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GEVREY LOCAL SOLVABILITY FOR SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we deal with a class of semilinear anisotropic partial differential equations. The nonlinearity is allowed to be Gevrey of a certain order both in x and $\partial^\alpha u$, with an additional condition when it is $G^{s_{\mathrm{cr}}}$ in the $(\partial^\alpha u)$-variables for a critical index s_{cr}. For this class of equations we prove the local solvability in Gevrey classes.

1. Introduction

In this paper we study the local solvability at the origin of the following class of semilinear partial differential equations:

$$P(x,D)u + G(x;\partial^\alpha u)|_{\langle\alpha,\sigma\rangle\le h^*} = \epsilon f(x), \tag{1.1}$$

$$P(x,D) = D^m_{x_n} - \sum_{\langle\alpha',\sigma'\rangle=m} b_{\alpha'}(x)D^{\alpha'}_{x'} + \sum_{k^*\le\langle\beta,\sigma\rangle<m} a_\beta(x)D^\beta_x. \tag{1.2}$$

In (1.1) and (1.2) we write $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_+$, $\sigma = (\sigma_1,\dots,\sigma_{n-1},1)$ with $\sigma_j \ge 1$ for every $j = 1,\dots,n-1$; we use the notation $\alpha' := (\alpha_1,\dots,\alpha_{n-1})$, and analogously $\sigma' = (\sigma_1,\dots,\sigma_{n-1})$; the numbers $h^*, k^* \in \mathbb{R}$ satisfy $0 \le h^* < k^* < m$, $m \in \mathbb{R}$, and $\langle\cdot,\cdot\rangle$ is the usual scalar product between two vectors, so $\langle\alpha,\sigma\rangle = \sum_{j=1}^{n-1}\alpha_j\sigma_j + \alpha_n$, $\langle\alpha',\sigma'\rangle = \sum_{j=1}^{n-1}\alpha_j\sigma_j$. Observe that we admit an

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anisotropy, in the sense that for every j the order of the x_j-derivative is always weighted by σ_j (in the sequel σ_j are supposed to be rational numbers).
As is well known, the main properties of the operators with multiple characteristics heavily depend on the lower order terms, so we have to take into account their influence, cf. the next Theorem 1.1.
Recall that, given an open set $\Omega \subset \mathbb{R}^n$, a function $f \in C^\infty(\Omega)$ is said to belong to the anisotropic space $G^\lambda(\Omega)$, $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_j \geq 1$ for $j = 1, \dots, n$, if for every compact set $K \subset \Omega$ there exists C such that

$$\max_{x \in K} |\partial^\alpha f(x)| \leq C^{|\alpha|+1} (\alpha!)^\lambda,$$

where $(\alpha!)^\lambda := \alpha_1!^{\lambda_1} \cdots \alpha_n!^{\lambda_n}$. If $\lambda_j > 1$ for every $j = 1, \dots, n$ we can consider $G_0^\lambda(\Omega)$, the set of all functions $f \in G^\lambda(\Omega)$ with compact support contained in Ω. When $\lambda_1 = \cdots = \lambda_n = s$ we obtain the usual isotropic Gevrey space $G^s(\Omega)$.
Many results on the theory of partial differential equations, also in Gevrey spaces, can be found for example in [7], [14], [11]. Concerning in particular the equation (1.1), it was studied in the case $n = 2$ in [3], where its local solvability and hypoellipticity are analyzed both in C^∞ and Gevrey classes; these results have been extended to the equation (1.1) in [2]. When proving the solvability in Gevrey classes, the nonlinearity $G(x; z)$, in [3] is assumed to be analytic in the z variable, whereas in the paper [2] $G(x; z)$ is regarded as $F(x; \Re z, \Im z)$, and the function $F(x; y)$ is admitted to be Gevrey in y, of order $s < s_{\rm cr}$ (see the next Theorem 1.1 for the precise expression of the critical index $s_{\rm cr}$). In the present paper we generalize this result: we admit here $F(x; y)$ to be Gevrey exactly of order $s_{\rm cr}$ in y, but we require the following additional condition:

$$|\mathcal{F}_{y \to \eta} F(x_0; \eta)| \leq C e^{-M|\eta|^{1/s_{\rm cr}} (\log|\eta|)^{\sigma_{\max}/\sigma_{\min}}}, \tag{1.3}$$

for a suitably large M and for all x_0 in a neighbourhood of the origin; we use the notation

$$\sigma_{\max} := \max_{j=1,\dots,n-1} \sigma_j, \quad \sigma_{\min} := \min_{j=1,\dots,n-1} \sigma_j; \tag{1.4}$$

$\mathcal{F}$ means the Fourier transform. Note that in particular all the functions $F(x; y)$ that are G_0^s in y for $s < s_{\rm cr}$, satisfy (1.3), and so we shall recapture here the result of [2].

Theorem 1.1. *Let us fix $\lambda = (\lambda_1, \dots, \lambda_n)$, $1 < \lambda_j < \frac{\sigma_j}{r}$ for $j = 1, \dots, n-1$, $\lambda_n > 1$, and $r \in \left(\frac{1}{2}, 1\right)$. We suppose that the coefficients of P in (1.2) and the datum f in (1.1) are in $G_0^\lambda(\Omega_\delta)$, $\Omega_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$. Assume that $h^* < m - 1 + r$, and the following holds:*

- $\Re \sum_{\langle \alpha', \sigma' \rangle = m} b_{\alpha'}(x) \xi'^{\alpha'} \neq 0$ *for* $\xi' := (\xi_1, \dots, \xi_{n-1}) \neq 0$;
- $\Im \sum_{k^* \leq \langle \beta, \sigma \rangle < m} a_\beta(x) \xi^\beta \xi_n^{m-1} \geq 0$ (*resp.* ≤ 0);
- $\Im \sum_{\langle \alpha', \sigma' \rangle = m} b_{\alpha'}(x) \xi'^{\alpha'} \xi_n^{m-1} \leq 0$ (*resp.* ≥ 0).

Regarding the nonlinearity, rewriting $G(x; z)$ *as* $F(x; \Re z, \Im z)$ *we suppose that:*

- $F(x; 0) = 0$;
- $F(x; y_0) \in G^\lambda(\Omega_\delta)$ *for every* $y_0 \in \mathbb{R}^N$;
- $F(x_0; y) \in G^s(\mathbb{R}^N)$, *for every* $x_0 \in \Omega_\delta$, $s \leq \frac{\sigma_{\min}}{r}$;
- *If* $s = \frac{\sigma_{\min}}{r}$ *in the previous requirement, we ask that the condition* (1.3) *is satisfied, with* $s_{\text{cr}} = \frac{\sigma_{\min}}{r}$, *for every* $x_0 \in \Omega_\delta$ *and* M *sufficiently large.*

Then the semilinear equation (1.1) *admits a classical solution in* Ω_δ, *for* δ *and* ϵ *sufficiently small.*

Regarding the linear equation $P(x, D)u = f$, we use the technique of the conjugation developed in [9] and then used in [8] for isotropic equations, and in [10], [3] in the anisotropic case: by this conjugation, cf. Section 2 below, we may transform the operator $P(x, D)$ into another one that we shall denote by $\widetilde{P}(x, D)$: we then obtain the C^∞ local solvability of $\widetilde{P}(x, D)$ by applying the techniques of [4], [5], cf. also [3]: this implies the Gevrey local solvability of $P(x, D)$ as in Theorem 1.1. The advantage of this procedure is that we can impose the hypotheses directly on the coefficients of the operator.
Regarding the nonlinearity $G(x; \partial^\alpha u)$, it is treated here with the technique developed in [1], see also [2]: this gives us the solvability result without any additional condition on the coefficients of the operator P. The local solution of the semilinear equation is then found, using standard procedure, by the Fixed Point Theorem, cf. for example [6].
Some examples of linear operators satisfying the hypotheses of Theorem 1.1 are given in [3], [2]. Here we fix attention on the non linear term: in particular we want to show an example of a Gevrey function of exact order s_{cr} satisfying (1.3).

Let us consider

$$F(y) = \mathfrak{F}^{-1}_{\eta \to y}\left(e^{-M[\eta]^{1/s_{\text{cr}}}(\log[\eta])^{\sigma_{\max}/\sigma_{\min}}}\right), \tag{1.5}$$

where $[\eta]$ *is a positive* C^∞ *function satisfying* $[\eta] = |\eta|$ *for large* $|\eta|$. *Then:*

(a) $F \in G^{s_{\rm cr}}(\mathbb{R}^N)$;

(b) F *satisfies* (1.3);

(c) $F \notin G^{s_{\rm cr}-\nu}(\mathbb{R}^N)$, *for every* $\nu > 0$.

In fact, since F *belongs to* $\mathcal{S}(\mathbb{R}^N)$ *and its Fourier transform satisfies* $|\widehat{F}(\eta)| \leq C\, e^{-M|\eta|^{1/s_{\rm cr}}}$, *we have* $F \in G^{s_{\rm cr}}(\mathbb{R}^N)$. *The point* (b) *is trivial, and* (c) *follows from the results of* [14], Section 3.2: *we can in fact regard* $\widehat{F}(\eta)$ *as a symbol in the class* $S^{\infty, s_{\rm cr}-\nu}(\mathbb{R}^N)$, *for every* $\nu > 0$. *Then* $(s_{\rm cr} - \nu)$-sing-supp $F = \{0\}$ *for* $\nu > 0$, *and so* $F \notin G^{s_{\rm cr}-\nu}(\mathbb{R}^N)$. *Observe that* F *is analytic outside the origin, cf.* [14], Remark 3.2.8.
In the Appendix we shall give an alternative direct proof of (c).

The structure of the paper is as follows: in Section 2 we analyze the linear equation $P(x,D)u = f$, recalling without proofs the results of [2]: in the linear case in fact Theorem 1.1 does not say anything new with respect to [2]. In Section 3 we deal with the nonlinearity, and in Section 4 we give the proof of Theorem 1.1.

2. The linear equation

In this section we recall some results showing the existence of a parametrix for the operator (1.2) in a suitable scale of Gevrey-Sobolev spaces $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$. A detailed exposition of this part is given in [2].

Definition 2.1. *Let us fix* $r \in (0,1)$ *and* $\delta > 0$; *a non-negative function* $\psi(x_n, \xi') \in C^\infty((-\delta,\delta) \times \mathbb{R}^{n-1})$ *is said to be a weight function of order* (r, σ') *if and only if for every* $(j, \beta') \in \mathbb{Z}_+ \times \mathbb{Z}_+^{n-1}$ *there exist* $C_{j\beta'}$ *and* L *such that*

$$\sup_{x_n \in (-\delta,\delta)} |D^j_{x_n} D^{\beta'}_{\xi'} \psi(x_n, \xi')| \leq C_{j\beta'} \langle \xi' \rangle_{\sigma'}^{r - \langle \beta', \sigma' \rangle},$$

for all $\langle \xi' \rangle_{\sigma'} \geq L$, *where* $\langle \xi' \rangle_{\sigma'} = \sum_{j=1}^{n-1} (1 + |\xi_j|^2)^{\frac{1}{2\sigma_j}}$.

Definition 2.2. *Let us now fix* $s > 0$, $\tau > 0$ *and a weight function* $\psi(x_n, \xi')$ *of order* (r, σ'). *Then* $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta)) := \{f \in L^2(\mathbb{R}^{n-1} \times (-\delta,\delta)) : \|f\|_{s,\sigma'} < \infty\}$, *where* $\|f\|_{s,\sigma'} := \|e^{\tau\psi(x_n, D')} f\|_{H^s_{1/\sigma'}}$ *is the norm in* $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta))$. *The space* $H^s_{1/\sigma'}(\mathbb{R}^{n-1} \times (-\delta,\delta))$ *is the Sobolev anisotropic space, defined by*

$$\|f\|_{H^s_{1/\sigma'}} := \left(\sum_{k=0}^{s} \int \langle \xi' \rangle_{\sigma'}^{2(s-k)} |D^k_{x_n} (\mathcal{F}_{x' \to \xi'} f)(x_n, \xi')|^2 \, d\xi' \, dx_n \right)^{1/2} < \infty$$

(*for s integer; this definition can be extended by interpolation to every $s > 0$*). *We write* $e^{\tau\psi(x_n,D')}f := (2\pi)^{1-n}\int e^{ix'\xi'}e^{\tau\psi(x_n,\xi')}(\mathcal{F}_{x'\to\xi'}f)(x_n,\xi')\,d\xi'$.

These spaces have been studied in detail in [8] in the isotropic case and in [10] in the anisotropic form of Definition 2.2. Recall that if $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfies $1 < \lambda_j < \frac{\sigma_j}{r}$ for every $j = 1, \dots, n-1$, $\lambda_n > 1$, then

$$G_0^\lambda(\mathbb{R}^{n-1} \times (-\delta,\delta)) \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1} \times (-\delta,\delta)). \tag{2.1}$$

As for the isotropic case we define for the operator P in (1.2) the anisotropic characteristic manifold

$$\overline{\Sigma} := \{(x,\xi) \in \Omega_\delta \times (\mathbb{R}^n \setminus \{0\}) : \xi_n^m - \sum_{\langle\alpha',\sigma'\rangle = m} b_{\alpha'}(x)\xi'^{\alpha'} = 0\}.$$

A quasi-conic set $\overline{\Lambda} \subset \mathbb{R}^n_\xi$ is a set containing the points $(t^{\sigma_1}\xi_1, \dots, t^{\sigma_{n-1}}\xi_{n-1}, t\xi_n)$ for every $t > 0$ whenever it contains the point ξ. We can then fix a quasi-conic neighbourhood $\overline{\Gamma} = \Omega_\delta \times \overline{\Lambda}$ of $\overline{\Sigma}$, where $\overline{\Lambda}$ is a quasi-conic set satisfying the condition $\langle\xi\rangle_\sigma \le c\langle\xi'\rangle_{\sigma'}$ for every $\xi \in \overline{\Lambda}$, c being a constant independent of ξ. Now we consider the conjugate operator $\widetilde{P}(x,D) := e^{\tau\psi(x_n,D')}P(x,D)e^{-\tau\psi(x_n,D')}$, and we fix $\psi(x_n,\xi') = \big(1 + \frac{x_n}{2\delta}\big)\langle\xi'\rangle^r_{\sigma'}$. By the general calculus in [10], Theorem 3.1, we have the following result.

Proposition 2.3. *Suppose that the coefficients of $P(x,D)$ are in $G_0^\lambda(\Omega)$, $1 < \lambda_j < \frac{\sigma_j}{r}$ for $j = 1, \dots, n-1$, $\lambda_n > 1$. Then the symbol $\tilde{p}(x,\xi)$ of $\widetilde{P}(x,D)$ is given by*

$$\tilde{p}(x,\xi) = p(x,\xi) + p_{m-1+r}(x,\xi) + \text{lower order terms},$$

where $p(x,\xi)$ is the symbol of $P(x,D)$ and $p_{m-1+r}(x,\xi)$ is given by

$$im\frac{\tau}{2\delta}\langle\xi'\rangle^r_{\sigma'}\xi_n^{m-1} - \tau\Big(1 + \frac{x_n}{2\delta}\Big)\sum_{\langle\alpha',\sigma'\rangle=m}\sum_{l=1}^{n-1} D_{x_l}b_{\alpha'}(x)\partial_{\xi_l}\big(\langle\xi'\rangle^r_{\sigma'}\big)\xi'^{\alpha'}.$$

A similar result holds if we choose $\psi(x_n,\xi') = \big(1 - \frac{x_n}{2\delta}\big)\langle\xi'\rangle^r_{\sigma'}$. Observe that for δ sufficiently small the symbol $p_{m-1+r}(x,\xi)$ in Proposition 2.3 is quasi elliptic in $\overline{\Gamma}$, i.e.

$$|p_{m-1+r}(x,\xi)| \ge C\langle\xi\rangle_\sigma^{m-1+r}$$

for every $(x,\xi) \in \overline{\Gamma}$. This fact, together with the other hypotheses of Theorem 1.1, allows us to construct a microlocal parametrix in $H^s_{1/\sigma'}$ for $\widetilde{P}(x,D)$ in $\overline{\Gamma}$, cf. [2] (see also the paper of De Donno submitted to the present proceedings, in which this point is treated in a more general situation). Then, by applying the inverse conjugation, we find a microlocal parametrix in $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ for $P(x,D)$ in $\overline{\Gamma}$. Since outside $\overline{\Gamma}$ the operator P is microlocally quasi elliptic, we have a microlocal parametrix in the complement of $\overline{\Gamma}$; by a standard procedure of patching together these parametrices, cf. for example [8], [10], [3], we can construct a local parametrix E in $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ ($E : \mathbb{H}^{s,\psi}_{\tau,\sigma',r} \to \mathbb{H}^{s+m-1+r,\psi}_{\tau,\sigma',r}$), and then deduce the solvability of the linear equation $P(x,D)u = f$ in $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\Omega_\delta)$; by the inclusion (2.1) we have the result of Gevrey local solvability of Theorem 1.1.

3. The nonlinearity

One of the fundamental properties of the space $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ is that there exists $s_{\text{alg}} > 0$ such that, for every $s > s_{\text{alg}}$, $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ is an algebra, provided the weight function satisfies the condition

$$(3.1) \qquad \psi(x_n,\xi') - \psi(x_n,\xi'-\eta') - \psi(x_n,\eta') \le -b\min\{\langle\xi'-\eta'\rangle_{\sigma'},\langle\eta'\rangle_{\sigma'}\}^r$$

for a constant $b > 0$. From this fact it is possible to prove the following result.

Proposition 3.1. *Let us fix $s > s_{\text{alg}}$ and suppose that the weight function $\psi(x_n,\xi')$ satisfies (3.1). Then there exist two constants c and a such that*

$$(3.2) \qquad \|e^{iu(x)} - 1\|_{s,\sigma'} \le \begin{cases} c\, e^{a\|u\|_{s,\sigma'}^{r/\sigma_{\min}} (\log\|u\|_{s,\sigma'})^{\sigma_{\max}/\sigma_{\min}}} & \text{if } \|u\|_{s,\sigma'} > 1 \\ c\,\|u\|_{s,\sigma'} & \text{if } \|u\|_{s,\sigma'} \le 1 \end{cases}$$

for every real-valued function $u \in \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$; the constants c and a depend only on n and s.

Proposition 3.1 is proved in the isotropic case in [1], with a slightly different norm; the proof in our case is given in [2].

Theorem 3.2. *Let us consider a function $F(x;y)$ satisfying the hypotheses of Theorem 1.1; choose $\mathbb{B} \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$ bounded with respect to $\|\cdot\|_{s,\sigma'}$, and take N real-valued functions $u_1(x),\dots,u_N(x) \in \mathbb{B}$. Let $\psi(x_n,\xi')$ satisfy (3.1) and assume $s > s_{\text{alg}}$. Then, writing $u(x) = (u_1(x),\dots,u_N(x))$, we have*

$$(3.3) \qquad \|F(x;u(x))\|_{s,\sigma'} \le \Psi(\|u_1\|_{s,\sigma'},\dots,\|u_N\|_{s,\sigma'}),$$

where Ψ is a continuous function satisfying $\Psi(y) \to 0$ as $y \to 0$. Taking $v_1(x),\dots,v_N(x) \in \mathbb{B}$ we also have

$$(3.4) \qquad \|F(x;u(x)) - F(x;v(x))\|_{s,\sigma'} \le C_{\mathbb{B}} \sum_{j=1}^{N} \|u_j - v_j\|_{s,\sigma'},$$

$C_{\mathbb{B}}$ *being a constant depending on* $\mathbb{B}$.

Before giving the proof of Theorem 3.2 we prove the following technical lemma.

Lemma 3.3. *For every* $a_1, \dots, a_N \in \mathbb{C}$ *we have:*

$$a_1 \cdots a_N - 1 = \sum_{\ell=1}^{N} \sum_{0<j_1<\cdots<j_\ell\le N} (a_{j_1} - 1) \cdots (a_{j_\ell} - 1) \tag{3.5}$$

Proof. We prove this identity by induction on N. If $N = 1$, (3.5) is trivial. Now we suppose that it is true for N and we consider $a_1 \cdots a_{N+1} - 1 = a_1 \cdots a_{N-1}(a_N a_{N+1}) - 1$; by the inductive hypothesis we have:

$$a_1 \cdots a_{N-1}(a_N\, a_{N+1}) - 1 = \sum_{\ell=1}^{N} \Big(\sum_{0<j_1<\cdots<j_\ell<N} (a_{j_1} - 1) \cdots (a_{j_\ell} - 1) + \sum_{0<j_1<\cdots<j_\ell=N} (a_{j_1} - 1) \cdots (a_{j_{\ell-1}} - 1)(a_N a_{N+1} - 1) \Big);$$

since $a_N a_{N+1} - 1 = (a_N - 1) + (a_{N+1} - 1) + (a_N - 1)(a_{N+1} - 1)$ we have

$$\begin{aligned} a_1 \cdots a_{N+1} - 1 = \sum_{\ell=1}^{N} \Big(& \sum_{0<j_1<\cdots<j_\ell<N} (a_{j_1} - 1) \cdots (a_{j_\ell} - 1) \\ & + \sum_{0<j_1<\cdots<j_{\ell-1}<N} \big[(a_{j_1} - 1) \cdots (a_{j_{\ell-1}} - 1)\big](a_N - 1) \\ & + \sum_{0<j_1<\cdots<j_{\ell-1}<N} \big[(a_{j_1} - 1) \cdots (a_{j_{\ell-1}} - 1)\big](a_{N+1} - 1) \\ & + \sum_{0<j_1<\cdots<j_{\ell-1}<N} \big[(a_{j_1} - 1) \cdots (a_{j_{\ell-1}} - 1)\big](a_N - 1)(a_{N+1} - 1) \Big) \\ = \sum_{\ell=1}^{N+1} & \sum_{0<j_1<\cdots<j_\ell\le N+1} (a_{j_1} - 1) \cdots (a_{j_\ell} - 1). \end{aligned}$$

So the proof is complete. □

Proof of Theorem 3.2. Observe at first that, since

$$\|u\|_{L^\infty} \le \sup_{x_n} \|\mathcal{F}_{x'\to\xi'} u(x_n, \xi')\|_{L^1_{\xi'}} \le C(\tau) \sup_{x_n} \|e^{\tau\psi(x_n, D')} u\|_{L^2_{x'}} \le C_1(\tau) \|u\|_{s,\sigma'},$$

$\mathbb{B}$ is also bounded in $L^\infty(\mathbb{R}^{n-1}\times(-\delta,\delta))$; moreover, we look for the local solvability at the origin of (1.1), and so we can suppose that $F(x;y)$ is compactly supported both in x and y. By the hypotheses on the Gevrey order of $F(x;y)$ and a standard procedure we can find two constants C and μ such that

$$\left|D^j_{x_n}\big(\mathcal{F}_{\substack{x'\to\xi'\\ y\to\eta}}F\big)(x_n,\xi';\eta)\right| \le C\, e^{-\mu\sum_{j=1}^{n-1}(1+|\xi_j|)^{1/\lambda_j}}\, e^{-M|\eta|^{r/\sigma_{\min}}(\log|\eta|)^{\sigma_{\max}/\sigma_{\min}}},$$

M being the constant of the estimate (1.3). By Definition 2.2 it follows that $\|e^{ix'\xi'}f(x)\|_{s,\sigma'} \le C\,e^{\langle\xi\rangle^r_{\sigma'}}\|f\|_{s,\sigma'}$, and so we have:

$$\begin{aligned}\|F(x;u(x))\|_{s,\sigma'} \le & C\int e^{\langle\xi'\rangle^r_{\sigma'}}\, e^{-\mu\sum_{j=1}^{n-1}(1+|\xi_j|)^{1/\lambda_j}}\, d\xi' \\ &\times\int \|e^{i\eta u(x)}-1\|_{s,\sigma'}\, e^{-M|\eta|^{r/(\sigma_{\min})}\,(\log|\eta|)^{\sigma_{\max}/\sigma_{\min}}}\, d\eta.\end{aligned} \tag{3.6}$$

Since $1<\lambda_j<\frac{\sigma_j}{r}$ for $j=1,\dots,n-1$, the integral in ξ' in (3.6) is finite. By Lemma 3.3 and the algebra property of $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ we have that for $s>s_{\mathrm{alg}}$

$$\|e^{i\eta u(x)}-1\|_{s,\sigma'} \le C\sum_{\ell=1}^{N}\sum_{0<j_1<\cdots<j_\ell\le N}\|e^{i\eta_{j_1}u_{j_1}(x)}-1\|_{s,\sigma'}\cdots\|e^{i\eta_{j_\ell}u_{j_\ell}(x)}-1\|_{s,\sigma'}. \tag{3.7}$$

So, by Proposition 3.1 and (3.7), if M is sufficiently large the integral in η in (3.6) is convergent, and so (3.3) is true. In particular, if $\|u_j\|_{s,\sigma'}\le 1$ for every j the function Ψ keeps the form $\Psi(\|u_j\|_{s,\sigma'}) = C\sum_{\ell=1}^{N}\sum_{0<j_1<\cdots<j_\ell\le N}\|u_{j_1}\|_{s,\sigma'}\cdots\|u_{j_\ell}\|_{s,\sigma'}$; then $\Psi(y)\to 0$ as $y\to 0$.

Now we want to prove (3.4). From the Taylor formula, stopped at the first term, we have:

$$\begin{aligned}F(x;u(x))-F(x;v(x)) &= \sum_{j=1}^{N}(u_j(x)-v_j(x)) \\ &\times\left[\int_0^1\big((\partial_{y_j}F)(x;u(x)+tv(x))-(\partial_{y_j}F)(x;0)\big)\,dt+(\partial_{y_j}F)(x;0)\right].\end{aligned}$$

Taking the $\mathbb{H}^{s,\psi}_{\tau,\sigma',r}$ norms, and applying (3.3) to the function $H(x;w(x)) := (\partial_{y_j}F)(x;w(x)) - (\partial_{y_j}F)(x;0)$ we deduce that

$$\|F(x;u(x)) - F(x;v(x))\|_{s,\sigma'} \leq C\sum_{j=1}^{N}\|u_j - v_j\|_{s,\sigma'}$$
$$\times\left[\int_0^1 \Psi(\|u_j + tv_j\|_{s,\sigma'})\,dt + \|(\partial_{y_j}F)(x;0)\|_{s,\sigma'}\right];$$

then (3.4) follows from the fact that $u_j(x)$ and $v_j(x)$ are in a bounded set and Ψ is continuous. $\square$

Remark 3.4. *Let us consider now* $J(u) := F(x;\Re(\partial^\alpha u), \Im(\partial^\alpha u))|_{\langle\alpha,\sigma\rangle\leq h^*}$, *cf.* (1.1). *We take* $u(x) \in \mathbb{B} \subset \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\mathbb{R}^{n-1}\times(-\delta,\delta))$, *where* $\mathbb{B}$ *is bounded as before; observe that* $\|\partial^\alpha u\|_{s,\sigma'} \leq C\|u\|_{s+h^*,\sigma'}$ *for every* α *satisfying* $\langle\alpha,\sigma\rangle \leq h^*$. *Then, by Theorem* 3.2 *we can find a continuous non-decreasing function* $\Phi : [0,+\infty) \to [0,+\infty)$, $\Phi(0) = 0$, *such that*

$$\|J(u)\|_{s,\sigma'} \leq \Phi(\|u\|_{s+h^*,\sigma'}); \tag{3.8}$$

moreover, if $u, v \in \mathbb{B}$ *we have:*

$$\|J(u) - J(v)\|_{s,\sigma'} \leq C_{\mathbb{B}}\|u - v\|_{s+h^*,\sigma'}. \tag{3.9}$$

4. Proof of Theorem 1.1

Using the tools of the preceding two sections we now prove the local solvability of the semilinear equation (1.1).

P r o o f of Theorem 1.1. We know from Section 2 that there exists a parametrix E of P: more precisely, we have that $P(x,D)E = \mathrm{Id} + R$, with $E : \mathbb{H}^{s,\psi}_{\tau,\sigma',r} \to \mathbb{H}^{s+m-1+r,\psi}_{\tau,\sigma',r} \hookrightarrow \mathbb{H}^{s+h^*,\psi}_{\tau,\sigma',r}$, and $R : \mathbb{H}^{s,\psi}_{\tau,\sigma',r} \to \mathbb{H}^{t,\psi}_{\tau,\sigma',r}$ for every $t > 0$. Then by arguments as in [6], see also [8], [10], [3], we can find a positive, continuous, non-decreasing function $\mathcal{L} : [0,\delta_0] \to [0,+\infty)$, $\mathcal{L}(0) = 0$ such that, defining

$$\mathcal{A}_s(\delta) := \sup_{\substack{v\neq 0\\ v\in\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\Omega_\delta)}} \frac{\|Ev\|_{s+h^*,\sigma'}}{\|v\|_{s,\sigma'}},\quad \mathcal{B}_s(\delta) := \sup_{\substack{v\neq 0\\ v\in\mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\Omega_\delta)}} \frac{\|Rv\|_{s,\sigma'}}{\|v\|_{s,\sigma'}},$$

we have $\mathcal{A}_s(\delta) \leq \mathcal{L}(\delta)$, $\mathcal{B}_s(\delta) \leq \mathcal{L}(\delta)$. We are looking for a solution of the form $u = Ev$, so the equation (1.1) becomes $v(x) = \mathcal{Q}(v(x)) + \epsilon f(x)$, where $\mathcal{Q}(v(x)) :=$

$-Rv(x) - F\big(x; \Re\left(\partial^\alpha(Ev)(x)\right), \Im\left(\partial^\alpha(Ev)(x)\right)\big)|_{\langle\alpha,\sigma\rangle\le h^*}$. We then have to find a fixed point of the operator $\mathcal{Q}(\cdot) + f$; we choose δ and ϵ such that the following conditions are satisfied:

$$\mathcal{B}_s(\delta)(1+\epsilon\|f\|_{s,\sigma'}) + \Phi\big(\mathcal{A}_s(\delta)(1+\epsilon\|f\|_{s,\sigma'})\big) \le 1 \tag{4.1}$$

$$\mathcal{B}_s(\delta) + \mathcal{A}_s(\delta)C_{\mathbb{B}} < 1, \tag{4.2}$$

where $\Phi(\cdot)$ and $C_{\mathbb{B}}$ are given by Remark 3.4, $\mathbb{B} := \{w \in \mathbb{H}^{s,\psi}_{\tau,\sigma',r}(\Omega_\delta) : \|w - \epsilon f\|_{s,\sigma'} \le 1\}$. Now, by (3.8) and (4.1) we have that $\mathcal{Q}(\cdot) + f : \mathbb{B} \to \mathbb{B}$; moreover (3.9) and (4.2) imply that $\mathcal{Q}(\cdot) + f$ is a contraction. We then obtain a solution as an application of the Fixed Point Theorem in the Banach space $\mathbb{B}$. Taking s sufficiently large the solution is classical. □

Appendix

In the Introduction we have analyzed the function $F(y)$ defined in (1.5). Now we give an alternative direct proof of the fact that $F \notin G^{s_{\rm cr}-\nu}(\mathbb{R}^N)$. We prove this fact ab absurdo: let us suppose that there exists $\nu > 0$ such that $F \in G^{s_{\rm cr}-\nu}(\mathbb{R}^N)$. We then choose an even function $\psi \in G_0^{s_{\rm cr}-\nu}(\mathbb{R}^N)$, $\psi \not\equiv 0$, and define $\varphi := \psi * \psi$. In this way we have $\varphi \in G_0^{s_{\rm cr}-\nu}(\mathbb{R}^N)$, as it is easy to deduce by the definition of the convolution product. Moreover, $\widehat{\varphi}$ is real-valued, and $\widehat{\varphi}(\eta) \ge 0$ for all $\eta \in \mathbb{R}^N$: in fact, $\widehat{\varphi}(\eta) = \mathfrak{F}_{y\to\eta}(\psi * \psi)(\eta) = \big(\widehat{\psi}(\eta)\big)^2$, and $\widehat{\psi}(\eta)$ is real-valued, ψ being even. Let us consider now $(\varphi F)(y)$; observe that $\varphi(y) \ne 0$ in a neighbourhood of $y = 0$ (we have already pointed out that the singularity of F is localized in $y = 0$). We have that $(\varphi F)(y) \in G_0^{s_{\rm cr}-\nu}(\mathbb{R}^N)$ and then, by the standard properties of the Fourier transform,

$$|\mathfrak{F}_{y\to\eta}(\varphi F)(\eta)| \le C\, e^{-\mu|\eta|^{1/(s_{\rm cr}-\nu)}} \tag{A.1}$$

for suitable C and μ; on the other hand,

$$\mathfrak{F}_{y\to\eta}(\varphi F)(\eta) = (2\pi)^{-N}\widehat{\varphi} * \widehat{F} = (2\pi)^{-N}\int \widehat{\varphi}(\xi)\widehat{F}(\eta-\xi)\,d\xi. \tag{A.2}$$

Now

$$\begin{aligned}\widehat{F}(\eta-\xi) &= e^{-M[\eta-\xi]^{1/s_{\rm cr}}(\log[\eta-\xi])^{\sigma_{\max}/\sigma_{\min}}} \\ &\ge e^{-M'[\eta]^{1/s_{\rm cr}}(\log[\eta])^{\sigma_{\max}/\sigma_{\min}}}\, e^{-M'[\xi]^{1/s_{\rm cr}}(\log[\xi])^{\sigma_{\max}/\sigma_{\min}}}.\end{aligned} \tag{A.3}$$

Since $\widehat{F}$ and $\widehat{\varphi}$ are real-valued non-negative functions, we have by (A.2) and (A.3) that

(A.4)
$$\begin{aligned}|\mathcal{F}_{y\to\eta}(\varphi F)(\eta)| &= (2\pi)^{-N}\int |\widehat{\varphi}(\xi)|\,|\widehat{F}(\eta-\xi)|\,d\xi \\ &\geq (2\pi)^{-N} e^{-M'[\eta]^{1/s_{\rm cr}}(\log[\eta])^{\sigma_{\max}/\sigma_{\min}}}\int |\widehat{\varphi}(\xi)|\,e^{-M'[\xi]^{1/s_{\rm cr}}(\log[\xi])^{\sigma_{\max}/\sigma_{\min}}}\,d\xi \\ &= C'\,e^{-M'[\eta]^{1/s_{\rm cr}}(\log[\eta])^{\sigma_{\max}/\sigma_{\min}}}.\end{aligned}$$

We finally deduce from (A.1) and (A.4) that

$$C'\,e^{-M'[\eta]^{1/s_{\rm cr}}(\log[\eta])^{\sigma_{\max}/\sigma_{\min}}} \leq |\mathcal{F}_{y\to\eta}(\varphi F)(\eta)| \leq C\,e^{-\mu|\eta|^{1/(s_{\rm cr}-\nu)}},$$

that cannot be true for large $|\eta|$. So $F \notin G^{s_{\rm cr}-\nu}(\mathbb{R}^N)$, for any $\nu > 0$.

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