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NONLINEAR ESTIMATES IN ANISOTROPIC GEVREY SPACES

Lucio Cadeddu*, Todor Gramchev*

Abstract. We introduce scales of Banach spaces of anisotropic Gevrey functions depending on multidimensional parameters. We prove estimates in such spaces for composition maps and nonlinearities in conservative forms. Applications for solvability and regularity of solutions of nonlinear PDEs are outlined.

1. Anisotropic Gevrey Spaces

Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $\vec{\sigma} = (\sigma_1, \dots, \sigma_n) \in [1, +\infty[^n$. We define $G^{\vec{\sigma}}_{un}(\Omega)$ - the spaces of the uniformly anisotropic $G^{\vec{\sigma}}(\Omega)$ Gevrey functions - as the set of all $f \in C^\infty(\Omega)$ such that there exists $C > 0$ satisfying

$$\sup_{x\in\Omega} |\partial_x^\alpha f(x)| \leq C^{|\alpha|+1} \alpha!^{\vec{\sigma}}, \qquad \alpha \in \mathbb{Z}^n_+. \tag{1.1}$$

where $\alpha! = \alpha_1! \dots \alpha_n!$, $\alpha!^{\vec{\sigma}} = \alpha!^{\sigma_1} \dots \alpha_n!^{\sigma_n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$. In particular, if $\sigma_1 = \dots \sigma_n = s$ we recover the well known space $G^s_{un}(\Omega)$ of uniformly Gevrey functions of index s. Local Gevrey spaces $G^s(\Omega)$ are defined in a natural way cf. [18], [16] for more details on Gevrey spaces. In particular, note that $\partial_x^\beta : G^{\vec{\sigma}}_{un}(\Omega) \mapsto G^{\vec{\sigma}}_{un}(\Omega)$ for all $\beta \in \mathbb{Z}^n_+$ and $G^{\vec{\sigma}}_{un}(\Omega) \hookrightarrow G^{\vec{\tau}}_{un}(\Omega)$ provided $\sigma_j \leq \tau_j$, $j = 1, \dots, n$.

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We point out that the Gevrey spaces are a natural framework for the study of PDEs with multiple characteristics and questions of regularity of solutions to evolution PDEs of Mathematical Physics. Typically, in the applications, one introduces scales of Banach spaces with norms depending on one parameter. The first type of norms is

$$\|u\|_{s,T} := \sum_{\alpha\in\mathbb{Z}^n_+} \frac{T^{|\alpha|}}{(\alpha!)^s}\|\partial^\alpha_x u\| < +\infty, \tag{1.2}$$

where $\|\cdot\|$ stands for the sup–norm in Ω or for some L^p based Sobolev norms $\|\cdot\|_{H^k_p}$, $1 < p < \infty$, $k \geq 0$. cf. [1], [10], [11] and the references therein. Another approach relies on the use of exponential norms by means of the Fourier transform cf. [2], [4], [5], [6], [7], [8], [10], [12], [14], [17]).

Given $f \in G^s_{un}(\mathbb{R}^n)$, in view of (1.1), we can define

$$\rho_s(f) = \sup\{T > 0 : \text{ such that (1.1) holds for } \sigma_1 = \ldots = \sigma_n = s.\} \tag{1.3}$$

and if $s = 1$, we obtain that every $f \in G^1_{un}(\mathbb{R}^n)$ is extended to a holomorphic function in $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; \max_{j=1,\ldots,n} |\mathrm{Im}\, z_j| < \rho_s(f)\}$. Clearly the definition of $\rho_s(f)$ does not take into account the different behaviour in the multidimensional case.

We introduce scales of Banach spaces $G^{\vec{\sigma}}(\Omega;\vec{T})$, $\vec{T} = (T_1, \ldots, T_n) \in]1, +\infty[^n$ defined as follows

$$|\!|\!|u|\!|\!|_{\vec{\sigma},\vec{T}} := \sum_{\alpha\in\mathbb{Z}^n_+} \frac{\vec{T}^\alpha}{\alpha!^{\vec{\sigma}}}|\partial^\alpha u|_\infty < +\infty, \tag{1.4}$$

where $|\cdot|_\infty$ stands for the sup–norm in Ω. If $\vec{\sigma} = (s, \ldots, s)$ we write

$$\|u\|_{s,\rho} := |\!|\!|u|\!|\!|_{s,(\rho,\ldots,\rho)}, \qquad G^s(\Omega;\rho) := G^s(\Omega;(\rho,\ldots,\rho)).$$

One readily gets that $G^{\vec{\sigma}}(\Omega;\vec{T})$ are Banach algebras (because $\sigma_j \geq 1$ for all $j = 1, \ldots, n$). We stress that, broadly speaking, when $T_1 = \ldots = T_n$, such type of spaces have been used for showing local Gevrey solvability and/or Gevrey well-posedness of the Cauchy problem for nonlinear evolution PDEs wih multiple characteristics cf. [2], [10], [5], [6], [12], [15]. In particular, [2], [10], [6] deal with some nonanalytic Gevrey perturbations.

For more details and geometrical features of $G^{\vec{\sigma}}(\Omega)$ type spaces, as well as applications to linear PDEs with multiple characteristics cf. [3] and the references therein (see also [13] where a rather complete theory of microlocal inhomogeneous Gevrey spaces, closely related to $G^{\vec{\sigma}}(\Omega)$, is developed).

We illustrate the advantage of the multi-scale type norms (1.4) even for the usual isotropic Gevrey space $G^s_{un}(\mathbb{R}^n)$ when $n > 1$. Let $f \in G^s(\mathbb{R}; \rho_0)$ for some $s \geq 1$, $\rho_0 > 0$. Consider a multi-rescaling action $g_\lambda(x_1, x_2) = f(x_1/\lambda_1)f(x_2/\lambda_2)$, where $\lambda = (\lambda_1, \lambda_2) \in]0, +\infty[^2$. Then $g_\lambda \in G^s(\mathbb{R}^2; \rho)$ for $\rho \leq \rho_0 \min\{\lambda_1, \lambda_2\}$. On the other hand, $g \in G^{(s,s)}(\mathbb{R}^2; (\rho_0\lambda_1, \rho_0\lambda_2))$. Next, consider the Cauchy problem

$$(1.5)\quad \partial_t u + \sum_{j=1}^{n} a_j x_j \partial_{x_j} u = 0, \quad u(0, x) = u_0(x) \in G^s(\mathbb{R}^n; \rho_0), \qquad t > 0, x \in \mathbb{R}^n,$$

where $a_j \in \mathbb{R}$, $j = 1, \ldots, n$, $s \geq 1$, $\rho_0 > 0$. The unique solution is given by $u(t, x) = u_0(x_1 e^{-a_1 t}, \ldots, x_n e^{-a_n t})$. If we use the one parameter scale of Banach spaces $G^s(\Omega; \rho)$ the solution $u(t, \cdot)$ of (1.5) satisfies $\|u(t, \cdot)\|_{s,\rho(t)} < +\infty$, where $\rho(t) = \rho_0 \exp(a_{min} t)$ and $a_{min} = \min\{a_1, \ldots, a_n\}$ while setting $\vec{T}(t) = (\rho_0 \exp(a_1 t), \ldots, \rho_0 \exp(a_n t))$ we get $\vert\vert\vert u(t, \cdot)\vert\vert\vert_{s,\vec{T}(t)} < +\infty$ for $t > 0$. Moreover, the following estimate holds: $\|u(t, \cdot)\|_{s,\rho(t)} \leq \vert\vert\vert u(t, \cdot)\vert\vert\vert_{s,\vec{T}(t)}$ for $t > 0$.

The first goal of our paper is motivated by results on nonlinear estimates in the framework of the isotropic G^s classes and their applications to PDEs cf. [2], [10]. Our first main result - see section 2 - is concerned with estimates of the action of composition maps in $G^{\vec{\sigma}}(\Omega; \vec{T})$. It may be also viewed as an analogue in $G^{\vec{\sigma}}$ classes of the Schauder type estimates for inhomogeneous Sobolev spaces in [9].

Secondly, we propose abstract energy estimates for nonlinearities in conservative forms in scales of suitable Hilbert spaces of $L^2(\Omega)$ based $G^{\vec{\sigma}}$ anisotropic functions, for $\Omega = \mathbb{R}^n$ and Ω being the n–dimensional torus. As particular cases we recover, when $\sigma_1 = \ldots = \sigma_n = s$, energy estimates in [7], [8], [14] for $s = 1$ (see also [11] for $s \geq 1$).

Our results will be applied for the study of solvability and regularity properties of solutions of Cauchy problems for abstract systems of nonlinear evolution PDEs in anisotropic Gevrey spaces. This will be done in another paper.

2. Nonlinear superposition estimates

Given $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n) \in]1, +\infty[^n$ we set $\sigma_0 = \min\{\sigma_1, \ldots \sigma_n\}$. Clearly, a smooth function g preserves $G^{\vec{\sigma}}(\Omega : \mathbb{R})$ iff $g \in G^{\sigma_0}(\mathbb{R})$. Our goal is to investigate the action of $(g \circ f)(x) := g(f(x))$ for $f \in G^{\vec{\sigma}}(\Omega; \vec{T})$, f real valued.

First, we show the following identity

Lemma 2.1. *Let $p \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ with $|\alpha| \geq 1$. Then*

$$\partial_x^\alpha (\prod_{\ell=1}^p h_\ell(x)) = \sum_{\substack{\alpha_{k;1}+\ldots+\alpha_{k;p}=\alpha_k \\ k=1,\ldots,n}} \prod_{j=1}^n \frac{\alpha_j!}{\alpha_{j;1}! \ldots \alpha_{j;p}!}$$

$$\times \prod_{\ell=1}^p \partial_{x_1}^{\alpha_{1;\ell}} \ldots \partial_{x_n}^{\alpha_{n;\ell}} h_\ell(x) \tag{2.6}$$

Proof. We shall proceed by induction with respect to n. The detailed proof of the case $n = 1$ can be found in [GR], [GB]. We observe that for $n \geq 2$

$$\partial_x^\alpha (\prod_{\ell=1}^p h_\ell(x)) = \sum_{\alpha_{n;1}+\ldots+\alpha_{n;p}=\alpha_n} \frac{\alpha_n!}{\alpha_{n;1}! \ldots \alpha_{n;p}!} \partial_{x'}^{\alpha'} \prod_{\ell=1}^p \partial_{x_n}^{\alpha_{n;\ell}} h_\ell(x), \tag{2.7}$$

where $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$, $x' = (x_1, \dots, x_{n-1})$. The inductive assumption for dimension $n-1$ and the Leibnitz rule for the action of $\partial_{x'}^{\alpha'}$ on the products of p functions complete the proof. □

Next we propose a refined version of Faà di Bruno type formulas

Proposition 2.2. *Let $\sigma = (\sigma_1, \dots, \sigma_n) \in]0, +\infty[^n$. Then for every $g \in C^\infty(\mathbb{R} : \mathbb{R})$, $f \in C^\infty(\mathbb{R}^n : \mathbb{R})$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ with $|\alpha| \geq 1$ the following identity is true*

$$\partial_x^\alpha (g(f(x)) = \alpha!^{\vec{\sigma}} \sum_{j=1}^{|\alpha|} \frac{g^{(j)}(f(x))}{(j!)^{\sigma_0}} \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\ldots,\alpha^j}$$

$$\times \prod_{\ell=1}^j \frac{\partial_{x_1}^{\alpha_1^\ell} \ldots \partial_{x_n}^{\alpha_n^\ell} f(x)}{(\alpha_1^\ell!)^{\sigma_1} \ldots (\alpha_n^\ell)^{\sigma_n}}, \tag{2.8}$$

where $\alpha^\ell = (\alpha_1^\ell, \dots, \alpha_n^\ell) \in \mathbb{Z}_+^n$ and

$$\mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\ldots,\alpha^j} = (j!)^{\sigma_0 - 1} \prod_{k=1}^n \left(\frac{\alpha_k^1! \ldots \alpha_k^j!}{\alpha_k!} \right)^{\sigma_k - 1}. \tag{2.9}$$

Proof. We have

$$\partial_x^\alpha (g(f(x)) = \sum_{j=1}^{|\alpha|} \frac{g^{(j)}(f(x))}{j!} \partial_y^\alpha ((f(y) - f(x))^j)|_{y=x}$$

$$= \sum_{j=1}^{|\alpha|} \frac{g^{(j)}(f(x))}{(j!)^{\sigma_0}} M^{\sigma_0}_{\alpha,j}[f], \tag{2.10}$$

where

$$(2.11) \quad M^{\sigma_0}_{\alpha,j}[f] = (j!)^{\sigma_0 - 1} \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \frac{\alpha!}{\alpha^1!\ldots\alpha^j!} \partial_x^{\alpha^1} f(x) \ldots \partial_x^{\alpha^j} f(x).$$

Dividing by $\alpha!^{\vec{\sigma}}(j!)^{\sigma_0-1}$ in (2.11) we obtain

$$(2.12) \quad \frac{M^{\sigma_0}_{\alpha,j}[f]}{\alpha!^{\vec{\sigma}}(j!)^{\sigma_0-1}} = \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \prod_{k=1}^{n} \left(\frac{\alpha_k^1!\ldots\alpha_k^j!}{\alpha_k!} \right)^{\sigma_k - 1} \prod_{\ell=1}^{j} \frac{f^{(\alpha^\ell)}(x)}{\alpha^\ell!^{\vec{\sigma}}}$$

which implies (2.9). $\square$

Set $\| f \|_{\vec{\sigma},\vec{T}} = | f |_{\vec{\sigma},\vec{T}} - |f|_\infty$. The main result on the action of nonlinear maps in $G^{\vec{\sigma}}(\Omega, \vec{T})$ is

Theorem 2.3. *Let $\vec{\sigma} \in [1, +\infty[^n$ and let $g \in G^{\sigma_0}_{un}(\mathbb{R})$, $g(0) = 0$, with $\sigma_0 = \min\{\sigma_1, \ldots, \sigma_n\}$. Since $g' \in G^{\sigma_0}_{un}(\mathbb{R})$ as well, we get by (1.3) that $\rho_0 := \rho_{\sigma_0}(g') > 0$. Then the following estimate holds*

$$(2.13) \qquad | g \circ f |_{\vec{\sigma},\vec{T}} \leq n^{\sigma_0 - 1} \|g'\|_{\sigma_0,\rho} | f |_{\vec{\sigma},\vec{T}}$$

for $\vec{T} \in]0, +\infty[^n$, $f \in G^{\vec{\sigma}}(\Omega; \vec{T})$, f being real valued, provided $\rho := n^{\sigma_0-1} \| f \|_{\vec{\sigma},\vec{T}} < \rho_0$.

Proof. We note that by $g(0) = 0$ we get $|g \circ f|_\infty \leq |g'|_\infty |f|_\infty$. In view of (2.8), (2.9), given $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \geq 1$, we have

$$\begin{aligned} \frac{\vec{T}^\alpha}{\alpha!^{\vec{\sigma}}} |\partial_x^\alpha (g(f(x)))| \leq & \sum_{j=1}^{|\alpha|} \frac{|g^{(j)}(f(x))|}{(j!)^{\sigma_0}} \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\ldots,\alpha^j} \\ (2.14) \qquad & \times \prod_{\ell=1}^{j} \frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}} |f^{(\alpha^\ell)}(x)| \end{aligned}$$

Therefore, summing (2.14) over $\alpha \in \mathbb{Z}_+^n$, $\alpha \neq 0$, we get

$$
\begin{aligned}
\| g \circ f \|_{\vec{\sigma},\vec{T}} &\leq \sum_{\alpha \in \mathbb{Z}_+^n \setminus 0} \sum_{j=1}^{|\alpha|} \frac{|g^{(j)} \circ f|_\infty}{(j!)^{\sigma_0}} \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\ldots,\alpha^j} \\
&\times \prod_{\ell=1}^{j} \frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}} |f^{(\alpha^\ell)}|_\infty \\
&\leq \sum_{j=1}^{\infty} \frac{|g^{(j)} \circ f|_\infty}{(j!)^{\sigma_0}} N_j[f;\vec{\sigma},\vec{T}] \qquad (2.15)
\end{aligned}
$$

where

$$
N_j[f;\vec{\sigma},\vec{T}] = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \geq j} \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\ldots,\alpha^j} \prod_{\ell=1}^{j} \frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}} |f^{(\alpha^\ell)}|_\infty \qquad (2.16)
$$

The final step depends on subtle combinatorial estimates.

Lemma 2.4. *Let* $\sigma_0 = \min\{\sigma_1, \ldots, \sigma_n\}$. *Then*

$$
\begin{aligned}
N_j[f;\vec{\sigma},\vec{T}] &\leq \sum_{j_1+\ldots+j_n=j} \left(\frac{j!}{j_1!\ldots j_n!} \right)^{\sigma_0-1} \\
&\times \sum_{\substack{\alpha^1+\ldots+\alpha^j=\alpha \\ |\alpha^\ell|\geq 1, \ell=1,\ldots,j}} \prod_{\ell=1}^{j} \frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}} |f^{(\alpha^\ell)}|_\infty \qquad (2.17)
\end{aligned}
$$

for all $j \in \mathbb{N}$.

Proof. Fix $\alpha^1, \ldots, \alpha^j \in \mathbb{Z}_+^n$ with $|\alpha^\ell| > 0$ for $\ell = 1, \ldots, j$. Let $k \in \{1, \ldots, n\}$ and we fix s_k to be a nonnegative integer such that at least s_k components of $(\alpha_k^1, \ldots, \alpha_k^j)$ are nonzero. Recall the combinatorial inequalities (cf. [10])

$$
\frac{m_1!\ldots m_q! q!}{(m_1+\ldots+m_q)!} \leq 1, \qquad q \in \mathbb{N}, m_1, \ldots, m_q \in \mathbb{N} \qquad (2.18)
$$

Hence, by (2.9), we get

$$\mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\dots,\alpha^j} = (j!)^{\sigma_0-1}\prod_{k=1}^{n}\left(\frac{1}{s_k!}\right)^{\sigma_k-1}\left(\frac{\alpha^1_k!\dots\alpha^j_k!s_k!}{\alpha_k!})\right)^{\sigma_k-1}$$

$$\leq \frac{(j!)^{\sigma_0-1}}{(s_1!)^{\sigma_1-1}\dots(s_n!)^{\sigma_n-1}} \leq \left(\frac{j!}{s_1!\dots s_n!}\right)^{\sigma_0-1} \tag{2.19}$$

Clearly, if $s_k = j$ for at least one k, the last inequality yields $\mathcal{P}^{\vec{\sigma},j}_{\alpha^1,\dots,\alpha^j} \leq 1$. Let now $s_k < j$ for $k = 1,\dots,j$. We note that the inequalities $|\alpha^\ell| \geq 1$, $\ell = 1,\dots,j$ imply $s_1 + \dots + s_n \geq j$. If $s_1 + \dots + s_n > j$, standard combinatorial arguments and the fact that the RHS of (2.19) increases if some s_ℓ decreases reduce to the case $s_1 + \dots + s_n = j$. The final step consists of bounding

$$\sum_{\substack{\alpha^1+\dots+\alpha^j=\alpha\\|\alpha^\ell|\geq 1,\ell=1,\dots,j}}\dots \leq \sum_{s_1+\dots+s_n=j}\left(\frac{j!}{s_1!\dots s_n!}\right)^{\sigma_0-1}\sum_{\substack{\alpha^1+\dots+\alpha^j=\alpha\\|\alpha^\ell|\geq 1,\ell=1,\dots,j}}^{j_1,\dots,j_n}\dots$$

where the internal summation in the RHS is over $\alpha^1,\dots\alpha^j$ with $s_1,\dots,s_j$ as above. The proof of (2.17) is complete. $\square$

Next, we plug (2.17) into (2.15) and obtain

$$\begin{aligned}
[\![g\circ f]\!]_{\vec{\sigma},\vec{T}} &\leq \sum_{j=1}^{\infty}\frac{|g^{(j)}\circ f|_\infty}{(j!)^{\sigma_0}}\sum_{j_1+\dots+j_n=j}\left(\frac{j!}{j_1!\dots j_n!}\right)^{\sigma_0-1}\\
&\times \sum_{\substack{\alpha^1+\dots+\alpha^j=\alpha\\|\alpha^\ell|\geq 1,\ell=1,\dots,j}}\prod_{\ell=1}^{j}\frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}}|f^{(\alpha^\ell)}|_\infty\\
&\leq \sum_{j=1}^{\infty}\frac{|g^{(j)}\circ f|_\infty}{(j!)^{\sigma_0}}\sum_{j_1+\dots+j_n=j}\left(\frac{j!}{j_1!\dots j_n!}\right)^{\sigma_0-1}\prod_{\ell=1}^{j}\sum_{\alpha^\ell\in\mathbb{Z}^n_+\setminus 0}\frac{\vec{T}^{\alpha^\ell}}{\alpha^\ell!^{\vec{\sigma}}}|f^{(\alpha^\ell)}|_\infty\\
&\leq \sum_{j=1}^{\infty}\frac{|g^{(j)}\circ f|_\infty}{(j!)^{\sigma_0}}([\![f]\!]_{\vec{\sigma},\vec{T}})^j\sum_{j_1+\dots+j_n=j}\left(\frac{j!}{j_1!\dots j_n!}\right)^{\sigma_0-1}\\
&\leq n^{\sigma_0-1}[\![f]\!]_{\vec{\sigma},\vec{T}}\sum_{j=1}^{\infty}\frac{|g^{(j)}\circ f|_\infty}{((j-1)!)^{\sigma_0}}(n^{\sigma_0-1}[\![f]\!]_{\vec{\sigma},\vec{T}})^{j-1}\\
&\leq n^{\sigma_0-1}[\![f]\!]_{\vec{\sigma},\vec{T}}\|g'\|_{\sigma_0,\rho}
\end{aligned} \tag{2.20}$$

provided $\rho := n^{\sigma_0-1}[\![f]\!]_{\vec{\sigma},\vec{T}} \leq \rho_0$. The proof of (2.13) is complete. $\square$

3. Gevrey energy estimates for nonlinear conservative terms

The main goal of this section is to derive energy estimates in anisotropic Gevrey spaces for nonlinear terms in a conservative form of the type $\nabla \cdot (K[\phi]\phi)$ where typically $K[\phi] = (K_1[\phi], \ldots, K_n[\phi])$, K_j, $j = 1, \ldots, n$ are linear continuous operators in Sobolev spaces (in some cases homogeneous Sobolev spaces). We assume that $u = K[v]$ is divergence free, i.e.,

$$\nabla \cdot K[v] = \sum_{j=1}^{n} \partial_{x_j} K_j[v] = 0. \tag{3.1}$$

Such nonlinear terms appear in evolution PDEs of Navier–Stokes type, Euler type or Kirchhoff type (see [8], [14], [11] and the references therein).

We shall treat (simultaneously) two cases: $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. The choice of the lattice $(2\pi\mathbb{Z}^n)$ instead of $\mathbb{Z}^n$ is not restrictive but it allows us to use unified notations. More precisely, both the continuous (on $\mathbb{R}^n$) and the discrete (on $\mathbb{T}^n$) Fourier transform are defined by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\Omega} \exp(-ix\xi)\, f(x)\, dx, \qquad x\xi = x_1\xi_1 + \ldots + x_n\xi_n,$$

identifying $\mathbb{T}^n = [0, 2\pi]^n$ in the case $\Omega = \mathbb{T}^n$. The inverse Fourier transform is written as $\mathcal{F}^{-1}_{\xi\to x} f = \int_{\mathbb{K}^n} \exp(ix\xi)\, f(\xi)\ d\!\!\!{}^-\xi$, $d\!\!\!{}^-\xi = (2\pi)^{-n} d\xi$, with $\mathbb{K} = \mathbb{R}$ if $\Omega = \mathbb{R}^n$ while in the case of $\Omega = \mathbb{T}^n$ $\mathbb{K} = \mathbb{Z}$ and $\int_{\mathbb{Z}^n} h(\xi)\ d\!\!\!{}^-\xi$ stands for $(2\pi)^{-n} \sum_{\xi\in\mathbb{Z}^n} h(\xi)$

Similarly, we set $L^p(\mathbb{K}^n) = \begin{cases} L^p(\mathbb{R}^n) & \text{if } \mathbb{K} = \mathbb{R} \\ \ell^p(\mathbb{Z}^n) & \text{if } \mathbb{K} = \mathbb{Z} \end{cases}, \qquad 1 \le p \le \infty.$

Let $\vec{\sigma} \in [1, +\infty[^n$, $\vec{\tau} \in [0, +\infty[^n$, and $r > 0$. We define:

$$< f, g >_{H^r} \ = \int_{\mathbb{K}^n} (\delta + |\xi|^r)^2 \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \tag{3.2}$$

$$\exp(\vec{\tau} < D >^{1/\vec{\sigma}})v(x) \ = \ \mathcal{F}^{-1}_{\xi\to x}(\exp(\sum_{j=1}^{n} \tau_j < \xi_j >^{\frac{1}{\sigma_j}})\hat{v}(\xi)) \tag{3.3}$$

$$< f, g >_{\vec{\sigma},\vec{\tau};H^r} \ = \ < \exp(\vec{\tau} < D >^{1/\vec{\sigma}})f, \exp(\vec{\tau} < D >^{1/\vec{\sigma}}\ g >_{H^r} \tag{3.4}$$

$$\|f\|_{\vec{\sigma},\vec{\tau};H^r} \ = \ \sqrt{< \exp(\vec{\tau} < D >^{1/\vec{\sigma}})f, \exp(\vec{\tau} < D >^{1/\vec{\sigma}}\ f >_{H^r}} \tag{3.5}$$

where $\delta = 1$ when $\Omega = \mathbb{R}^n$ while $\delta = 0$ if $\Omega = \mathbb{T}^n$, and in that case we consider functions on $\mathbb{T}^n$ with mean value zero, i.e. $\hat{f}(0) = 0$. In view of (3.4), (3.5) we introduce in a natural way the Hilbert space $G^{\vec{\sigma}}(\Omega; \vec{\tau}, H^r)$ as the set of all functions $f \in G^{\vec{\sigma}}_{un}(\Omega)$ such that $\|f\|_{\vec{\sigma},\vec{\tau};H^r} < +\infty$.

We are interested in estimating the commutator

$$\begin{aligned} \mathcal{C}[\phi] &= < \exp(\vec{\tau} < D >^{1/\vec{\sigma}}) K[\phi] \cdot \nabla\phi), \exp(\vec{\tau} < D >^{1/\vec{\sigma}})\phi >_{H^r} \\ &- < \nabla \cdot K[\phi] \exp(\vec{\tau} < D >^{1/\vec{\sigma}})\phi), \exp(\vec{\tau} < D >^{1/\vec{\sigma}})\phi >_{H^r} \end{aligned} \tag{3.6}$$

where $\phi \in G^{\vec{\sigma}}(\Omega; \vec{\tau}, H^{\tilde{r}})$ for some $\tilde{r} > r$.

We set, as in section 2, $\sigma_0 = \min\{\sigma_1, \ldots, \sigma_n\} \geq 1$.

Theorem 3.1. *Suppose that $r > r_0 := n/2 + 1 + 1/(2\sigma_0)$. Then for every $\varepsilon > 0$ there exist two positive constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$ such that*

$$\begin{aligned} |\mathcal{C}[\phi]| &\leq C_1 \|\phi\|_{H^r} (\|u\|_{H^r} \|\phi\|_{H^{r_0+\varepsilon}} + \|\phi\|_{H^r} \|u\|_{H^{r_0+\varepsilon}}) \\ &+ C_2 \sum_{j=1}^n \tau_j \|\phi\|_{\vec{\sigma},\vec{\tau};H^{r+1/(2\sigma_0)}} (\|\phi\|_{\vec{\sigma},\vec{\tau};H^{r+1/(2\sigma_0)}} \|u\|_{\vec{\sigma},\vec{\tau};H^{r_0+\varepsilon}} \\ &+ \|u\|_{\vec{\sigma},\vec{\tau};H^{r+1/(2\sigma_0}} \|\phi\|_{\vec{\sigma},\vec{\tau};H^{r_0+\varepsilon}}). \end{aligned} \tag{3.7}$$

Proof. Set $u := K[\Phi]$. Note that $\nabla \cdot u = 0$ iff $\hat{u}(\zeta) \cdot \zeta \equiv 0$. Set

$$f_{\vec{\sigma}}(\xi; \vec{\tau}) = < \xi >^r \exp(\sum_{j=1}^n \tau_j < \xi_j >^{1/\sigma_j}), \tag{3.8}$$

$$Q_{\vec{\sigma}}(\xi, \eta; \vec{\tau}) = f_{\vec{\sigma}}(\xi; \vec{\tau}) - f_{\vec{\sigma}}(\eta; \vec{\tau}) \tag{3.9}$$

By the Parseval identity, $\mathcal{C}[\phi]$ can be rewritten as follows

$$\mathcal{C}[\phi] = i \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_\eta} Q_{\vec{\sigma}}(\xi, \eta; \vec{\tau}) (\hat{u}(\xi - \eta) \cdot \eta) f_{\vec{\sigma}}(\xi; \vec{\tau}) \overline{\phi(\xi)} \phi(\eta)\, d\xi d\eta \tag{3.10}$$

We have

$$\begin{aligned} |Q_{\vec{\sigma}}(\xi, \eta; \vec{\tau})| &\leq C|\xi - \eta|(< \xi - \eta >^{r-1} + < \eta >^{r-1}) \\ &+ C|\xi - \eta| \sum_{j=1}^n \tau_j (< \xi - \eta >^{r-1} + < \eta >^{r-1}) < \eta_j >^{1/\sigma_j} f_{\vec{\sigma}}(\eta; \vec{\tau}) \\ &+ C \sum_{j=1}^n \tau_j |\xi_j - \eta_j|^{1/\sigma_j} (< \xi - \eta >^r + < \eta >^r) f_{\vec{\sigma}}(\xi - \eta; \vec{\tau}) f_{\vec{\sigma}}(\eta; \vec{\tau}). \end{aligned} \tag{3.11}$$

Indeed, in view of the inequalities $\exp(|z|) \leq 1+|z|\exp(|z|)$, $|t+s|^\rho \leq |t|^\rho+|s|^\rho$, $||t|^\rho - |s|^\rho| \leq |t-s|^\rho$, provided $0 < \rho \leq 1$, we get

$$\begin{aligned} |Q_{\vec{\sigma}}(\xi,\eta;\vec{\tau})| &\leq |<\xi>^r - <\eta>^r| f_{\vec{\sigma}}(\eta;\vec{\tau}) + <\xi>^r |f_{\vec{\sigma}}(\xi;\vec{\tau}) - f_{\vec{\sigma}}(\eta;\vec{\tau})| \\ &\leq \sum_{j=1}^n \tau_j (<\xi-\eta>^{r-1} + <\eta>^{r-1}) <\eta_j>^{1/\sigma_j} f_{\vec{\sigma}}(\eta;\vec{\tau}) \\ &+ \sum_{j=1}^n \tau_j |\xi_j - \eta_j|^{1/\sigma_j} (<\xi-\eta>^r + <\eta>^r) f_{\vec{\sigma}}(\xi-\eta;\vec{\tau}) f_{\vec{\sigma}}(\eta;\vec{\tau}) \end{aligned}$$

which implies (3.11).

Set

$$[\phi]_r(\xi) = <\xi>^r |\hat{\phi}(\xi)|, \qquad N^\rho_{\vec{\sigma},\vec{\tau}}[\phi](\xi) = <\xi>^\rho \exp(\sum_{j=1}^n \tau_j <\xi_j>^{1/\sigma_j}) |\hat{\phi}(\xi)|.$$

In view of (3.10) and (3.11) we have $|\mathcal{C}[\phi]| \leq \sum_{k=1}^4 \Theta_k[\phi]$, where

$$\Theta_1[\phi] = \int_{\mathbb{K}^n} \int_{\mathbb{K}^n} [\phi]_r(\xi)[u]_r(\xi-\eta)[\phi]_1(\eta)\, d\!\!\!{}^-\xi\, d\!\!\!{}^-\eta,$$

$$\Theta_2[\phi] = \int_{\mathbb{K}^n} \int_{\mathbb{K}^n} [\phi]_r(\xi)[u]_1(\xi-\eta)[\phi]_r(\eta)\, d\!\!\!{}^-\xi\, d\!\!\!{}^-\eta,$$

$$\Theta_3[\phi] = \sum_{j=1}^n \tau_j \int_{\mathbb{K}^n} \int_{\mathbb{K}^n} N^r_{\vec{\sigma},\vec{\tau}}[\phi](\xi) N^{r+1/(2\sigma_j)}_{\vec{\sigma},\vec{\tau}}[u](\xi-\eta) N^{1+1/(2\sigma_j)}_{\vec{\sigma},\vec{\tau}}[\phi](\eta)\, d\!\!\!{}^-\xi\, d\!\!\!{}^-\eta,$$

$$\Theta_4[\phi] = \sum_{j=1}^n \tau_j \int_{\mathbb{K}^n} \int_{\mathbb{K}^n} N^{r+1/(2\sigma_j)}_{\vec{\sigma},\vec{\tau}}[\phi](\xi) N^{1+1/(2\sigma_j)}_{\vec{\sigma},\vec{\tau}}[u](\xi-\eta) N^{r+1/(2\sigma_j)}_{\vec{\sigma},\vec{\tau}}[\phi](\eta)\, d\!\!\!{}^-\xi\, d\!\!\!{}^-\eta.$$

We need

Lemma 3.2. *Let $g_j : \mathbb{K}^n \mapsto \mathbb{R}$, $j = 1,2,3$ satisfy the following property: there exist $j_1 \in \{1,2,3\}$ and $\mu > n/2$ such that*

$$\text{(3.12)} \quad \| <\cdot>^\mu g_{j_1}\|_{L^2(\mathbb{K}^n)} := \left(\int_{\mathbb{K}^n} <\xi>^{2\mu} \|g_{j_1}(\xi)\|^2\, d\xi\right)^{1/2} < +\infty$$

$$\text{(3.13)} \quad \|g_{j_p}\|_{L^2(\mathbb{K}^n)} < +\infty \quad \textit{for } j_p \in \{1,2,3\} \setminus \{j_1\},\ p = 2,3.$$

Then

$$\begin{aligned} I[g_1, g_2, g_3] &= \int_{\mathbb{K}^d}\int_{\mathbb{K}^d} |g_1(\xi-\eta)g_2(\xi)g_3(\eta)|\, d\xi d\eta \\ &\leq \kappa_\mu \| < \cdot >^\mu g_{j_1}\|_{L^2}\|g_{j_2}\|_{L^2}\|g_{j_3}\|_{L^2} \end{aligned} \tag{3.14}$$

where $\kappa_\mu = \left(\int_{\mathbb{K}^n} < \xi >^{-2\mu}\, d\xi\right)^{1/2} < +\infty$.

P r o o f. We observe that by $\mu > n/2$ and the Schwartz inequality we get

$$\|g_{j_1}\|_{L^1} \leq c\| < \cdot >^\mu g_{j_1}\|_{L^2} < +\infty \tag{3.15}$$

$$\| < \cdot >^\mu g_{j_p}\|_{L^2} < +\infty, \qquad \text{for } p = 2, 3 \tag{3.16}$$

where $c = \left(\int_{\mathbb{K}^n} < \xi >^{-2\mu}\, d\xi\right)^{1/2} < +\infty$. We complete the proof of (3.14) by applying the Schur lemma and the above estimates. □

We apply the above lemma to $\Theta_k[\phi]$, with the choice of g_{j_1} being the term with Sobolev index $1 + 1(2\sigma_j)$, $k = 1, 2, 3, 4$, and obtain the desired estimate (3.7). □

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Dipartimento di Matematica e Informatica
Università di Cagliari
via Ospedale 72, 09124 Cagliari
Italia
cadeddu@unica.it, todor@unica.it