^j - d\lambda \sum_{\substack{i,j\geq 0 \\ i+j> nd+s-1 \\ i+j\equiv s-1[d]}} T^i h \circ \alpha T^j$$

Consequently,

$$\|P^{nd+s} - \Pi_s\| << \rho^{nd} + \sum_{k=0}^{n} (kd+s)\beta^{d(n-k)}\rho^{kd} + \sum_{k=n+1}^{+\infty} (kd+s)\rho^{kd}$$

But $\rho \leq \beta$ and $\sum_{k=0}^{n} (kd+s)a^{d(n-k)}b^{kd} << n^2(\max(a,b))^{nd}$, for $0 \leq a \leq 1$, $0 \leq b \leq 1$, the theorem is proved. □

Corollary 1. *Under the conditions of the theorem (2), $P^{nd+s} \longrightarrow \Pi_s$ when $n \longrightarrow +\infty$ with respect to the endowed norm. $s = 1, 2, \ldots, d$.*

Theorem 3. *Let X be a Markov chain satisfies the previous conditions A), B), C) of section 1. Then, this chain is uniformly ergodic with respect to the endowed norm, and*

$$\lim_{t\to+\infty} \|t^{-1} \sum_{n=1}^{t} P^n - \Pi\| = 0$$

where $\Pi = d^{-1} \sum_{s=1}^{d} \Pi_s = \lambda \sum_{i \geq 0} \sum_{j \geq 0} T^i h \circ \alpha T^j = \mathbf{1} \circ \pi$, *with* π *is the unique invariant measure of the kernel* P.

Proof.

$$\|\frac{1}{t}\sum_{k=1}^{t} P^k - \Pi\| = \frac{1}{t}\| \sum_{\substack{1 \leq s \leq d \\ 1 \leq k \leq t \\ k \equiv s[d]}} (P^k - \Pi_s)\| + O(\frac{1}{t})$$

But,

$$\| \sum_{\substack{1 \leq s \leq d \\ 1 \leq k \leq t \\ k \equiv s[d]}} (P^k - \Pi_s)\| \ll \sum_{k=1}^{t} k^2 \beta^k << \sum_{k=1}^{\infty} k^2 \beta^k << 1$$

Therefore,

$$\|\frac{1}{t}\sum_{k=1}^{t} P^k - \Pi\| << \frac{1}{n}$$

The proof is achieved. □

Theorem 4. *Let* X *be a Markov chain satisfies the previous conditions* A), B) *and* C) *of section 1, and* $d = 1$. *Then* $\sum_{n=0}^{\infty} \|P^n - \Pi\| \leq M$, *where*

$$M = \frac{c}{1-\rho} + \frac{\bigwedge H \beta c}{(1-\beta)(1-\rho)^2} + \frac{\lambda H (1+\rho) c}{(1-\rho)^3}$$

Proof. The proof is established by using the inequalities, $\|T^n\| \leq c\rho^n$ and $|\lambda(n) - \lambda| \leq \bigwedge \beta^n \quad \forall n \geq 0$. □

5. Estimates of the strong stability

We consider, in the sequel of this section, an other homogeneous Markov chains $Y = (Y_n, n \in \mathbb{N})$ with a transition kernel Q admitting an invariant probability measure ν. In order to establish the main results in this section we need the following lemmas.

Lemma 4. *Let* X *be a strongly* v*-stable Markov chain, which verifies the conditions* $d = 1$, $\|T\| = \rho < 1$, *and let* Y *be a Markov chain having a transition kernel* Q *such that* $\epsilon = \|P - Q\| = \Delta < 1 - \rho$, *then, we have the inequality*

$$\sup_{i \geq 0} \|\alpha Q^i\| \leq \frac{\|\alpha\|}{1 - \rho - \epsilon}$$

P r o o f. Let us prove firstly that $|\alpha Q^i h| \leq 1$, for all i.

Effectively, $\alpha Q^i h$ is positive, and $\alpha Q^i h = \int\limits_E \int\limits_E \alpha(dx) Q^i(x,dy) h(y)$. But, $P(y,A) \geq h(y)\alpha(A)\, \forall\, A \in \mathcal{E}$ because $T \geq 0$. Therefore

$$P(x,E) = 1 \geq h(y)\alpha(E)$$

which implies that $h(y) \leq 1/\alpha(E)$ and $|\alpha Q^i h| \leq 1$ for all i.

Secondly, we remark that $\alpha Q^i = \alpha Q^{i-1}(\Delta + T) + (\alpha Q^{i-1} h)\alpha$ for $i \geq 2$, then

$$\|\alpha Q^i\| \leq \|\alpha Q^{i-1}\| (\epsilon + \rho) + \|\alpha\| \quad i \geq 2$$

By the recursive procedure, we obtain,

$$\|\alpha Q^i\| \leq \|\alpha\| \left(1 + (\epsilon + \rho) + \ldots + (\epsilon + \rho)^i\right) \leq \frac{\|\alpha\|}{1 - \rho - \epsilon}$$

The proof is achieved. $\square$

Lemma 5. *Under conditions of the lemma (4), we have the inequality*

$$\sup_{i \geq 1} \|Q^i\| \leq 1 + \frac{\Upsilon}{(1 - \epsilon - \rho)^2}$$

where $\Upsilon = \|\alpha\| \,.\, \|h\|$.

P r o o f. $Q^i = (\Delta + T)Q^{i-1} + h \circ \alpha Q^{i-1}$, $i \geq 2$, then

$$\|Q^i\| \leq (\rho + \epsilon)\|Q^{i-1}\| + \frac{\Upsilon}{1 - \rho - \epsilon} \quad \text{for} \quad i \geq 2$$

By induction, we obtain $\|Q\| \leq (\rho + \epsilon) + \Upsilon$ and

$$\|Q^i\| \leq (\rho + \epsilon)^i + \frac{\Upsilon}{(1 - \rho - \epsilon)^2} \quad \text{for} \quad i \geq 2$$

The result is established. $\square$

Theorem 5. *Let X be a strongly v-stable Markov chain, which verifies the conditions $d = 1$, $\|T\| = \rho < 1$, and let Y be a Markov chain having a transition kernel Q such that $\epsilon = \|P - Q\| = \Delta < 1 - \rho$, then*

$$\sup_{t \geq 0} \|Q^t - P^t\| \leq M \left(1 + \frac{\Upsilon}{(1 - \rho - \epsilon)^2}\right) \|Q - P\| \tag{4}$$

where M is a constant introduced in theorem (4).

Proof. We set $\Delta_t = Q^t - P^t, \quad t \geq 0$. We have

$$\Delta_t = Q(Q^{t-1} - P^{t-1}) + (Q - P)P^{t-1} = Q\Delta_{t-1} + \Delta_1 P^{t-1}.$$

But,

$$\Delta\Pi = (Q - P)\Pi = (Q - P)(1 \circ \pi) = ((Q - P)\mathbf{1})\,\pi = 0$$

because $Q(x, E) - P(x, E) = 0$. Therefore,

$$\begin{aligned} \Delta_t &= Q\Delta_{t-1} + \Delta(P^{t-1} - \Pi) \\ &= \Delta(P^{t-1} - \Pi) + Q\left(Q(Q^{t-2} - P^{t-2}) + (Q - P)P^{t-2}\right) \\ &= \Delta(P^{t-1} - \Pi) + Q\Delta(P^{t-2} - \Pi) + Q^2\Delta_{t-2} \end{aligned}$$

By induction, we obtain,

$$\Delta_t = \Delta(P^{t-1} - \Pi) + \ldots + Q^{t-2}\Delta(P - Q) + Q^{t-1}\Delta$$

This implies that

$$\|\Delta_t\| \leq \|\Delta\| \sup_{i\geq 1} \|Q^i\| \sum_{i=0}^{\infty} \|P^i - \Pi\|$$

From the lemma (5) and the theorem (4), the theorem is proved. □

6. Application for the waiting process

Those results are applied for a random walk process described by a chain $(X_n)_{n\geq 0}$ defined by $X_{n+1} = (X_n + \xi_{n+1})^+$, $n \geq 0$ which taking values in $\mathbb{R}^+$. where $(\xi_n)_{n\geq 0}$ is a sequence of independent random variables identically distributed with a common distribution function F. This exemple was studied by N.V. Kartashov in [6].

We set $h(x) = P(\xi_1 + x \leq 0)$ and $\alpha(dy) = \delta_0(dy)$, where δ_0 is Dirac distribution concentred on origine. We consider the distribution

$$P(x, A) = P\left((x + \xi_1)^+ \in A\right) = P(X_1 \in A / X_0 = x)$$

It is easy to observe that

$$P(x, A) = P(0 < x + \xi_1 \in A) + h(x).\alpha(A) \tag{5}$$

Let us put $T(x, A) = P(0 < x + \xi_1 \in A)$ and Choose a test function v defined as below

$$v = \exp(\gamma x), x \in \mathbb{R}^+ \tag{6}$$

Know we consider a distribution $p_n = \alpha T^{n-1} h$, $n \geq 1$.

Lemma 6. *Let* $\tau = \inf(n : \xi_1 + \ldots + \xi_n \leq 0)$, *then* $P(\tau = n) = p_n$.

Proof. We have

$$\begin{aligned} p_n = \alpha T^{n-1} h &= \int T^{n-1}(0, dy).h(y) \\ &= P\left(0 < \xi_1, \ldots, 0 < \xi_1 + \ldots + \xi_{n-1}, \xi_1 + \ldots + \xi_n \leq 0\right) \end{aligned}$$

which achieves a proof. □

The class of norms defined on $m\mathcal{E}$ by the relation (1), for the particular choosing of v, have the following form

$$\|\mu\|_\gamma = \|\mu\|_v = \int_0^{+\infty} \exp(\gamma x)|\mu|(dx)$$

The correspondant norms in the spaces $\mathcal{J}$ and $\mathcal{B}$ have the forme

$$\|f\|_\gamma = \sup_{x\geq 0} \exp(-\gamma x)|f(x)|$$

and

$$\|Q\|_\gamma = \sup_{x\geq 0} \exp(-\gamma x) \int_0^{+\infty} |Q|(x, dy) \exp(\gamma y)$$

From this, we deduce that

$$\|P^t - \Pi\|_\gamma = \sup_{x\geq 0} \exp(-\gamma x) \int_0^{+\infty} |P^t(x, dy) - \pi(dy)| \exp(\gamma y)$$

with $P^t(x, dy) = P(X_t \in dy|X_0 = x)$, for all $(x, dy) \in E \times \mathcal{E}$.

Theorem 6 (Kartashov [6) *]Let* $E\xi_1 < 0$ *and* $\forall\, \delta \geq 0$, $E[\exp(\delta\xi_1)] < \infty$. *Then, for all* γ *such that* $\rho(\gamma) = E[\exp(\gamma\xi_1)] < 1$, *the Markov chain* X *is aperiodic and strongly* v*-stable, where* v *was defined in (6).*

Under conditions of the previous theorem, we have the following result.

Theorem 7. *Under conditions of theorem (6) and* $d = 1$. *Then, we have*

$$\sum_{n=0}^{\infty} \|P^n - \Pi\| \leq \frac{1}{1-\rho} + \frac{\bigwedge \beta}{(1-\beta)(1-\rho)^2} + \frac{(1+\rho)\pi(\{0\})}{(1-\rho)^3} \tag{7}$$

Proof.

We have to compute λ. Effectively, we have

$$\lambda = (\pi h)(\alpha \mathbf{1}) = 1. \int_E \pi(dx).P(x + \xi_1 \leq 0) = \pi(\{0\})$$

The proof follows directly from theorem (4), the values of λ, $\|\alpha\|$ and an estimation of $\|h\|$. $\square$

Remark 3. *The condition $P(\xi_1 \leq 0) > 0$ is sufficient in order to have $d = 1$.*

Let us consider, the chain Y with transition kernel Q, then we remark that

$$\varepsilon = \|Q - P\|_\gamma = \sup_{x \geq 0} \exp(-\gamma x) \int_0^{+\infty} |Q(x, dy) - P(x, dy)| \exp(\gamma y) \tag{8}$$

In the same way, we have

$$\sup_{t \geq 0} \|Q^t - P^t\|_\gamma = \sup_{t \geq 0} \sup_{x \geq 0} \exp(-\gamma x) \int_0^{+\infty} |Q^t(x, dy) - P^t(x, dy)| \exp(\gamma y) \tag{9}$$

Consequently, we have the following theorem.

Theorem 8. *Under the same conditions of theorem (6) and for all $\Upsilon < 1$ such that $\rho\Upsilon < 1$, we have*

$$\sup_{t \geq 0} \|Q^t - P^t\|_\gamma \leq \hbar \, \|Q - P\|_\gamma$$

where,

$$\hbar = \left(1 + \frac{\Upsilon}{(1 - \rho - \epsilon)^2}\right) \left(\frac{1}{1 - \rho} + \frac{\bigwedge \beta}{(1 - \beta)(1 - \rho)^2} + \frac{(1 + \rho)\pi(\{0\})}{(1 - \rho)^3}\right)$$

Proof.

The proof follows directly from the inequality (4) of theorem (5), the values of λ, $\|\alpha\|$, an estimation of $\|h\|$. Consequently, the result is immediately obtained. $\square$

Conclusion

Contrary to others methods, the estimates obtained by strong stability method, depends only in some constants directly closed for models parameters and with the possibility of an exact computation of them. As a general rule, the initial parameters values of systems has approximately known (they are defined on basis statistics methods), which drives about errors for the calculus of research characteristics for each studied system. Hence a possibilities to use inequalities of stability which obtained in this paper to estimate numerically the error of définition for concerned characteristics, for a small perturbations of system's parameters. Some other estimates are obtained and that will a subject for an other next paper.

Acknowledgement

The authors would like to thank E.V.Podsipanin (St-Petersburg) for some helpful suggestions.

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