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# CRITERION OF NORMALITY OF THE COMPLETELY REGULAR TOPOLOGY OF SEPARATE CONTINUITY

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Abstract. For given completely regular topological spaces $X$ and $Y$, there is a completely regular space $X \mathbin{\widetilde{\otimes}} Y$ such that for any completely regular space $Z$ a mapping $f : X \times Y \to Z$ is separately continuous if and only if $f : X \widetilde{\otimes} Y \to Z$ is continuous. We prove a necessary condition of normality, a sufficient condition of collectionwise normality, and a criterion of normality of the products $X \mathbin{\widetilde{\otimes}} Y$ in the case when at least one factor is scattered.

Let $X$, $Y$ and $Z$ be arbitrary topological spaces. Then for a mapping $f : X \times Y \to Z$ there appears double notion of continuity: continuity in all variables jointly (or joint continuity) and continuity in each variable separately (or separate continuity).

It is well known (see e.g. [1, 17.D] or [5]) that we can define a topological space $X \otimes Y$ on the product set $X \times Y$ with the property that for any space $Z$ a mapping $f : X \times Y \to Z$ is separately continuous if and only if $f : X \otimes Y \to Z$ is continuous. However this topology is inconvenient for investigating separately

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continuous mappings since in many important cases $X \otimes Y$ fails to be completely regular. In particular, the results of papers [5] and [4] imply that the spaces $\mathbb{R} \otimes \mathbb{R}$, $\mathbb{R} \otimes \mathbb{A}$, $\mathbb{R} \otimes \alpha\Gamma$, $\mathbb{A} \otimes \mathbb{A}$ and $\mathbb{A} \otimes \alpha\Gamma$ are not completely regular. Here $\mathbb{R}$ denotes the real line, $\mathbb{A}$ — the double arrow (the product $[0;1] \times \{0;1\}$ ordered lexicographically), and $\alpha\Gamma$ — the one-point compactification of a discrete infinite space $\Gamma$.

In [6] on the set $X \times Y$, where $X$ and $Y$ are completely regular spaces, a new topological space $X \mathbin{\widetilde{\otimes}} Y$ was constructed. This space satisfies the following conditions: $X \mathbin{\widetilde{\otimes}} Y$ is completely regular and for any completely regular space $Z$ a mapping $f : X \times Y \to Z$ is separately continuous if and only if $f : X \mathbin{\widetilde{\otimes}} Y \to Z$ is continuous.

The problem of normality of the spaces $X \mathbin{\widetilde{\otimes}} Y$ comes quite naturally. In [5] and [6] Knight, Moran and Pym obtained sufficient conditions of normality of the products $X \mathbin{\widetilde{\otimes}} Y$ only in the case when at least one factor is locally countable. By using these results it is easy to see normality of the spaces $\mathbb{R} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ and $\mathbb{A} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ for countable $\Gamma$. In [3] the author proved a sufficient condition of normality of the spaces $X \mathbin{\widetilde{\otimes}} Y$ that have at least one scattered factor. It follows from this condition that the spaces $\mathbb{R} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ and $\mathbb{A} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ are normal for arbitrary $\Gamma$. Moreover, established in [6] and [3] necessary conditions of normality of the products of metrizable spaces indicate that the space $\mathbb{R} \mathbin{\widetilde{\otimes}} \mathbb{R}$ is not normal.

In this paper the results of works [6] and [3] are generalized. In particular, it is shown that the spaces $\mathbb{R} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ and $\mathbb{A} \mathbin{\widetilde{\otimes}} \alpha\Gamma$ are collectionwise normal (Theorem 7), but the space $\mathbb{R} \mathbin{\widetilde{\otimes}} \mathbb{A}$ is not normal (Theorem 4, see also [6, 8.4]). The main result of the paper is a criterion of normality of the completely regular topology of separate continuity for a rather large class of spaces (Theorem 9).

### Necessary condition of normality.

**Lemma 1.** *Any Čech-complete non-scattered space contains a compact that can be mapped irreducibly onto the segment* $[0;1]$.

P r o o f. Let $X$ be a Čech-complete non-scattered space. Then there exist a non-empty perfect subset $Z$ in $X$ ([2], 1.7.10) and open in $\beta Z$ sets $G_n$ such that $Z = \bigcap\limits_{n=1}^{\infty} G_n$. By standard tree arguments for any finite binary sequence we may determine an open in $\beta Z$ set $U_{(i_1,\dots,i_n)}$ so that: a) $\overline{U}_{(i_1,\dots,i_{n-1},0)} \cap \overline{U}_{(i_1,\dots,i_{n-1},1)} = \emptyset$; b) $\overline{U}_{(i_1,\dots,i_{n-1},0)} \cup \overline{U}_{(i_1,\dots,i_{n-1},1)} \subset U_{(i_1,\dots,i_{n-1})}$; c) $\overline{U}_{(i_1,\dots,i_n)} \subset G_n$.

Further, we put $K = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{(i_1,\dots,i_n)\in\{0;1\}^n} \overline{U}_{(i_1,\dots,i_n)}$ and construct the function $f : K \to [0;1]$ transforming points from $K$ to real numbers from $[0;1]$ written in the binary form $0.i_1 \dots i_n \dots$. Since $f$ is continuous and $K$ is compact, there is a closed subset $M \subset K$ such that the restriction $g$ of $f$ on $M$ is irreducible and $g(M) = [0;1]$ ([2], 3.1.C). $\square$

**Lemma 2.** *For an irreducible closed mapping the image of an isolated point is isolated point and the preimage of a dense set is a dense set.*

P r o o f. Let $g : M \to N$ be an irreducible closed mapping, a point $m$ be isolated in $M$, and a set $P$ be dense in $N$. As $g(M \setminus \{m\}) \neq [0;1]$ the set $g(M \setminus \{m\}) = [0;1] \setminus g(m)$ is closed. Hence $g(m)$ is an isolated point in $N$.

To prove the second claim we denote $Q = g^{-1}(P)$. Then the set $N \setminus g(\overline{Q})$ is open in $N$ and disjoint with $P$. Therefore $N \setminus g(\overline{Q}) = \emptyset$, and by virtue of irreducibility of $g$ we obtain $\overline{Q} = M$. $\square$

Let $X$ and $Y$ be topological spaces. A mapping $f : X \to Y$ is called an F-refinement if $f$ is a continuous mapping with the finite preimages of points. A space $X$ is said to F-refine into a space $Y$ if there exists an F-refinement $f : X \to Y$.

**Lemma 3.** *Let $X \times Y$ be a hereditarily normal space, $M$ be a closed subset of $X$, and $f : M \to Y$ be an F-refinement. Then the set $D = \{(m, f(m));\ m \in M\}$ is discrete in $X \mathbin{\widetilde{\otimes}} Y$.*

P r o o f. Choose a point $m_0 \in M$, and let U be any its neighborhood such that $\{m \in M;\ f(m) = f(m_0)\} \cap \overline{U} = \emptyset$. Since the sets $D \setminus \{(m_0, f(m_0))\}$ and $E = (\{m_0\} \times Y) \cup (\overline{U} \times \{f(m_0)\}) \setminus \{(m_0, f(m_0))\}$ are closed in $X \times Y \setminus \{(m_0, f(m_0))\}$, there is a continuous function $h : X \times Y \setminus \{(m_0, f(m_0))\} \to [0;1]$ such that $h(D \setminus \{(m_0, f(m_0))\}) = \{1\}$ and $h(E) = \{0\}$. We extend $h$ to $X \times Y$ by defining $h(m_0, f(m_0)) = 0$. Then $h$ is separately continuous or, in other words, continuous with respect to the topology of the space $X \mathbin{\widetilde{\otimes}} Y$. Additionally $h(m_0, f(m_0)) = 0$ and $h(D \setminus \{(m_0, f(m_0))\}) = \{1\}$. $\square$

**Theorem 4.** *Let a space $X$ contain a Čech-complete non-scattered subspace that F-refines into a space $Y$, and let the space $X \times Y$ be perfectly normal. Then the space $X \mathbin{\widetilde{\otimes}} Y$ is not normal.*

P r o o f. We suppose that $Z \subset X$ is a Čech-complete non-scattered space. By Lemma 1 there exist a compact $M \subset Z$ and an irreducible function $g : M \to [0;1]$, and by Lemma 2 the sets $T = g^{-1}([0;1] \cap \mathbb{Q})$ and $M \setminus T = g^{-1}([0;1] \setminus \mathbb{Q})$ are dense in $M$.

Now we consider the mapping $D : M \to M \times Y$ given by the rule $D(m) = (m, f(m))$, where $f : M \to Y$ is an F-refinement, and denote $F_0 = D(T)$ and $F_1 = D(M \setminus T)$. By Lemma 3 the sets $F_0$ and $F_1$ are closed in $X \mathbin{\widetilde{\otimes}} Y$. Our goal is to show that it is impossible to separate the sets $F_0$ and $F_1$ by neighborhoods in the space $X \mathbin{\widetilde{\otimes}} Y$.

Let $G_0$ and $G_1$ be arbitrary neighborhoods of the sets $F_0$ and $F_1$ respectively. Since $X \times Y$ is perfectly normal, we have that $D(M)$ is equal to intersection of some decreasing sequence of open sets $\{U_j\}_{j\in\mathbb{N}}$. Therefore for each point $m \in M$ we can find a natural number $j(m)$ such that $\big(U_{j(m)} \cap (M \times \{f(m)\})\big) \cup \big(U_{j(m)} \cap (\{m\} \times f(M))\big) \subset G_i$, where $i = 0$ or $i = 1$.

For a natural number $j$, we put $M_j = \{m \in M;\ j(m) \leqslant j\}$. Clearly $M = \bigcup\limits_{j=1}^{\infty} M_j$. Then it follows from the Baire category theorem that for some $j_0$ the set $M_{j_0}$ is not nowhere dense, i.e. there is an open in $M$ set $W_0$ such that $W_0 \subset \overline{M}_{i_0}^{M}$, and besides we may assume that $W_0 \times f(W_0) \subset U_{i_0}$. The three alternatives are possible for the set $W = W_0 \cap M_{j_0}$: a) $W \cap T = \emptyset$; b) $W \cap (M \setminus T) = \emptyset$; c) $W \cap T \neq \emptyset$ and $W \cap (M \setminus T) \neq \emptyset$.

We shall consider all these alternatives.

a) Let $m_1 \in W_0 \cap T$ and $m_2 \in W \cap \{m \in M;\ (m, f(m_1)) \in U_{j(m_1)}\}$. Then $j(m_2) \leqslant j_0 < j(m_1)$ and $(m_2, f(m_1)) \in U_{j(m_1)} \subset U_{j(m_2)}$. Consequently $G_0 \cap G_1 \neq \emptyset$.

b) Let $m_1 \in W_0 \cap (M \setminus T)$ and $m_2 \in W \cap \{m \in M;\ (m, f(m_1)) \in U_{j(m_1)}\}$. Then $j(m_2) \leqslant j_0 < j(m_1)$ and $(m_2, f(m_1)) \in U_{j(m_1)} \subset U_{j(m_2)}$. Consequently $G_0 \cap G_1 \neq \emptyset$.

c) Let $m_1 \in W \cap T$ and $m_2 \in W \cap (M \setminus T)$. Then $\max\{j(m_1), j(m_2)\} \leqslant j_0$ and $(m_1, f(m_2)) \in W_0 \times f(W_0) \subset U_{j_0} \subset U_{j(m_1)} \cap U_{j(m_2)}$. Consequently $G_0 \cap G_1 \neq \emptyset$. □

**Sufficient condition of normality.**

**Lemma 5.** *Let $Y$ be a paracompact, and assume that a space $X$ contains a point $\infty$ such that $(X \setminus \{\infty\}) \,\widetilde{\otimes}\, Y$ is collectionwise normal. Then the space $X \,\widetilde{\otimes}\, Y$ is collectionwise normal too.*

P r o o f. Let $\{F_s\}_{s\in S}$ be a discrete family of closed sets in the space $X \,\widetilde{\otimes}\, Y$. First, we shall prove that one can separate the sets $F_s$ by neighborhoods in the case when the set $S$ is divided into subsets $S_1$ and $S_2$ such that $A = \bigcup\limits_{s\in S_1} F_s \subset (X \setminus \{\infty\}) \times Y$ and $B = \bigcup\limits_{s\in S_2} F_s \subset \{\infty\} \times Y$. Obviously, in this case it suffices to separate the sets $A$ and $B$.

Denote $Z = \{y \in Y;\ (\infty, y) \in B\}$. For each point $z \in Z$ there is an open in $X \,\widetilde{\otimes}\, Y$ set $U_z$ such that $(\infty, z) \in U_z \subset \overline{U}_z \subset (X \times Y) \setminus A$. Then the paracompact set $B$ has a locally finite open cover $\{\{\infty\} \times V_t\}_{t\in T}$ inscribed in the cover $\{U_z \cap B\}_{z\in Z}$. For each index $t \in T$ we fix a point $z(t)$ such that $\{\infty\} \times V_t \subset U_{z(t)}$ and put $W_t = (X \times V_t) \cap U_{z(t)}$. The family $\{W_t\}_{t\in T}$ is locally finite in $X \,\widetilde{\otimes}\, Y$. Hence $B \subset \bigcup\limits_{t\in T} W_t \subset \overline{\bigcup\limits_{t\in T} W_t} = \bigcup\limits_{t\in T} \overline{W}_t \subset \bigcup\limits_{t\in T} \overline{U}_{z(t)} \subset (X \times Y) \setminus A$.

Now we are ready to prove the statement of the lemma in general case. In view of collectionwise normality of $Y$ we can find disjoint open in $Y$ sets $U_s$ such that $F_s \cap (\{\infty\} \times Y) \subset U_s$. By using the above considered case we obtain that $F_s \setminus (X \times U_s) \subset V_s^1$ and $F_s \cap (\{\infty\} \times Y) \subset V_s^2$ for some disjoint open in $X \widetilde{\otimes} Y$ sets $V_s^1$ and $V_s^2$. Since $(X \setminus \{\infty\}) \widetilde{\otimes} Y$ is collectionwise normal, there are disjoint open in $X \widetilde{\otimes} Y$ sets $W_s^1$ and $W_s^2$ such that $F_s \setminus (V_s^1 \cup V_s^2) \subset W_s^1$ and $F_s \setminus (X \times U_s) \subset W_s^2$. Thus it is easy to check that the sets $\big((W_s^1 \cup V_s^1 \cup V_s^2) \cap (X \times U_s)\big) \cup \big(W_s^2 \cap V_s^1\big)$ are disjoint neighborhoods of the sets $F_s$. $\square$

We recall that a normal space $X$ is called strongly zero-dimensional if for any closed set $F \subset X$ and for any its neighborhood $U$ there exists a clopen set $H$ such that $F \subset H \subset U$ ([2, 6.2]).

**Lemma 6.** *In any open cover of a strongly zero-dimensional paracompact one can inscribe a disjoint open cover.*

Proof. Let $X$ be a strongly zero-dimensional paracompact, and let $\{U_t\}_{t\in T}$ be an open cover of the space $X$. Regularity and paracompactness of $X$ enable us, in an obvious way, to inscribe combinatorially with closure an open cover $\{V_t\}_{t\in T}$ in the cover $\{U_t\}_{t\in T}$.

By the definition of strong zero-dimensionality, for each $t \in T$ there is a clopen set $H_t$ such that $\overline{V}_t \subset H_t \subset U_t$. We may assume that the set $T$ is well ordered and put $W_t = H_t \setminus \bigcup\limits_{t'<t} H_{t'}$. Then $\{W_t\}_{t\in T}$ is the required cover. $\square$

We recall that an ordinal $\mathrm{ht}(X) = \min\{\alpha;\ X^{(\alpha)} = \emptyset\}$ is called scattered height of the space $X$. Here $X^{(\alpha)}$ is the $\alpha$-th Cantor-Bendixson derivative of $X$.

**Theorem 7.** *Let $X$ be a scattered strongly zero-dimensional paracompact, and $Y$ be a paracompact. Then the space $X \widetilde{\otimes} Y$ is collectionwise normal.*

Proof. A) Let $\mathrm{ht}(X) = \alpha + 1$ be an isolated ordinal. We suppose that for all spaces $\widetilde{X}$ with the property $\mathrm{ht}(\widetilde{X}) \leqslant \alpha$ the statement of the theorem is true. Since the space $X^{(\alpha)}$ is discrete for each point $x \in X^{(\alpha)}$ there is an open in $X$ set $U_x$ such that $U_x \cap X^{(\alpha)} = \{x\}$. And also choose arbitrary neighborhoods $U_x$ in the space $X \setminus X^{(\alpha)}$ for all remaining points $x \in X \setminus X^{(\alpha)}$. By Lemma 6, in the open cover $\{U_x\}_{x\in X}$ we can inscribe an open disjoint cover $\{V_t\}_{t\in T}$. Then $X \widetilde{\otimes} Y = \bigoplus\limits_{t\in T} (V_t \widetilde{\otimes} Y)$. By Lemma 5 and the inductive assumption, all the spaces $V_t \widetilde{\otimes} Y$ are collectionwise normal. Hence the space $X \widetilde{\otimes} Y$ is collectionwise normal too.

B) Let $\mathrm{ht}(X) = \alpha$ be a limit ordinal. We suppose that for all spaces $\widetilde{X}$ with the property $\mathrm{ht}(\widetilde{X}) < \alpha$ the statement of the theorem is true. For each point $x \in X$ we fix an ordinal $\beta_x < \alpha$ such that $x \notin X^{(\beta_x)}$ and take an arbitrary neighborhood $U_x$ of the point $x$ in the space $X \setminus X^{(\beta_x)}$. By Lemma 6 in open cover $\{U_x\}_{x\in X}$ we can inscribe an open disjoint cover $\{V_t\}_{t\in T}$. Then by inductive assumption the space $V_t \widetilde{\otimes} Y$ is collectionwise normal for any $t \in T$. Hence the space $X \mathbin{\widetilde{\otimes}} Y = \bigoplus_{t\in T} (V_t \mathbin{\widetilde{\otimes}} Y)$ is collectionwise normal too. □

### Criterion of normality.

**Lemma 8.** *Any locally compact paracompact scattered space is strongly zero-dimensional.*

Proof. Indeed a scattered space is hereditarily disconnected, and in the class of locally compact paracompact spaces hereditary disconnectedness is equivalent to strong zero-dimensionality ([2], 6.2.9). □

**Theorem 9.** *Let a locally compact paracompact space $X$ $F$-refine into a paracompact space $Y$, and let the space $X \times Y$ be perfectly normal. Then the space $X \mathbin{\widetilde{\otimes}} Y$ is normal if and only if $X$ is scattered.*

Proof. Theorem 4 and Čech-completeness of the locally compact space $X$ imply necessity, and Lemma 8 and Theorem 7 imply sufficiency. □

**Corollary 10.** *Let $X$ be a locally compact paracompact space, and let the space $X \times X$ be perfectly normal. Then the space $X \mathbin{\widetilde{\otimes}} X$ is normal if and only if $X$ is scattered.*

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