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A MODIFIED MODEL OF RISK BUSINESS

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We consider the risk model in which the claim counting process $\{N(t)\}$ is a modified stationary renewal process. $\{N(t)\}$ is governed by a sequence of independent and identically distributed inter-occurrence times with a common exponential distribution function with mass at zero equal to $\rho > 0$. The model is called a Pólya - Aeppli risk model. The Cramér - Lundberg approximation and the martingale approach of the model are given.

1. Introduction

Assume that the standard model of an insurance company, called risk process $\{X(t), t \geq 0\}$ is given by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k, \quad (\sum_{1}^{0} = 0). \tag{1}$$

Here c is a positive real constant representing the risk premium rate. The sequence $\{Z_k\}_{k=1}^{\infty}$ of mutually independent and identically distributed random variables with common distribution function F, $F(0) = 0$, and finite mean value μ is independent of the counting process $N(t), \quad t \geq 0$. The process $N(t)$ is interpreted as the number of claims on the company during the interval $[0, t]$. In

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the classical risk model the process $N(t)$ is a homogeneous Poisson process, see for instance [1] and [6].

The Pólya - Aeppli distribution is a generalization of the classical $Po(\lambda)$ distribution, by adding a new parameter ρ, see [2]. It appears in [4] and [5] as a compound Poisson distribution. The additional parameter ρ is called an inflation parameter. The Pólya - Aeppli process as a generalization of the Poisson process is defined in [3].

We will suppose that $N(t)$ is described by the Pólya - Aeppli distribution with mean function $\frac{\lambda}{1-\rho}t$, i.e.

$$(2)\qquad P(N(t)=n) \;=\; \begin{cases} e^{-\lambda t}, & n=0 \\ e^{-\lambda t}\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{[\lambda(1-\rho)t]^i}{i!}\rho^{n-i}, & n=1,2,\ldots. \end{cases}$$

In this section we will discus briefly the basic properties of the Pólya - Aeppli process.

In [3] is proved that the non-negative random variables $T_1, T_2, \ldots$, representing the inter- arrival times are mutually independent. The time T_1 until the first epoch is exponentially distributed with parameter λ. $T_2, T_3, \ldots$ are identically distributed as a random variable T_2. Moreover T_2 is zero with probability ρ, and with probability $1-\rho$ exponentially distributed with parameter λ, i.e. T_2 is exponentially distributed with mass at zero equal to ρ. The probability distribution function is given by

$$F_{T_2}(t) = 1-(1-\rho)e^{-\lambda t},\ t \geq 0.$$

The mean value is $ET_2 = \frac{1-\rho}{\lambda}$.

The process, described above is a delayed renewal process. It is easy to verify that the probability distribution functions of the delay T_1 and the inter - arrival times satisfy the following relation

$$F_{T_1} = \frac{1}{ET_2}\int_0^t [1-F_{T_2}(u)]du.$$

In this case the delayed renewal process is the only stationary renewal process. So, the Pólya - Aeppli process is a homogeneous process and if $\rho = 0$ it becomes a homogeneous Poisson process.

In this paper we need also the probability generating function (pgf) of the Pólya - Aeppli process. It is given by

$$P_{N(t)}(s) = e^{-\lambda t\frac{1-s}{1-\rho s}}.$$

The Pólya - Aeppli risk model is defined in [3]. The probability of ruin and the Cramér - Lundberg approximation are derived. In this paper the martingale approach is given.

2. The Pólya - Aeppli risk model

We consider the risk process $X(t)$, defined by (1), where $N(t)$ is the Pólya - Aeppli process, independent of the claim sizes $Z_k, k = 1, 2, \dots$. This process is called a Pólya - Aeppli risk model.

The relative safety loading θ is defined by

$$\theta = \frac{c(1-\rho) - \lambda\mu}{\lambda\mu} = \frac{c(1-\rho)}{\lambda\mu} - 1, \tag{3}$$

and in the case of positive safety loading $\theta > 0$, $c > \frac{\lambda\mu}{1-\rho}$.

Let

$$F_I(x) = \frac{1}{\mu}\int_0^x [1 - F(z)]dz$$

be the integrated tail distribution of the claim size. Let us define the function

$$H(z) = \rho F(z) + \frac{\lambda\mu}{c} F_I(z) \tag{4}$$

and note that

$$H(\infty) = \rho F(\infty) + \frac{\lambda}{c}\int_0^\infty [1 - F(z)]dz = \rho + \frac{\lambda\mu}{c} < 1.$$

Denote by

$$\tau(u) = \inf\{t > 0, u + X(t) \le 0\}$$

the time to ruin of a company having initial capital u. We let $\tau = \infty$, if for all $t > 0$ $u + X(t) > 0$.

The probability of ruin in the infinite horizon case is

$$\Psi(u) = P(\tau(u) < \infty)$$

and in the finite horizon case

$$\Psi(u, t) = P(\tau(u) \le t).$$

Assume that there exists a constant $R > 0$ such that

$$\int_0^\infty e^{Rz} dH(z) = 1, \tag{5}$$

where $H(z)$ is given by (4), and denote $h(R) = \int_0^\infty e^{Rz} dF(z) - 1$.

The following proposition, proved in [3] is an analogue of the corresponding approach to the classical risk model.

Proposition 1. *Let, for the Pólya-Aeppli risk model, the Cramér condition (5) holds and $h'(R) < \infty$. Then*

$$\lim_{u\to\infty} \Psi(u)e^{Ru} = \frac{\mu\theta}{A^2(\mu,\theta,R,\rho)h'(R) - \mu(1+\theta)}, \tag{6}$$

where

$$A(\mu,\theta,R,\rho) = \frac{1 - [1-\mu(1+\theta)R]\rho}{1-\rho}.$$

The relation (5) is known as the Cramér condition and the constant R, if it exists, as the adjustment coefficient or Lundberg exponent for the Pólya-Aeppli risk model.

If $\rho = 0$, $A(\mu,\theta,R,0) = 1$ and (6) coincides with the Cramér - Lundberg approximation for the classical risk model [1].

3. Martingales for the Pólya-Aeppli risk model

Let us denote by $(\mathcal{F}_t^X)$ the natural filtration generated by any stochastic process $X(t)$. $(\mathcal{F}_t^X)$ is the smallest complete filtration to which $X(t)$ is adapted.

Let us denote by $LS_Z(r) = \int_0^\infty e^{-rx} dF(x)$ the Laplace-Stieltjes transform (LS-transform) of any random variable Z with distribution function $F(x)$.

Lemma 1. *For the Pólya-Aeppli risk model*

$$Ee^{-rX(t)} = e^{g(r)t},$$

where

$$g(r) = \frac{1}{1-\rho LS_Z(-r)}[\rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr].$$

P r o o f. Let us consider the random sum from the right hand side of (1)

$$S_t = \sum_{k=1}^{N(t)} Z_k,$$

where $N(t)$ is a Pólya-Aeppli process, independent of $Z_k, k = 1, 2, \ldots$. S_t is a compound Pólya-Aeppli process and the LS- transform is given by

$$LS_{S_t}(r) = P_{N(t)}(LS_Z(r)) = e^{-\lambda t \frac{1-LS(r)}{1-\rho LS(r)}}.$$

For the LS-transform of $X(t)$ we have the following

$$LS_{X(t)}(r) = Ee^{-rX(t)} = Ee^{-r[ct-S_t]} = e^{-rct}Ee^{rS_t} =$$

$$= e^{-rct}P_{N(t)}(LS_Z(-r)) = e^{-rct}e^{-\lambda t \frac{1-LS_Z(-r)}{1-\rho LS_Z(-r)}} = e^{g(r)t}.$$

□

From the martingale theory we get the following

Lemma 2. *For all $r \in R$ the process*

$$M_t = e^{-rX(t)-g(r)t}, \ t \geq 0$$

is an $\mathcal{F}_t^X$-martingale, provided that $LS_Z(-r) < \infty$.

4. Martingale approach to the Pólya-Aeppli risk model

Using the martingale properties of M_t, we will give some useful inequalities for the ruin probability.

Proposition 2. *Let $r > 0$. For the ruin probabilities of the Pólya-Aeppli risk model we have the following results*

i) $\Psi(u,t) \leq e^{-ru} \sup_{0\leq s\leq t} e^{g(r)s}, 0 \leq t < \infty$

ii) $\Psi(u) \leq e^{-ru} \sup_{s\geq 0} e^{g(r)s}$.

iii) If the Lundberg exponent R exists, then R is the unique strictly positive solution of

$$\rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr = 0 \tag{7}$$

and $\Psi(u) \leq e^{-Ru}$.

P r o o f. i) For any $t_0 < \infty$, the martingale stopping time theorem yields the following

$$1 = M_0 = EM_{t_0\wedge\tau} = E[M_{t_0\wedge\tau}, \tau \leq t] + E]M_{t_0\wedge\tau}, \tau > t] \geq$$

$$\geq E[M_{t_0\wedge\tau}, \tau \leq t] = E[e^{-rX(\tau)-g(r)\tau}|\tau \leq t]P(\tau \leq t),$$

from which

$$P(\tau \leq t) = \frac{e^{-ru}}{E[e^{-g(r)\tau}|\tau \leq t]}.$$

The statement i) follows from the above relation.

ii) follows immediately from i) when $t \to \infty$.

iii) The Cramér condition (5) becomes

$$\rho \int_0^\infty e^{rx} dF(x) + \frac{\lambda}{c} \int_0^\infty e^{rx}(1 - F(x))dx = 1.$$

Using

$$\int_0^\infty e^{rx}(1-F(x))dx = \int_0^\infty \int_x^\infty e^{rx} dF(y)dx = \int_0^\infty \int_0^y e^{rx} dx dF(y) = \frac{1}{r}[LS_Z(-r)-1],$$

it can be written as

$$\rho LS_Z(-r) + \frac{\lambda}{cr}(LS_Z(-r) - 1) = 1.$$

This is equivalent to the equation (7).

Let us denote

$$f(r) = \rho cr LS_Z(-r) + \lambda(LS_Z(-r) - 1) - cr.$$

So R is a positive solution of $f(r) = 0$. Because $f(0) = 0, f'(0) = \lambda\mu - (1 - \rho)c < 0$ and $f''(r) = (\rho cr + \lambda)LS_Z''(-r) + 2\rho c LS_Z'(-r) > 0$ there is at most one strictly positive solution. □

Remark 1. The equation (7) is equivalent to $g(r) = 0$.

Remark 2. The above inequalities are well known for the classical risk model. In the case of $\rho = 0$ the Pólya - Aeppli risk model becomes the classical risk model. The Cramér condition (5) and the function $g(r)$ are the same, see [7].

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