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AN APPLICATION OF THE PRINCIPLE OF TOPOLOGICAL INDUCTION TO THE EXTREME POINTS THEOREM

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The V-convexity introduced in this paper uses all translations of a convex set V in a linear topological space L, instead of all half-spaces in the usual convexity. The notion of a V-extreme point is introduced and a Krein-Milman type theorem is proved using the general Principle of Topological Induction of Y. Tagamlitzki.

Let L be a linear locally convex topological space and V be a closed convex neighbourhood of the origin [1].

For the set $M \subset L$ we define the hull

$$\langle M \rangle = \bigcap \{x+V;\ x \in L,\ x+V \supset M\},$$

where the right-hand side denotes L, if no x in L satisfies $x+V \supset M$.

Definition 1. *The point $x \in M$ is called V-extreme, if for every two different points a, $b \in M$ holds $x \notin \langle\{a,b\}\rangle \setminus \{a,b\}$.*

We denote by $\mathring{V}$ the interior of V.

Definition 2. *We say that the convex neighbourhood of 0 does not contain infinite points, if there is no such $y \in L$ that $ly \in V$ for every $l \geqq 0$.*

Theorem 1. *Let M be a compact subset of L and V be a convex closed neighbourhood of 0. Let the boundary ∂V of V does not contain line segments and V does not contain infinite points.*

If there exists a point $x_0 \in L$, such that $x_0 + \mathring{V} \supset M$, then the set E of V-extreme points of M is not empty and the equality $\langle E \rangle = \langle M \rangle$ is true.

The proof consist of an application of the Principle of Topological Induction [2, 3], which we present as one theorem:

Theorem 2. (Principle of Topological Induction), 1. *Let X and Y be topological spaces. Let on X a quasi-order ($\geq$) be given, i. e. a transitive and reflexive relation, and every monotonically increasing generalized sequence $x_g \in X$, $g \in G$, in it be convergent (here G is a segment of ordinals, see* [4, Ch. 2] *for the notion of generalized sequences). Let Y be compact.*

2. *A multivalued map $f: X \to Y$ is given, which is monotonic, i. e. $x_1 \geq x_2$ implies $f(x_1) \supset f(x_2)$.*

With the symbol $\complement$ we denote the complement to a set.

For every subset $S \subset Y$ we define the parenthesis $(S) \subset Y$ in the following way:

$$(S) = \{y \in Y;\ \text{(i) for every } x \in X\ f(x) \supset S \text{ implies } y \in f(x).$$

(ii) if for some $x_1 \in X$ we have $f(x_1) \cap S \neq \emptyset$ and $\complement f(x_1) \cap S \neq \emptyset$, then there is an $x_2 \in X$, $x_2 \geqq x_1$, such that $y \in f(x_2)$ and $\complement f(x_2) \cap S \neq \emptyset\}$.

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In the space Y we define a relation $(\leqq)$: for $y_1, y_2 \in Y$, $y_1 \leqq y_2$, if and only if there exists a set $S \subset Y$ such that $y_1 \in S$ and $y_2 \in \langle S\rangle$.

It is proved that this relation is a quasi-order [2].

The quasi-order in X and the map f are supposed to satisfy the following properties:

3. The sets $\{x \in X; x \geqq a\}$ are closed for every $a \in X$.

4. For every $x \in X$ and every $y \in Y$ the sets $f(x)$ and $f^{-1}(y)$ are open.

The basic statement of the theorem is that the set of minimal elements Ex of the ordered space Y is not empty and has the following properties:

1. If $p \in \mathrm{Ex}$, then $y \leqq p$ implies $y \approx p$, i. e. for every $x \in X$ $y \in f(x)$, if and only if $p \in f(x)$.

2. The inclusion $f(x) \supset \mathrm{Ex}$ implies $f(x) \supset Y$.

Proof of Theorem 1. According to the assumptions of the theorem, we fix the point x_0 with the property $x_0 + \mathring{V} \supset M$. To apply Theorem 2, we define the space X to be the topological space L itself.

The order $(\leqq)$ in X is defined in the following way: for $x_1, x_2 \in X$ we say that $x_1 \leqq x_2$, if and only if $x_2 - x_0 = s(x_1 - x_0)$ for some number $s: 0 < s \leqq 1$. It is evident that a monotonically increasing sequence $x_g \in X$, $g \in G$, is convergent since it lies on a ray l with endpoint x_0, which is maximal element in l.

The space Y is defined as the set M with the topology induced by L. The multivalent function f is defined as follows: $f(x) = (x + \mathring{V}) \cap M$ for every $x \in X$.

It is evident that the sets $f(x)$ and $f^{-1}(y)$ are open for every $x \in X$ and $y \in Y$. Let us check up the rest of the conditions of Theorem 2. The set $\{x \in X; x \geqq a\}$ for a given $a \in X$ is closed since it is in fact the line segment with endpoints x_0 and a.

For the proof of Theorem 1 we need the following lemmas:

Lemma 1. *For every closed set $S \subset Y$ it is true that*

$$\langle S\rangle = \cap \{x + \mathring{V};\ \ x + \mathring{V} \supset S,\ \ x \in L\}.$$

Proof. Let us denote the set defined in the right-hand side of (1) with T. We shall prove that $T \supset \langle S\rangle$. Let us suppose that there is $c \in L$ such that $c \in \langle S\rangle$ but $c \notin T$. This means that there is an $x_1 \in L$ for which $x_1 + \mathring{V} \supset S$ and $c \notin x_1 + \mathring{V}$. Since S is a compact set, it follows that there is a sufficiently small positive number s such that $c \notin x_1 + s(x_1 - c) + V$ and $x_1 + s(x_1 - c) + V \supset S$. This contradicts $c \in \langle S\rangle$.

Let us prove that $T \subset \langle S\rangle$. If there exists $c \in L$ such that $c \in T$ but $c \notin \langle S\rangle$, it follows that there is $x_1 \in L$ for which $x_1 + V \supset S$ and $c \notin x_1 + V$. One of the conditions of Theorem 1 is that for some $x_0 \in L$, $x_0 + \mathring{V} \supset M \supset S$. Since V is a closed convex set, it follows that for a sufficiently small positive number s we have $x_1 + s(x_0 - x_1) + \mathring{V} \supset S$ and $c \notin x_1 + s(x_0 - x_1) + V$.

The proof is finished.

Lemma 2. *If $a, b \in x_0 + V$, then the parenthesis defined in the text of Theorem 2 is represented as follows*: $(\{a, b\}) = \langle\{a, b\}\rangle \setminus \{a, b\}$.

Proof. Let $a \in f(u)$ but $b \notin f(u)$ for some $u \in X$. Let us consider the set $A = x_0 + s(u - x_0) + \mathring{V}$, where $s = \max\{t; (x_0 + t(u - x_0) + V) \supset \{a, b\}, 0 \leqq t \leqq 1\}$.

Since V is convex, $a \in f(u)$ and $a \in x_0 + \mathring{V}$, it follows that $a \in A$. Indeed, $a \in f(u)$ implies $a = u + v_1$ for some $v_1 \in \mathring{V}$, and $a = x_0 + v_2$ for some

$v_2 \in \overset{\circ}{V}$. This gives $a = x_0 + s(u - x_0) + v_2 + s(v_1 - v_2)$. This proves that $a \in A$. Evidently, we have $b \in \partial A$.

We shall prove that $\langle\{a, b\}\rangle \cap \partial A = \{b\}$. Let us suppose that there is $z \neq b$, $z \in \langle\{a, b\}\rangle \cap \partial A$. To get a contradiction, consider the plane L_1 incident with the points a, b, z and let $A_1 = A \cap L_1$. Since ∂A_1 does not contain line segments (∂V is such!), there exist exactly two points $p, q \in L_1$ such that $p+a, p+b$, $q+a, q+b \in \partial A_1$.

Consider the arcs arc $(p+a, p+b) \subset \partial A_1$ and arc $(q+a, q+b) \subset \partial A_1$. Then the set $\langle\{a, b\}\rangle \cap L_1$ is contained in the figure surrounded by the following translations of these arcs: arc $(p+a, p+b) - p$ and arc $(q+a, q+b) - q$. Now, recalling again that ∂A_1 does not contain line segments and that $z \in \partial A_1$, we get $z \notin \partial (\langle\{a, b\}\rangle \cap L_1)$.

This contradiction proves that $\langle\{a, b\}\rangle \cap \partial A = \{b\}$.

If z is a point such that $z \in \langle\{a, b\}\rangle \setminus \{a, b\}$, then the above implies that the point $x = x_0 + s(u - x_0)$ is greater than the point u like an element of X, $z \in f(x)$ and $\complement f(x) \cap \{a, b\} = \{b\} \neq \emptyset$.

This proves that $(\{a, b\}) \supset \langle\{a, b\}\rangle \setminus \{a, b\}$.

The inverse inclusion follows easily from the definition of the parenthesis of a set and Lemma 1.

Lemma 3. *If the points $a, b \in x_0 + \overset{\circ}{V}$ and $a \neq b$, then there is some $x_1 \in L$ for which $a \in x_1 + \overset{\circ}{V}$ but $b \notin x_1 + V$.*

Proof. Since V is a convex set and does not contain infinite points, there exists a number $s > 0$ such that $(a+b)/2 \in (x_0 + s(a-b) + \partial V)$. Then, the relations $a \in (x_0 + s(a-b) + \overset{\circ}{V})$ and $b \notin (x_0 + s(a-b) + V)$ hold, which proves the lemma.

Now, let us continue the proof of Theorem 1.

Theorem 2 states that Ex is a nonempty set. We shall prove that every point $p \in \mathrm{Ex}$ is V-extreme, i. e. $p \in E$. Suppose that the opposite is true: $p \in \mathrm{Ex}$, but for some different points a, $b \in M$, holds $p \in \langle\{a, b\}\rangle \setminus \{a, b\}$. The last, according to Lemma 2, means that $p \in (\{a, b\})$, i. e. $a \leq p$, $b \leq p$. Theorem 2 implies that $a \approx p \approx b$. This means that for every $x \in X$, $a \in f(x)$ implies $b \in f(x)$. This contradicts the separation Lemma 3.

Finally, let us prove the basic statement of Theorem 1. According to Lemma 1, it suffices to prove that, if for some $x \in L$ $x + \overset{\circ}{V} \supset E$, then $x + \overset{\circ}{V} \supset M$. We proved that $\mathrm{Ex} \subset E$. Theorem 2 may be applied now to $x \in X$, which completes the proof of Theorem 1.

Aknowledgement. The present work was done in 1978 and was an answer to the question asked by the late Ivan Prodanov as to whether it is possible to prove Theorem 1 through the general principle of topological induction.

In the original version of the paper the construction of the space X in the proof of Theorem 1 was rather heavy. The present construction was proposed to the author by Dimiter Skordev, for which the author would like to thank him.

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