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CONCERNING TRIVIAL MAXIMAL ABELIAN SUBALGEBRAS OF $B(X)$

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To the memory of Y. A. Tagamlitzki

We call a complex or real Banach algebra trivial, if it is either a Banach space with trivial (zero) multiplication or it is the unitization of such an algebra. Thus a trivial algebra is always commutative and in the case of an algebra with unit element it is a local ring, i. e. it has exactly one maximal ideal equal to its radical. In this paper we prove that for any real or complex Banach space X the algebra $B(X)$ of all its continuous endomorphisms has always a trivial maximal Abelian subalgebra and we give description of all such subalgebras.

Let X be a real or complex Banach space. For a non-void subset S of $B(X)$ denote by S' its commutant, i. e. the set

$$S'=\{T\in B(X): TA=AT \text{ for all } A \text{ in } S\}.$$

It is a closed subalgebra of $B(X)$ containing its unity I, and in case when S consists of mutually commuting operators we have

$S'=\bigcup\{\mathcal{A}:\mathcal{A}$ is a maximal Abelian subalgebra of $B(X)$ with $S\subset\mathcal{A}\}$.

This implies that S' is a maximal Abelian subalgebra of $B(X)$, provided it is commutative. This simple remark will be used in the proof of our theorem.

In the sequel we denote by X^* the conjugate space of a Banach space X and by T^* the conjugate operator of an element T in $B(X)$. We put also $\operatorname{rad}\mathcal{A}$ for the radical of a commutative Banach algebra $\mathcal{A}$. Thus in case of a trivial algebra $\mathcal{A}$ with unit element we have $\mathcal{A}=\operatorname{rad}\mathcal{A}\oplus KI$, where K is the field of scalars ($K=\mathbb{C}$ or $K=\mathbb{R}$) and KI is the one-dimensional subspace of $\mathcal{A}$ spanned by the unit element I.

Since for $\dim X\leq 1$ the whole algebra $B(X)$ is commutative and trivial, we assume in our result that $\dim X>1$. In this case we say that a closed linear subspace X_0 of X is proper, of $\{0\}\neq X_0\neq X$. For an operator A in $B(X)$ denote by $\ker A$ its kernel and by $\operatorname{im} A$ its range, i. e. the sets $\ker A=\{x\in X: Ax=0\}$ and $\operatorname{im} A=\{Ax: x\in X\}$. Our result reads as follows

Theorem. *Let X be a real or complex Banach space with* $\dim X>1$ *and let X_0 be a proper closed linear subspace of X. Then the set*

$$(1)\qquad \{A\in B(X): \operatorname{im} A\subset X_0 \text{ and } X_0\subset\ker A\}$$

is a trivial Abelian subalgebra of $B(X)$ and its unitization $\mathcal{A}$ is a trivial maximal Abelian subalgebra of $B(X)$.

Conversely, if $\mathcal{A}$ is a trivial maximal Abelian subalgebra of $B(X)$, then its radical $\operatorname{rad}\mathcal{A}$ is of the form (1), where

$$(2)\qquad X_0=\bigcap\{\ker A: A\in\operatorname{rad}\mathcal{A}\}.$$

Proof. Denote by M the set (1). Obviously, it is a trivial subalgebra of $B(X)$. Let T be an operator in the commutant M'. For a functional f in X^*

with $X_0 \subset \ker f$ and for an element z in X_0 denote by $A(f, z)$ the one-dimensional operator given by $A(f, z)x = f(x)z$, this operator is clearly in M and so it commutes with T. Thus, for all x in X, all z in X_0 and all f in X^* with $X_0 \subset \ker f$ we have

$$f(Tx)z = f(x)Tz. \tag{3}$$

Choosing $f_0 \neq 0$ with $X_0 \subset \ker f_0$ and substituting for x in (3) an element x_0 in X with $f_0(x_0) = 1$ we obtain

$$Tz = \alpha_T z$$

for all z in X_0, where α_T is the scalar given by $\alpha_T = f_0(Tx_0)$. Put $T_1 = T - \alpha_T I$. We have $T_1 \in M'$ and $X_0 \subset \ker T_1$. We shall show that the operator T_1 is in M, i. e. $\mathrm{im}\, T_1 \subset X_0$. If not, then there is an element u_0 in X with $T_1 u_0 \notin X_0$ and we can find an element A in M with $AT_1 u_0 \neq 0$ (A can be chosen to be of the form $A(f, z)$). But this is impossible, since $AT_1 u_0 = T_1 A u_0$ and $Au_0 \in X_0 \subset \ker T_1$. Thus, T_1 is in M and so T is in its unitization $\mathscr{A}$ which is a commutative algebra and thus a maximal Abelian subalgebra of $B(X)$ since it equals M'.

Conversely, suppose that $\mathscr{A}$ is a trivial maximal Abelian subalgebra of $B(X)$ and put $M = \mathrm{rad}\, \mathscr{A}$. For any two operators T_1 and T_2 in M we have $\mathrm{im} T_1 \subset \ker T_2$, and so $\mathrm{im}\, T_1 \subset X_0$, where X_0 is given by (2). Since $X_0 \subset \ker T_1$ and T_1 is an arbitrary element of M, it follows that M is contained in the set (1). By the maximality of $\mathscr{A}$ M equals to this set, and so $\mathrm{rad} \mathscr{A}$ is of the form (1). The conclusion follows.

Corollary 1. Any subset S of $B(X)$ consisting of mutually annihilating operators (i. e. $T_1 T_2 = 0$ for all T_i in S, $i = 1, 2$) is contained in some trivial maximal Abelian subalgebra of $B(X)$. In particular, any trivial subalgebra of $B(X)$ is contained in a trivial maximal Abelian subalgebra of $B(X)$.

Denote by $\mathscr{A}(X_0)$ the trivial maximal Abelian subalgebra of $B(X)$ whose radical is (1).

Corollary 2. The algebra $\mathscr{A}(X_0)$ is isomorphic as a Banach space to the space $B(X/X_0, X_0) \oplus K$, where $B(U, V)$ denotes the Banach space of all continuous linear operators from a Banach space U to a Banach space V and K is the field of scalars (the one-dimensional Banach space).

Examples. Taking as X_0 any subspace of X of codimension one, we obtain a trivial maximal Abelian subalgebra $\mathscr{A}(X_0)$ isomorphic as a Banach space to the space X. Its radical consists of one-dimensional operators of the form $A(f_0, z)$, where f_0 is a fixed functional in X with $\ker f_0 = X_0$ and $z \in X_0$. The isomorphism between $\mathscr{A}(X_0)$ and X is given by

$$A(f_0, z) + \lambda I \leftrightarrow z + \lambda e_0,$$

where e_0 is a fixed element in X with $f_0(e_0) = 1$.

Similarly, taking as X_0 a linear subspace of X of dimension one $X_0 = Kx_0$ with $x_0 \in X$ and $\|X_0\| = 1$, we obtain an algebra $\mathscr{A}(X_0)$ isomorphic as a Banach space to the conjugate space X^*. It consists of all operators of the form $A(f, x_0) + \lambda I$, where $f \in X^*$ with $x_0 \in \ker f$ and $\lambda \in K$. The Banach space isomorphism between $\mathscr{A}(x_0)$ and X^* is given by

$$A(f, x_0) + \lambda I \leftrightarrow f + \lambda f_0,$$

where f_0 is a fixed element in X^* with $f_0(x_0) = 1$.

In case when the space X has a direct sum decomposition $X = X_0 \oplus X_1$, where X_1 is isomorphic to X_0, then the algebra $\mathscr{A}(X_0)$ is isomorphic as a Banach space to the space $B(X_0)$. In particular when $X = H$ — an infinitedimensional Hilbert space, then H can be orthogonally decomposed as $H=H_0 \oplus H_1$, where H_0 and H_1 are isometrically isomorphic to H. In this case the algebra $\mathscr{A}(H_0)$ is isomorphic as a Banach space to the space $B(H)$. It can be proved that in this case the operators in $\mathscr{A}(H_0)$ are of the following form. Let R be a partial isometry on H, which maps H_1 isometrically onto H_0 and maps H_0 onto $\{0\}$. Then

$$\mathscr{A}(H_0)=\{RA+AR:\ A\in B(H) \text{ and } RAR=\alpha(A)R\},$$

where $\alpha(A)$ is a scalar depending upon A. It can be shown that if $RA+AR=RA_1+A_1R$, then $\alpha(A)=\alpha(A_1)$ and so it defines on $\mathscr{A}(H_0)$ a functional f given by $f(RA+AR)=\alpha(A)$. It is a multiplicative linear functional on $\mathscr{A}(H_0)$ and its kernel equals to rad$\mathscr{A}(H_0)$ (we have $\alpha(R^*)=1$ and $R^*R+RR^*=I$).

If $\dim X=n<\infty$, then by Corollary 2 the possible dimensions of algebras $\mathscr{A}(X_0)$ are $(n-k).k+1$, $k=1, 2, \ldots, n-1$, and so there are $[\frac{n}{2}]$ non-isomorphic trivial maximal Abelian subalgebras of $B(X)$, where $[r]$ is the integral part of a number r. The largest possible dimension of $\mathscr{A}(X_0)$ is in this case $[\frac{n^2}{4}]+1$ and the smallest dimension is n. All these results in the case of finite dimensional spaces are well known even for more general scalars (cf. [2, Chapt. 2, § 3]), however, in the case of real or complex scalars our reasoning seems to be shorter. In case when X is a Hilbert space the maximal Abelian subalgebras of $B(X)$ which are local rings are known in the literature (cf. [1], or [3, p. 81, proposition 4.4]), however, the existence of such trivial algebras seems to be new and somewhat surprising. We finish this paper with some simple results on invariant subspaces for algebras $\mathscr{A}(X_0)$. In the sequel we denote by lin (X) the family of all closed linear subspaces of a Banach space X, and for a subset S of $B(X)$ we denote by lat (S) the set (it has a structure of a lattice) of all subspaces in lin (X) which are invariant with respect to all operators in S. In case when S consists of a single operator T we simply write lat (T).

Proposition 1. *Let X be a Banach space with* $\dim X>1$. *Then*

$$\text{lat}(\mathscr{A}(X_0))=\{Y\in \text{lin}(X):\ \textit{either } X_0\subset Y, \textit{ or } Y\subset X_0\}, \tag{4}$$

where X_0 is a proper linear subspace of X.

Proof. It is clear that all subspaces in the family (4) are invariant with respect to all operators in $\mathscr{A}(X_0)$. On the other hand, if Y is a closed linear subspace of X which contains some element $x_0\notin X_0$ and does not contain some element $z_0\in X_0$, then it cannot be invariant with respect to all elements in $\mathscr{A}(X_0)$, since there always exists an operator of the form $A(f, z_0)$ which sends x_0 to z_0. The conclusion follows.

A subalgebra $\mathscr{A}$ of $B(X)$ is said to be reflexive (sf. [3]), if the condition lat$(\mathscr{A})\subset$lat(T) implies $T\in\mathscr{A}$.

Proposition 2. *Let H be a Hilbert space,* $\dim H>1$, *then no trivial maximal Abelian subalgebra of $B(H)$ is reflexive.*

Proof. For a closed proper linear subspace H_0 of H denote by $P(H_0)$ the orthogonal projection of H onto H_0. Clearly, we have lat$(\mathscr{A}(H_0))\subset$lat$(P(H_0))$ and $P(H_0)\notin(H_0)$. The conclusion follows.

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