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PIECEWISE CONVEX CURVES AND THEIR INTEGRAL REPRESENTATION

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Abstract. A convex arc in the plane is introduced as an oriented arc Γ satisfying the following condition: For any three of its points $c^1 < c^2 < c^3$ the triangle $c^1c^2c^3$ is counter-clockwise oriented. It is proved that each such arc Γ is a closed and connected subset of the boundary of the set Φ_Γ being the convex hull of Γ. It is shown that the convex arcs are rectifyable and admit a representation in the natural parameter by the Riemann-Stieltjes integral with respect to an increasing, nonnegative and continuous from the right function s^+. Further it is shown that the obtained representation relates to the support function of the set Φ_Γ. Concerning the reverse question, namely what can be said for the curves that admit such representation, it is shown that they are exactly the curves that can be decomposed into finitely many convex arcs. This result suggests the name piecewise convex curves. In particular, the class of piecewise convex curves contains the convex curves being boundary sets of convex figures, therefore the results from the paper can be used as a tool for studying convex curves.

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1. Introduction. The convex sets and convex curves in two dimensions are an important part of convex set theory. This paper is some contribution to this topic. The notion of a convex arc is generalized to a piecewise convex curve. The convex curves being boundaries of convex figures are particular cases of piecewise convex curves. Hence, the presented here results can be useful in studying convex curves.

A convex arc is defined as an oriented arc Γ satisfying the condition: For any three of its points $c^1 < c^2 < c^3$ the triangle $c^1c^2c^3$ is counter-clockwise oriented. It is proved that each convex arc Γ is a closed and connected subset of the boundary of the set Φ_Γ being the convex hull of Γ (see below Theorem 1, where it is also emphasized when Γ reduces to a segment or degenerates to a point). It is shown that the convex arcs are rectifyable and admit representation in the natural parameter by the Riemann-Stieltjes integral with respect to an increasing, nonnegative and continuous from the right function $s^+ : [\theta_a, \theta_b] \to \mathbb{R}$. It is well known [8] that such a function generates a measure on the interval $[\theta_a, \theta_b]$ and one can identify s^+ with this measure.

The class of the considered curves is extended to all those, which admit the proposed integral representation. It is shown that this class coincides with the curves, which can be decomposed into finitely many convex arcs, and on this base they are named piecewise convex curves. The convex curves are a particular case of piecewise convex curves.

For a convex curve the measure generated by s^+ occurs in principle in Vitale [10]. (Vitale proves nonconstructively the existence of a measure in terms of which the support function can be expressed. His measure could be the one generated by the considered in this paper functions s^+ and s^-, or any intermediate function.) The usefulness of such a measure when studying approximation of convex sets has been shown by several authors, e. g. McClure, Vitale [7], Nedelcheva [9], Ludwig [6], Ligun, Shumeĭko [5].

Several advantages of the presented in the paper approach can be mentioned. In opposite to Vitale a straightforward meaning of the obtained measure in terms of lengths is given, which leads immediately to the equation of the curve in the natural parameter. As a corollary an integral representation of the support function of a piecewise convex curve is obtained. Actually, the main task of the Vitale's paper is to establish such a representation for the particular case of a convex curve.

The proposed integral representation can be useful in investigation of variety of problems concerning convex curves and piecewise convex curves. Our intention is to apply this tool to explain phenomena which occur in the approximation of convex curves by polygonal curves, some of them described in [2], [3] and [12].

The structure of the paper is the following. Section 2 defines convex arcs and investigating the equations in the natural parameter of convex arcs derives their integral representations. Section 3 studying the curves, which admit an integral representation of the obtained type, leads to the notion of a piecewise convex curve. It is shown there that the integral representation characterizes the piecewise convex curves and it is studied when a given integral representation corresponds to a convex arc.

2. Convex arcs. All considerations in this paper concern the Euclidean plane $\mathbb{R}^2$. The points in $\mathbb{R}^2$ and their radius-vectors are identified with pairs of reals. We make use of the transformations $T^+ : \mathbb{R}^2 \to \mathbb{R}^2$, $a = (a_1, a_2) \mapsto T^+a = (-a_2, a_1)$ and $T^- : \mathbb{R}^2 \to \mathbb{R}^2$, $a = (a_1, a_2) \mapsto T^-a = (a_2, -a_1)$ being in fact rotations on right angle respectively in counter-clockwise and clockwise directions. For any two points c^1, $c^2 \in \mathbb{R}^2$ we denote by $\overline{c^1c^2}$ the segment with initial point c^1 and final point c^2. We denote also by $e_\theta = (\cos\theta, \sin\theta)$ the unit vector concluding with the x-axis an angle with measure θ.

We call an arc each set $\Gamma \subset \mathbb{R}^2$ homeomorphic to a compact interval $[\alpha, \beta] \subset \mathbb{R}$. In this convention the points in $\mathbb{R}^2$ are also arcs, since each point is homeomorphic to a degenerate interval. Each homeomorphism $h : [\alpha, \beta] \to \Gamma$ determines an ordering relation called usually orientation on Γ with the agreement $h(t_1) \le h(t_2)$ if $t_1 \le t_2$. We write also $h(t_1) < h(t_2)$ if $t_1 < t_2$. The arc Γ together with a given orientation is called an oriented arc. It is well known that each arc admits only two orientations. Further from the context will be clear when Γ denotes an arc or oriented arc.

The points $h(\alpha)$ and $h(\beta)$ do not depend on the concrete homeomorphism $h : [\alpha, \beta] \to \Gamma$ determining the orientation of the oriented arc Γ and are called correspondingly the initial and the final points of the oriented arc Γ.

We call the oriented arc Γ a convex arc if for any three points $c^i = (c^i_1, c^i_2) \in \Gamma$, $i = 1, 2, 3$, such that $c^1 < c^2 < c^3$, it holds

$$(1)\qquad (c^1, c^2, c^3) := \begin{vmatrix} 1 & c^1_1 & c^1_2 \\ 1 & c^2_1 & c^2_2 \\ 1 & c^3_1 & c^3_2 \end{vmatrix} \ge 0\,.$$

The above determinant gives the doubled value of the oriented area of the oriented triangle $c^1c^2c^3$, hence it is nonnegative if this triangle is counter-clockwise oriented. Roughly speaking, we call the oriented arc convex if it is counter-clockwise curved.

The convention in analysis is that for a planar set with a boundary being a simple closed curve usually an orientation is introduced leaving the set on the left side, see e. g. the Green formula. Further in Theorem 1 it is shown that a convex arc is a proper subset of a convex curve, that is of the boundary of a convex figure. Once we agree that the convex arc should be an oriented arc, as far as the orientation plays an important role in the sequel, for the sake of compatibility with the natural orientation of the convex curves we should consider as natural the counter-clockwise orientation. This explains the given definition of a convex arc. Obviously, similar results can be obtained for arcs being clockwise curved. The paper does not deal with this case.

Here there are some examples of convex arcs. Each point in $\mathbb{R}^2$ is a convex arc. The segment $\overline{ab}$ is a convex arc. The graph of a continuous convex function of one variable defined on a compact interval (with orientation corresponding to the increasing of the argument) is a convex arc.

For any two points c^1, $c^2 \in \mathbb{R}^2$, $c^1 \neq c^2$, we introduce the notations:

$$\begin{aligned}
p_{c^1c^2} &= \{c \in \mathbb{R}^2 \mid (c^1, c^2, c) = 0\}\,, \\
E^+_{c^1c^2} &= \{c \in \mathbb{R}^2 \mid (c^1, c^2, c) \geq 0\}\,, \\
\breve{E}^+_{c^1c^2} &= \{c \in \mathbb{R}^2 \mid (c^1, c^2, c) > 0\}\,, \\
E^-_{c^1c^2} &= \{c \in \mathbb{R}^2 \mid (c^1, c^2, c) \leq 0\}\,, \\
\breve{E}^-_{c^1c^2} &= \{c \in \mathbb{R}^2 \mid (c^1, c^2, c) < 0\}.
\end{aligned}$$

Here $p_{c^1c^2}$ is the straight line through the points c^1 and c^2, and $E^+_{c^1c^2}$, $\breve{E}^+_{c^1c^2}$, $E^-_{c^1c^2}$, $\breve{E}^-_{c^1c^2}$ are the half-planes that it determines.

Since the determinant in (1) alternates its sign when permuting two of its rows, we see that the value (c^1, c^2, c^3) alternates its sign by a permutation of any two of the points. Therefore condition (1) means that $c^2 \in E^-_{c^1c^3}$ for any three points $c^1 < c^2 < c^3$ in Γ. The property that Γ is a convex arc means that $c \in E^-_{c^1c^2}$ for $c^1 < c < c^2$ and $c \in E^+_{c^1c^2}$ for $c < c^1 < c^2$ or $c^1 < c^2 < c$.

In this section we denote by Γ a convex arc, and by a and b its initial and final points. Given any two points $c^1 < c^2$ in Γ, then we put $\Gamma_{c^1c^2} = \{c \in \Gamma \mid$

$c^1 \leq c \leq c^2\}$. Since $\Gamma_{c^1c^2}$ is the image of the restriction of the homeomorphism determining Γ on a compact interval, we see that $\Gamma_{c^1c^2}$ is a convex arc.

We will use the notion of a convex figure. Following [11] we call a convex figure any convex compact set in the plane with nonempty interior.

Next we determine the structure of the convex arc Γ.

Theorem 1. *Let Γ be a convex arc with a and b being its initial and final points. Then the following cases may occur:*

a) If $a = b$, then Γ degenerates to a point.

b) If $a \neq b$ and $(c^1, c^2, c^3) = 0$ for any three points $c^1 < c^2 < c^3$ of Γ, then Γ is the segment $\overline{ab}$.

c) If $a \neq b$ and $(c^1, c^2, c^3) > 0$ for at least one triple of points $c^1 < c^2 < c^3$ of Γ, then $\Gamma \cup \overline{ba}$ is the boundary of a convex figure.

P r o o f. a) If Γ is a homeomorphic image of an interval with coinciding images of its end points, then this interval degenerates to a point and also Γ degenerates to a point.

b) Since $(a, b, c) = 0$ for any $c \in \Gamma$, we get $\Gamma \subset p_{ab}$. Since Γ is connected and has a and b as an initial and final points, we get $\Gamma = \overline{ab}$.

c) *At first we prove that $\Gamma \cup \overline{ab}$ is a simple closed curve.*

Obviously Γ does not possess multiple points as a homeomorphic image of an interval. It remains to show that Γ does not intersect the relative interior of $\overline{ab}$.

Take the points $c^1 < c^2 < c^3$, such that $(c^1, c^2, c^3) > 0$. Then for at least one of the points c^i, $i = 1, 2, 3$, it holds $(a, c^i, b) > 0$. Otherwise we would have $c^i \in p_{ab}$, $i = 1, 2, 3$, whence $(c^1, c^2, c^3) = 0$, a contradiction. Thus, there exists a point $\bar{c} \in \Gamma$, such that $(a, \bar{c}, b) > 0$. Then $\Gamma_{a\bar{c}} \subset E^-_{a\bar{c}}$ and $\Gamma_{\bar{c}b} \subset E^-_{\bar{c}b}$. Therefore $\Gamma = \Gamma_{a\bar{c}} \cup \Gamma_{\bar{c}b} \subset E^-_{a\bar{c}} \cup E^-_{\bar{c}b}$. Assume that there exists a point $c^* \in \Gamma$, such that $c^* \in \overline{ab}$ and $c^* \neq a$, $c^* \neq b$. It follows $c^* = \lambda a + (1 - \lambda)b$ for some $0 < \lambda < 1$. We get from here after short transformations

$$(a, \bar{c}, c^*) = (a, \bar{c}, \lambda a + (1 - \lambda)b) = (1 - \lambda)(a, \bar{c}, b) > 0\,.$$

Therefore $c^* \in \breve{E}^+_{a\bar{c}}$. We can get in a similar way $c^* \in \breve{E}^+_{\bar{c}b}$, whence $c^* \in \breve{E}^+_{a\bar{c}} \cap \breve{E}^+_{\bar{c}b}$. At the same time

$$c^* \in \Gamma \subset E^-_{a\bar{c}} \cup E^-_{\bar{c}b} = \mathbb{R}^2 \setminus \left(\breve{E}^+_{a\bar{c}} \cap \breve{E}^+_{\bar{c}b}\right)\,,$$

a contradiction.

Thus $\overline{ba}$ and Γ do not possess common points different from a and b. Therefore $\Gamma \cup \overline{ba}$ is a simple closed curve, whence it is the boundary of a compact set Φ_Γ. We have shown also that $\operatorname{co}\{a, \bar{c}, b\} \subset \Phi_\Gamma$, which shows directly that Φ_Γ has a nonempty interior.

We claim that Φ_Γ is a convex figure.

We will show that each straight line p passing through an arbitrary point c^0 from the interior of Φ_Γ intersects the boundary $\Gamma \cup \overline{ba}$ in exactly two points, whence it would follow that Φ_Γ is a convex figure (compare with Yaglom, Boltyanski [11], page 17, problem 5).

Let p be arbitrary straight line passing through the point c^0 from the interior of Φ_Γ. Obviously, p intersects the boundary of Φ_Γ in at least two points. At least one point lays on each of the two rays in which c^0 splits p, a consequence of Φ_Γ bounded and c^0 in the interior of Φ_Γ. We will show that these intersecting points are at most two, whence it would follow that their number is exactly two.

Let us note that p cannot intersect the boundary of Φ_Γ in a segment. To prove this we observe that p intersects $\overline{ab}$ in at most one point. Otherwise $\overline{ab} \subset p$ and because $\Gamma \subset E^-_{ab}$ and consequently $\Phi_\Gamma \subset E^-_{ab}$ it follows that p contains only boundary points of Φ_Γ and therefore it cannot pass through the interior point c^0, a contradiction. If we assume that $p \cap \Gamma = \overline{\bar{a}\bar{b}}$, where $\bar{a} < \bar{b}$, then from the convexity of Γ we would have $c \in E^+_{\bar{a}\bar{b}}$ for all $c \in \Gamma$, i. e. $\Phi_\Gamma \subset E^+_{\bar{a}\bar{b}}$. This means that p contains only boundary points of Φ_Γ and therefore it cannot pass through the interior point c^0, a contradiction.

We consider the cases:

1^0. *Let p do not intersect $\overline{ba}$.* Assume that p intersects Γ in at least three points $c^1 < c^2 < c^3$. We have the possibilities:

1^0 a. *The point c^2 is between c^1 and c^3 on the line p.* The segments $\overline{c^1c^2}$ and $\overline{c^2c^3}$ have the same directions. Then there exists a point $c^* \in \Gamma$ for which $c^1 < c^* < c^2 < c^3$ and $(c^1, c^*, c^2) > 0$ (otherwise we would have that $\overline{c^1c^2} \subset \Gamma \cap p$, that is $\Gamma \cap p$ contains a segment, which as it was shown is impossible). Therefore $c^* \in \breve{E}^-_{c^1c^2} = \breve{E}^-_{c^2c^3}$ and in consequence $(c^*, c^2, c^3) < 0$, a contradiction with the convexity of Γ.

1^0 b. *The point c^3 is between c^1 and c^2 on the line p.* The segments $\overline{c^1c^3}$ and $\overline{c^2c^3}$ are with opposite directions. Then there exists a point $c^* \in \Gamma$, for which $c^1 < c^2 < c^* < c^3$ and $(c^2, c^*, c^3) > 0$, in other words $c^* \in \breve{E}^-_{c^2c^3} = \breve{E}^+_{c^1c^3}$. Therefore $(c^1, c^3, c^*) > 0$, which contradicts to the convexity of Γ.

1^0 c. *The point c^1 is between c^2 and c^3 on the line p.* The segments $\overline{c^1c^2}$ and $\overline{c^1c^3}$ are with opposite directions. Then there exists a point $c^* \in \Gamma$, such that $c^1 < c^* < c^2 < c^3$ and $(c^1, c^*, c^2) > 0$. This means $c^* \in \breve{E}^-_{c^1c^2} = \breve{E}^+_{c^1c^3}$. Therefore $(c^1, c^3, c^*) > 0$, which contradicts to the convexity of Γ.

We have shown, that the intersecting points of p with Γ are exactly two when p does not intersect $\overline{ab}$.

2^0. *Let p intersect $\overline{ba}$.* The intersecting point of p with $\overline{ab}$ is only one, denote it by $c = p \cap \overline{ab}$. We will show, that p intersects Γ in exactly one point.

Assume that there exist at least two intersecting points c^1 and c^2 of p with Γ, for which $c^1 < c^2$. Then either $a \in \breve{E}^-_{c^1c^2}$ or $b \in \breve{E}^-_{c^1c^2}$, since a and b are in different half-planes with respect to p. Let $a \in \breve{E}^-_{c^1c^2}$ (the case $b \in \breve{E}^-_{c^1c^2}$ is similar). Therefore $(c^1, c^2, a) < 0$. On the other hand $a < c^1 < c^2$ and from the convexity of Γ it follows that $(c^1, c^2, a) \geq 0$, a contradiction.

From 1^0 and 2^0 it follows that p intersects the boundary of Φ_Γ in exactly two points. Therefore Φ_Γ is a convex figure. $\square$

In the proof of Theorem 1 c) we introduced the set Φ_Γ for which it holds $\Phi_\Gamma = \operatorname{co}\Gamma$. We will use the same notation also in the cases a) and b). Then Φ_Γ is a point in case a), a segment in case b), and a convex figure in case c). In each case Φ_Γ is a compact convex set in the plane. Also in the sequel we use the notation Φ_Γ for the convex hull of Γ.

Recall that the boundary of a convex figure is usually called a convex curve [11]. Therefore, Theorem 1 shows that each convex arc is either a point, or a segment, or a connected and closed proper subset of the convex curve being the boundary of Φ_Γ.

It is shown in [1] that each convex figure possesses a perimeter, that is each convex curve is rectifyable. Consequently, each convex arc Γ is rectifyable and therefore it admits an equation in natural parameter

$$\Gamma: \quad r = f(s)\,, \quad 0 \leq s \leq L\,, \tag{2}$$

where the natural parameter s is the length of the arc from the initial point a to the current point. Here L is the length of Γ. The function f is continuous. Moreover, it is well known that f is Lipschitz with constant 1. Since Γ has no multiple points, the function f is injective. Each continuous and injective mapping with domain a compact set is a homeomorphism. Therefore (2) is a homeomorphic representation of the arc Γ. Further it can be shown that the passing from a parameter t determining the convex arc Γ to the natural parameter is realized by an increasing function $s = s(t)$. This shows that the natural parameter s determines the same ordering on Γ as the parameter t, that is property (1) holds with respect to the ordering determined by the parameter s. Therefore (2) is a representation of Γ as a convex arc, which can be referred to as representation by natural parameter.

Our main purpose is to describe the function f in (2) in terms of a parameter θ being connected with the support function of Φ_Γ in direction e_θ, which is done in Theorem 2. The support functions are important tools when treating problems concerning convex figures. The representation obtained in Theorem 2 could play similar role when studying convex arcs. In the next section we define piecewise convex curves as a generalization of both the convex arcs and the convex curves, and extend the representation from Theorem 2 to piecewise convex curves. The study of piecewise convex curves and in particular of convex curves can be based on the obtained representation.

We need first the following notations.

Let K be a convex set in $\mathbb{R}^2$. We call a support function of K the function

$$\Lambda : \mathbb{R} \to \mathbb{R}\,, \quad \Lambda(\theta) = \sup\{r \cdot e_\theta \mid r \in K\}\,.$$

Here $r \cdot e_\theta$ denotes the scalar product of the radius-vector r and the vector e_θ.

The straight line $p_\theta : r \cdot e_\theta = \Lambda(\theta)$ is said to be a support line of K in direction e_θ. We will consider p_θ as an axis with orientation determined by the vector $T^+ e_\theta$ being colinear to p_θ.

Suppose that Γ is a convex arc with initial point a, final point b and parametric representation in natural parameter given by (2). When $a \neq b$ we denote by γ a real number, for which $e_\gamma = T^-(a-b)/\|a-b\|$. Here $\|\cdot\|$ denotes the Euclidean norm. When $a = b$, which according to Theorem 1 has place only if Γ degenerates to a point, we denote by γ any real number.

Let $\theta \in [\gamma, \gamma + 2\pi]$ and p_θ be the support line of Φ_Γ in direction e_θ. Let $\Phi_\Gamma \cap p_\theta$ be the segment (possibly degenerated to a point) with end points $r^-(\theta)$ and $r^+(\theta)$ where the direction from $r^-(\theta)$ to $r^+(\theta)$ coincides with the orientation on p_θ.

We put

$$c^-(\theta) = \begin{cases} a\,, & \theta = \gamma, \\ r^-(\theta)\,, & \gamma < \theta \le \gamma + 2\pi, \end{cases} \qquad c^+(\theta) = \begin{cases} r^+(\theta)\,, & \gamma \le \theta < \gamma + 2\pi, \\ b\,, & \theta = \gamma + 2\pi. \end{cases}$$

We determine the functions $s^-, s^+ : [\gamma, \gamma + 2\pi] \longrightarrow \mathbb{R}$ by

$$f(s^-(\theta)) = c^-(\theta), \quad f(s^+(\theta)) = c^+(\theta),$$

where f is the function from the representation (2) of Γ with natural parameter. In fact $s^-(\theta)$ gives the length of $\Gamma_{ac^-(\theta)}$ and $s^+(\theta)$ gives the length of $\Gamma_{ac^+(\theta)}$.

For each pair of points $c^1 < c^2$ from Γ we determine the number $\vartheta = \vartheta(c^1, c^2) \in [\gamma, \gamma + 2\pi]$ by $e_\vartheta = T^-(c^2 - c^1)/\|c^2 - c^1\|$.

If $c \in \Gamma$ and $a < c < b$ we put

$$\vartheta^-(c) = \sup\{\vartheta(c^1, c) \mid c^1 \in \Gamma, c^1 < c\},$$

$$\vartheta^+(c) = \inf\{\vartheta(c, c^2) \mid c^2 \in \Gamma, c < c^2\}.$$

For $c = a$ we put $\vartheta^-(c) = \gamma$ and determine $\vartheta^+(c)$ from the above equalities, for $c = b$ we put $\vartheta^+(c) = \gamma + 2\pi$ and determine $\vartheta^-(c)$ from the above equalities.

In Theorem 1 below discusses the representation of convex arcs. As a preparation we need the following lemma.

Lemma 1. *Let Γ be a convex arc with initial point a and final point b. Let $\delta > 0$ and $[\theta_1, \theta_2]$ be a subinterval of $[\gamma, \gamma + 2\pi]$ such that $0 < \theta_2 - \theta_1 = 2\sigma \le 2\delta$. Denote $c^i = c^+(\theta_i)$ for $i = 1, 2$. The following estimations have place:*

$$\| c^2 - c^1 \| \le s^+(\theta_2) - s^+(\theta_1) \le \frac{\| c^2 - c^1 \|}{\cos \frac{1}{2}(\theta_2 - \theta_1)}. \tag{3}$$

$$|\, \| c^2 - c^1 \| - \big(s^+(\theta_2) - s^+(\theta_1)\big) \,| \le \big(s^+(\theta_2) - s^+(\theta_1)\big) \left(\frac{1}{\cos \delta} - 1 \right). \tag{4}$$

$$\| e_{\theta_2} - e_{\theta_1} \| \le 2 \sin \delta\,. \tag{5}$$

P r o o f. We prove (3). Denote the intersection point of the support lines p_{θ_1} and p_{θ_2} with p. The chord $\overline{c^1c^2}$ has length not greater than the length of the convex arc $\Gamma_{c^1c^2}$, which proves the left inequality. In turn the length of the convex arc $\Gamma_{c^1c^2}$ is not greater than the length of the broken line $\overline{c^1p} \cup \overline{p\,c^2}$, whose sides $\overline{c^1p}$ and $\overline{p\,c^2}$ constitute an angle $\theta_2 - \theta_1$. The length of the broken line does not exceed the last term in (3). To prove this we put $\| c^2 - c^1 \| = \hat{c}$, $\|p - c^1\| = \hat{a}$, $\|p - c^2\| = \hat{b}$. Obviously

$$s^+(\theta_2) - s^+(\theta_1) \leq \hat{a} + \hat{b}. \tag{6}$$

We will show that

$$\hat{a} + \hat{b} \leq \frac{\hat{c}}{\cos\sigma}. \tag{7}$$

In fact, from the law of sines for the triangle $c^1 p\, c^2$, denoting by R the radius of the circumscribed circle, and by α and β the angles respectively at the vertices c^1 and c^2, we get $\hat{a} = 2R\sin\alpha$, $\hat{b} = 2R\sin\beta$, $\hat{c} = 2R\sin(\pi - 2\sigma)$. Therefore (7) is equivalent to the following inequality

$$\sin\alpha + \sin\beta \leq \frac{\sin(\pi - 2\sigma)}{\cos\sigma} = \frac{\sin 2\sigma}{\cos\sigma} = 2\sin\sigma,$$

which in turn is equivalent to

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \leq 2\sin\sigma$$

and hence to the obvious inequality

$$\cos\frac{\alpha-\beta}{2} \leq 1.$$

This verifies (7). Now (6) and (7) imply the claimed inequality.

The proof of inequality (4) follows from

$$\begin{aligned} |\|c^2 - c^1\| - \big(s^+(\theta_2) - s^+(\theta_1)\big)\,| &= \big(s^+(\theta_2) - s^+(\theta_1)\big) - \|c^2 - c^1\| \\ &\leq \frac{\|c^2 - c^1\|}{\cos\frac{1}{2}(\theta_2 - \theta_1)} - \|c^2 - c^1\| = \|c^2 - c^1\| \left(\frac{1}{\cos\sigma} - 1\right) \\ &\leq \big(s^+(\theta_2) - s^+(\theta_1)\big)\left(\frac{1}{\cos\delta} - 1\right). \end{aligned}$$

For the proof of (5) consider the rhomboid determined by e_{θ_2} and e_{θ_1} and observe that the length d of one of its diagonals is

$$\| e_{\theta_2} - e_{\theta_1} \| = d 2\sin\sigma \leq 2\sin\delta. \qquad \square$$

Theorem 2. *Let Γ be a convex arc with initial point a and final point b. We put $\theta_a = \gamma$ and $\theta_b = \gamma + 2\pi$. Now $a = c^-(\theta_a)$ and $s^-(\theta_a) = 0$.*

a) The following integral representation has place:

$$
(8)\qquad
\begin{aligned}
&c^+(\theta) = c^-(\theta_0) + T^+e_{\theta_0}\,(s^+(\theta_0) - s^-(\theta_0)) + \int_{\theta_0}^{\theta} T^+e_\lambda\, ds^+(\lambda)\,,\\
&c^-(\theta) = c^-(\theta_0) + \int_{\theta_0}^{\theta} T^+e_\lambda\, ds^-(\lambda)\,,
\end{aligned}
$$

for all $\theta_0 \in [\theta_a, \theta_b)$ *and* $\theta \in [\theta_0, \theta_b]$ *(the integrals are in the sense of Riemann-Stieltjes.*

b) The function f *from the representation (2) in natural parameter is given by*

$$
(9)\ f(s) = \begin{cases} c^-(\theta)\,, & s = s^-(\theta),\\ c^+(\theta)\,, & s = s^+(\theta),\\ c^-(\theta)\dfrac{s^+(\theta) - s}{s^+(\theta) - s^-(\theta)} + c^+(\theta)\dfrac{s - s^-(\theta)}{s^+(\theta) - s^-(\theta)}\,, & s^-(\theta) < s < s^+(\theta). \end{cases}
$$

c) The support function Λ *of* Φ_Γ *satisfies*

$$
\Lambda(\theta) = e_\theta \cdot c^-(\theta) = e_\theta \cdot c^+(\theta)\,, \quad \theta_a \le \theta \le \theta_b\,.
$$

P r o o f. a) We prove the first of the equalities (8). Let $\theta_0 < \theta_1 < \ldots < \theta_n = \theta$. Let $c^i = c^+(\theta_i)$, $i = 0, 1, 2, \ldots, n$. This means

$$
c^n = c^0 + \sum_{i=1}^{n} \|c^i - c^{i-1}\| \cdot T^+e_{\vartheta(c^{i-1},c^i)}
$$

(if for some i the points c^{i-1} and c^i coincide, we put $e_{\vartheta(c^{i-1},c^i)} = e_{\vartheta^+(c^{i-1})}$).

Let $\delta > 0$ and denote by $L\Gamma_{c^u c^v}$ the length of the arc $\Gamma_{c^u c^v}$. Choose a partition $\{\theta_i\}$ such that $0 < \theta_i - \theta_{i-1} \le 2\delta$, $i = 0, 1, 2, \ldots, n$. Put

$$
\begin{aligned}
A &= c^n - (c^+(\theta_0) + \int_{\theta_0}^{\theta} T^+e_\lambda\, ds^+(\lambda)\,)\\
&= \sum_{i=1}^{n} (\|c^i - c^{i-1}\|\, T^+e_{\vartheta(c^{i-1},c^i)} - \int_{\theta_{i-1}}^{\theta_i} T^+\, e_\lambda\, ds^+(\lambda)\,)\,.
\end{aligned}
$$

The following estimations have place:

$$
\begin{aligned}
|A| \;\le\; & \sum_{i=1}^{n} \Big|\, \big(s^+(\theta_i) - s^+(\theta_{i-1})\big)\, T^+ e_{\vartheta(c^{i-1},c^i)} - \int_{\theta_{i-1}}^{\theta_i} T^+ e_\lambda \, ds^+(\lambda) \,\Big| \\
& + \sum_{i=1}^{n} \Big|\, \|c^i - c^{i-1}\| - \big(s^+(\theta_i) - s^+(\theta_{i-1})\big) \,\Big| \\
\le\; & \sum_{i=1}^{n} \int_{\theta_{i-1}}^{\theta_i} \|T^+ e_{\vartheta(c^{i-1},c^i)} - T^+ e_\lambda\| \, ds^+(\lambda) \\
& + \sum_{i=1}^{n} \big(s^+(\theta_i) - s^+(\theta_{i-1})\big) \left(\frac{1}{\cos\delta} - 1\right) \\
\le\; & \sum_{i=1}^{n} \int_{\theta_{i-1}}^{\theta_i} \|T^+ e_{\theta_i} - T^+ e_{\theta_{i-1}}\| \, ds^+(\lambda) \; + L\Gamma_{c^0 c^n} \left(\frac{1}{\cos\delta} - 1\right) \\
\le\; & L\Gamma_{c^0 c^n} \left(2\sin\delta + \frac{1}{\cos\delta} - 1\right) \longrightarrow 0 \quad \text{for} \quad \delta \to 0\,.
\end{aligned}
$$

Therefore $A \to 0$ for $\max|\theta_i - \theta_{i-1}| \to 0$, which proves the first equality (8). The second equality is derived in a similar way. In the proof we have used Lemma 1.

b) The claim follows easily from the following equalities valid by definition

$$
s^-(\theta) = s(c^-(\theta)) \quad \text{and} \quad s^+(\theta) = s(c^+(\theta))\,.
$$

c) The claimed equality has place, since both $c^+(\theta)$ and $c^-(\theta)$ belong to the support line p_θ. $\square$

We conclude this section with two examples of convex arcs. They illustrate the passing from the parameter θ to the natural parameter. The next section also refers to these examples.

Example 1. *Let $K_0 = \mathrm{co}\,\{(0,\,1),\,(0,\,-1)\}$, $K_1 = \{r \in \mathbb{R}^2 \mid \|r\| \le 1\}$ and $K = K_0 + K_1$, where the sum is understood as the Minkowski sum of convex sets. Define the convex arc Γ to be the counter-clockwise oriented part of the boundary of K from the point $(1,\,1)$ to the point $(1,\,-1)$. We have $a = (1,\,1)$, $\gamma = 0$, and $s^-,\, s^+ : [0,\,2\pi] \to \mathbb{R}$ are given by*

$$
s^-(\theta) = \begin{cases} \theta\,, & 0 \le \theta \le \pi\,, \\ 2+\theta\,, & \pi < \theta \le 2\pi\,, \end{cases} \qquad s^+(\theta) = \begin{cases} \theta\,, & 0 \le \theta < \pi\,, \\ 2+\theta\,, & \pi \le \theta \le 2\pi\,. \end{cases}
$$

consequently the same convex arc Γ will be obtained if in the definition of Γ instead of the given functions s^-, s^+ we apply their restrictions to a smaller interval $[\theta_a,\, \theta_b]$ with any $\theta_a \in [0,\, 3\pi/4)$ and $\theta_b \in (5\pi/4,\, 2\pi]$. While Theorem 2 was formulated with interval $[\theta_a, \theta_b]$ with length 2π, Example 2 shows that the conclusions are valid sometimes with smaller intervals. Let us mention that in Example 1 the interval $[0,\, 2\pi]$ cannot be diminished.

3. Piecewise convex curves. Formula (8) with $\theta_0 = \theta_a$ transforms into

$$\begin{aligned} c^+(\theta) &= a + T^+ e_{\theta_a}\, s^+(\theta_a) + \int_{\theta_a}^{\theta} T^+ e_\lambda \, ds^+(\lambda)\,, \\ c^-(\theta) &= a + \int_{\theta_a}^{\theta} T^+ e_\lambda \, ds^-(\lambda)\,, \end{aligned} \tag{10}$$

true for all $\theta \in [\theta_a, \theta_b]$. This can be considered as an integral representation of the convex arc Γ, since in virtue of (9) once we have got the functions c^- and c^+, we can restore Γ. Let us underline that the essential information in (10) is the knowledge of the initial point a, the interval $[\theta_a, \theta_b]$ and the function $s^+ : [\theta_a, \theta_b] \to \mathbb{R}$, which is increasing, nonnegative, and continuous from the right. The latter is seen from the next Theorem 3, where it is shown that the function s^- can be expressed by s^+. Pay attention there, that the knowledge of only s^- is not enough to restore s^+, for the value $s^+(\theta_b)$ cannot be obtained by s^-.

Let us underline that the Riemann-Stieltjes integral from a continuous function with respect to an increasing function exists always [4]. The function $\lambda \to T^+ e_\lambda = (-\sin\lambda, \cos\lambda)$ is continuous. Therefore, the integrals in (10) exist always.

Theorem 3. *Let $s^+ : [\theta_a, \theta_b] \to \mathbb{R}$ be increasing, nonnegative, and continuous from the right function and $a \in \mathbb{R}^2$. Determine the function $s^- : [\theta_a, \theta_b] \to \mathbb{R}$ from*

$$s^-(\theta) = \begin{cases} 0\,, & \theta = \theta_a\,, \\ \lim\limits_{\theta_1 \to \theta^-} s^+(\theta_1) = s^+(\theta - 0)\,, & \theta_a < \theta \le \theta_b\,. \end{cases} \tag{11}$$

Determine $c^+(\theta)$ and $c^-(\theta)$ from (10) *for all $\theta \in [\theta_a, \theta_b]$. Then it holds:*

Formula (8) *with* $\theta_0 = 0$ *gives*

$$c^-(\theta) = \begin{cases} (\cos\theta,\, 1+\sin\theta), & 0 \le \theta \le \pi\,, \\ (\cos\theta,\, -1+\sin\theta), & \pi < \theta \le 2\pi\,, \end{cases}$$

$$c^+(\theta) = \begin{cases} (\cos\theta,\, 1+\sin\theta), & 0 \le \theta < \pi\,, \\ (\cos\theta,\, -1+\sin\theta), & \pi \le \theta \le 2\pi\,. \end{cases}$$

Formula (9) *gives for the representation* (2) *with natural parameter the function*

$$f(s) = \begin{cases} (\cos s,\, 1+\sin s), & 0 \le s \le \pi\,, \\ (-1,\, \pi+1-s), & \pi \le s \le \pi+2\,, \\ (\cos(s-2),\, -1+\sin(s-2)), & \pi+2 \le s \le 2\pi+2\,. \end{cases}$$

Example 2. *Let* K *be the triangle* $K = \{(x,y) \mid -1 \le x \le 0,\ -1-x \le y \le 1+x\}$. *Define the convex arc* Γ *to be the counter-clockwise oriented part of the boundary of* K *from the point* $(0,\, 1)$ *to the point* $(0,\, -1)$. *We have* $a = (0,\, 1)$, $\gamma = 0$, *and* $s^-,\, s^+ : [0,\, 2\pi] \to \mathbb{R}$ *are given by*

$$s^-(\theta) = \begin{cases} 0\,, & 0 \le \theta \le 3\pi/4\,, \\ \sqrt{2}\,, & 3\pi/4 < \theta \le 5\pi/4\,, \\ 2\sqrt{2}\,, & 5\pi/4 < \theta \le 2\pi\,. \end{cases} \qquad s^+(\theta) = \begin{cases} 0\,, & 0 \le \theta < 3\pi/4\,, \\ \sqrt{2}\,, & 3\pi/4 \le \theta < 5\pi/4\,, \\ 2\sqrt{2}\,, & 5\pi/4 \le \theta \le 2\pi\,. \end{cases}$$

Formula (8) *gives*

$$c^-(\theta) = \begin{cases} (0,\, 1), & 0 \le \theta \le 3\pi/4\,, \\ (-1,\, 0), & 3\pi/4 < \theta \le 5\pi/4\,, \\ (0,\, -1), & 5\pi/4 < \theta \le 2\pi\,, \end{cases}$$

$$c^+(\theta) = \begin{cases} (0,\, 1), & 0 \le \theta < 3\pi/4\,, \\ (-1,\, 0), & 3\pi/4 \le \theta < 5\pi/4\,, \\ (0,\, -1), & 5\pi/4 \le \theta \le 2\pi\,. \end{cases}$$

For the function f *we get*

$$f(s) = \begin{cases} (-s/\sqrt{2},\, 1-s/\sqrt{2})\,, & 0 \le s \le \sqrt{2}\,, \\ (-2+s/\sqrt{2},\, 1-s/\sqrt{2})\,, & \sqrt{2} \le s \le 2\sqrt{2}\,. \end{cases}$$

Concerning Example 2 we see that the functions s^- and s^+ are constants on the intervals $[0,\, 3\pi/4)$ and $(5\pi/4,\, 2\pi]$. As a result the same function f and