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SPECTRA OF RUELLE TRANSFER OPERATORS FOR CONTACT FLOWS

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Dedicated to Vesselin Petkov on the occasion of his 65th birthday

Abstract. In this survey article we discuss some recent results concerning strong spectral estimates for Ruelle transfer operators for contact flows on basic sets similar to these of Dolgopyat obtained in the case of Anosov flows with C^1 stable and unstable foliations. Some applications of Dolgopyat's results and the more recent ones are also described.

1. Introduction.

Let M be a C^2 complete (not necessarily compact) Riemann manifold and let $\phi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ be a C^2 *contact flow* on M, i.e. there exists a smooth (C^2) flow-invariant one-form ω on M such that $\omega \wedge (d\omega)^n \neq 0$ on M, where $\dim(M) = 2n + 1$. A subset Λ of M is called a *basic set* for ϕ_t if Λ is a locally maximal compact invariant hyperbolic subset of M which is not a single closed orbit and ϕ_t is transitive on Λ.

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Let $\|\cdot\|$ be the *norm* on T_xM determined by the Riemann metric on M and let $E^u(x)$ and $E^s(x)$ ($x \in \Lambda$) be the tangent spaces to the strong unstable and stable manifolds $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$, respectively (see Sect. 2).

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t near Λ consisting of rectangles $R_i = [U_i, S_i]$, where U_i (resp. S_i) are subsets of $\Lambda \cap W^u_\epsilon(z_i)$ and $\Lambda \cap W^s_\epsilon(z_i)$, respectively, for some $z_i \in \Lambda$ and $\epsilon > 0$ (see Sect. 2 for details). Assuming that the local stable and unstable laminations over Λ are Lipschitz, the *first return time function* $\tau : R = \cup_{i=1}^k R_i \longrightarrow [0, \infty)$ and the standard Poincaré map $\mathcal{P} : R \longrightarrow R$ are (essentially) Lipschitz. Setting $U = \cup_{i=1}^k U_i$, the *shift map* $\sigma : U \longrightarrow U$ is defined by $\sigma = p \circ \mathcal{P}$, where $p : R \longrightarrow U$ is the projection along the leaves of local stable manifolds. Given a Lipschitz real-valued function f on U, set $g = f - P\tau$, where $P \in \mathbb{R}$ is the unique number such that the topological pressure of g with respect to σ is zero (cf. e.g. [37]).

For $a, b \in \mathbb{R}$, one defines the *Ruelle operator* $L_{g-(a+\mathbf{i}b)\tau} : C^{\mathrm{Lip}}(U) \longrightarrow C^{\mathrm{Lip}}(U)$ in the usual way (cf. Sect.2.). Here $C^{\mathrm{Lip}}(U)$ is the space of Lipschitz functions $g : U \longrightarrow \mathbf{C}$. By $\mathrm{Lip}(g)$ we denote the Lipschitz constant of g and by $\|g\|$ the *standard* sup *norm* of g. Given $b \in \mathbb{R} \setminus \{0\}$, consider the norm

$$\|h\|_{\mathrm{Lip},b} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|b|},$$

on $C^{\mathrm{Lip}}(U)$.

In his PhD thesis [14] and subsequently in [15] Dmitry Dolgopyat considered the case of a transitive Anosov flow on a compact manifold (i.e. the case $\Lambda = M$) with C^1 jointly non-integrable stable and unstable foliations (see Sect. 2 for the definitions). Instead of $C^{\mathrm{Lip}}(U)$, Dolgopyat considered the space $C^1(U)$ with the norm $\|h\|_{1,b} = \|h\|_0 + \|dh\|/|b|$. The first of his results stated below concerns the potential $f = 0$ which corresponds to the so called Sinai-Bowen-Ruelle measure on M (see Sect. 2 for details).

Theorem 1.1 ([15]). *Let $\phi_t : M \longrightarrow M$ be a transitive Anosov flow on a compact Riemann manifold with C^1 jointly non-integrable stable and unstable foliations and let $g = -P\,\tau$ (i.e. $f = 0$). Then for every $\epsilon > 0$ there exist constants $\rho \in (0,1)$, $a_0 \in (0,1)$ and $C > 0$ such that for every integer $m > 0$ and every $h \in C^1(U)$, if $a, b \in \mathbb{R}$, are such that $|a| \leq a_0$, $|b| \geq 1/a_0$, then*

$$\|L^m_{-(P+a+\mathbf{i}b)\tau} h\|_{1,b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{1,b}.$$

In particular, the spectral radius of $L_{-(P+a+\mathbf{i}b)\tau}$ in $C^1(U)$ does not exceed $\rho < 1$.

An important case when the assumptions in the above theorem are fulfilled is that of the geodesic flow on a compact Riemann manifold X whose sectional curvature is negative and satisfies the so called 1/4-*pinching condition*, i.e. the sectional curvature is always in $[-k, -k/4)$ for some constant $k > 0$. By a well-known result of Hirsch and Pugh [22], in this case the stable and unstable foliations are always C^1. In fact, when $\dim(X) = 2$ the latter is true without assuming the 1/4-pinching condition ([22]).

Theorem 1.2 ([14], [15]). *Let X be a compact surface of negative curvature and let $\phi_t : M = S(X) \longrightarrow M$ be the geodesic flow on its sphere bundle. For any $f \in C^1(U)$ and any $\epsilon > 0$ there exist constants $\rho \in (0,1)$, $a_0 \in (0,1)$ and $C > 0$ such that for every integer $m > 0$ and every $h \in C^1(U)$, if $a, b \in \mathbb{R}$, are such that $|a| \leq a_0$, $|b| \geq 1/a_0$, then*

$$\|L^m_{f-(P+a+\mathbf{i}b)\tau} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\mathrm{Lip},b}.$$

In particular, the spectral radius of $L_{-(P+a+\mathbf{i}b)\tau}$ in $C^1(U)$ does not exceed $\rho < 1$.

As consequences of these results Dolgopyat [14], [15] obtained exponential decay of correlations for the flow $\phi_t : M \longrightarrow M$; see Sect. 2 below for details. In fact, these consequences appear to be the main motivation for the work of Dolgopayt and are stated as the main results in [14] and [15].

Dolgopyat's results on decay of correlations were preceded by these of Chernov [10] establishing sub-exponential decay of correlations for certain Anosov flows. Further results in this direction were established by Young [61], [62], Liverani [31] ond others – see Sect. 3 below for some other references. Liverani [31] proved exponential decay of correlations for any Hölder continuous potentials for contact Anosov flows on compact Riemann manifolds. To my knowledge this has been the strongest and most general result for exponential decay of correlations for flows so far.

For general information on various types of decay of correlations, the reader is referred to the survey article of Baladi [3] (cf. also the Addendum by Dolgopyat and Pollicott [17]).

Unlike the applications of Theorems 1.1 and 1.2 concerning decay of correlations, there are other applications for which it is not known whether they can be replaced by something else. For example, it has been well known since Dolgopyat's paper [15] that the strong spectral estimates for Ruelle transfer operators in Theorems 1.1 and 1.2 lead to deep results concerning zeta functions and related topics which are difficult to obtain by other means. For example, Theorem 1.1 is fundamental in the works [43] and [45] of Pollicott and Sharp dealing with geodesic flows on compact surfaces of negative curvature. These applications are described in Sect. 3 below.

Sect. 2 contains some basic definitions and terminology. As mentioned above, Sect. 3 describes some applications of Theorems 1.1 and 1.2. Sect. 4 is devoted to some more recent results similar to Dolgopyat's however concerning flows on basic sets which are not necessarily manifolds. There we first deal with open billiard flows in $\mathbb{R}^n \setminus K$, where K is a finite disjoint union of strictly convex compact bodies with smooth boundaries satisfying an additional (no eclipse) condition. This is followed by a short discussion of the case of geodesic flows on manifolds of constant negative curvature. Finally, we describe some recently obtained ([58]) analogue of Theorem 1.1. for contact flows on basic sets satisfying some additional conditions – this is Theorem 4.4 which is the main point in Sect. 4.

Open billiard flows appear naturally in [41], where a link is established between the existence of analytic continuations (in certain related regions in the complex plane) of a zeta function on one side and the cut-off resolvent of the Dirichlet Laplacian in $\mathbb{R}^n \setminus K$ on the other. The main result of [41] is described in Sect. 5 below. The strong spectral estimates in Theorem 4.4 (applied to open billiard flows) play a crucial role there.

Sect. 6 contains a sketch of the proof of Theorem 4.4.

2. Markov families and Gibbs measures. Throughout this paper M denotes a C^2 complete (not necessarily compact) Riemann manifold, and $\phi_t : M \longrightarrow M$ $(t \in \mathbb{R})$ a C^2 flow on M. A ϕ_t-invariant closed subset Λ of M is called *hyperbolic* if Λ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ such that there exists a $d\phi_t$-invariant decomposition $T_xM = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of T_xM $(x \in \Lambda)$ into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at x, $\|d\phi_t(u)\| \leq C\,\lambda^t\,\|u\|$ for all $u \in E^s(x)$ and $t \geq 0$, and $\|d\phi_t(u)\| \leq C\,\lambda^{-t}\,\|u\|$ for all $u \in E^u(x)$ and $t \leq 0$.

The flow ϕ_t is called an *Axiom A flow* on M if the non-wandering set of ϕ_t (see e.g. [28]) is a disjoint union of a finite set consisting of fixed hyperbolic points and a compact hyperbolic subset containing no fixed points in which the periodic points are dense.

A non-empty compact ϕ_t-invariant hyperbolic subset Λ of M which is not a single closed orbit is called a *basic set* for ϕ_t if ϕ_t is transitive on Λ (that is, ϕ_t has a dense orbit in Λ) and Λ is locally maximal, i.e. there exists an open neighbourhood V of Λ in M such that $\Lambda = \cap_{t\in\mathbb{R}}\phi_t(V)$. According to Smale's spectral decomposition theorem (see e.g. [28]), if the non-wandering set Ω of an

Axiom A flow ϕ_t is compact and F is the (finite) set of fixed hyperbolic points of the flow, then $\Omega \setminus F$ is a finite disjoint union of basic sets. If M is compact and is itself a basic set, then ϕ_t is called an *Anosov flow*.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W_\epsilon^s(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } \ t \geq 0,\ d(\phi_t(x), \phi_t(y)) \to_{t\to\infty} 0\},$$

$$W_\epsilon^u(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } \ t \leq 0,\ d(\phi_t(x), \phi_t(y)) \to_{t\to-\infty} 0\}$$

be the (strong) *stable* and *unstable manifolds* of size ϵ.

In general $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ depend Hölder continuously on $x \in \Lambda$, however throughout we always **assume that $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ are Lispchitz** in $x \in \Lambda$ (which is a consequence of the condition (P) anyway; see [23] or [46]).

For any $A \subset M$ and $I \subset \mathbb{R}$ denote $\phi_I(A) = \{\ \phi_t(y) : y \in A, t \in I\ \}$.

It follows from the hyperbolicity of Λ (cf. [28]) that if $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W_\epsilon^s(x)$ and $\phi_{[-\epsilon,\epsilon]}(W_\epsilon^u(y))$ intersect at exactly one point $[x, y] \in \Lambda$. That is, there exists a unique $t \in [-\epsilon, \epsilon]$ such that $\phi_t([x, y]) \in W_\epsilon^u(y)$. Setting $\Delta(x, y) = t$, defines the so called *temporal distance function* ([27], [10], [15]). For $x, y \in \Lambda$ with $d(x, y) < \delta$, define

$$\pi_y(x) = [x, y] = W_\epsilon^s(x) \cap \phi_{[-\epsilon,\epsilon]}(W_\epsilon^u(y)).$$

Thus, for a fixed $y \in \Lambda$, $\pi_y : W \longrightarrow \phi_{[-\epsilon,\epsilon]}(W_\epsilon^u(y))$ is the *projection* along local stable manifolds defined on a small open neighbourhood W of y in Λ.

The stable and unstable laminations on Λ (or foliations in the case $\Lambda = M$) are called *jointly non-integrable* if there exists $z \in \Lambda$ such that Δ is not constantly zero on $(\Lambda \cap W_\epsilon^u(z)) \times (\Lambda \cap W_\epsilon^s(z))$.

Given $A \subset \Lambda$ we will denote by $\mathrm{Int}_\Lambda(A)$ and $\partial_\Lambda A$ the *interior* and the *boundary* of the subset A of Λ in the topology of Λ. We will say that A is an *admissible subset* of $W_\epsilon^u(z)$ ($z \in \Lambda$) if A coincides with the closure of its interior in $W_\epsilon^u(z)$. Admissible subsets of $W_\epsilon^s(z)$ are defined similarly. Following [15], a subset R of Λ will be called a *rectangle* if it has the form $R = [U, S] = \{[x, y] : x \in U, y \in S\}$, where U and S are admissible subsets of $W_\epsilon^u(z)$ and $W_\epsilon^s(z)$, respectively, for some $z \in \Lambda$. For such R, given $\xi = [x, y] \in R$, we will denote $W_R^u(\xi) = \{[x', y] : x' \in U\}$ and $W_R^s(\xi) = \{[x, y'] : y' \in U\} \subset W_{\epsilon'}^s(x)$.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a family of rectangles with $R_i = [U_i, S_i]$, $U_i \subset W_\epsilon^u(z_i) \cap \Lambda$ and $S_i \subset W_\epsilon^s(z_i) \cap \Lambda$, respectively, for some $z_i \in \Lambda$. Set $R = \cup_{i=1}^k R_i$. The family $\mathcal{R}$ is called *complete* if there exists $T > 0$ such that for every $x \in \Lambda$, $\phi_t(x) \in R$ for some $t \in (0, T]$. Thus, $\tau(x) > 0$ is the smallest positive time with $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, and $\mathcal{P} : R \longrightarrow R$ is the *Poincaré map* related to the family

$\mathcal{R}$. The function $\tau : R \longrightarrow [0, \infty)$ is called the *first return time* associated with $\mathcal{R}$. Notice that τ is constant on the set $W^s_{R_i}(x)$, $x \in R_i$. A complete family $\mathcal{R} = \{R_i\}_{i=1}^k$ of rectangles in Λ is a *Markov family* of size $\chi > 0$ for the flow ϕ_t if $\mathrm{diam}(R_i) \leq \chi$ for all i and: (a) for any $i \neq j$ and any $x \in R_i \cap \mathcal{P}^{-1}(R_j)$ we have $\mathcal{P}(W^s_{R_i}(x)) \subset W^s_{R_j}(\mathcal{P}(x))$ and $\mathcal{P}(W^u_{R_i}(x)) \supset W^u_{R_j}(\mathcal{P}(x))$; (b) for any $i \neq j$ at least one of the sets $R_i \cap \phi_{[0,\chi]}(R_j)$ and $R_j \cap \phi_{[0,\chi]}(R_i)$ is empty.

The existence of a Markov family $\mathcal{R}$ of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [6] (cf. also Ratner [48]).

From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for ϕ_t. Set

$$U = \cup_{i=1}^k U_i.$$

The *shift map* $\sigma : U \longrightarrow U$ is given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves.

Denote by $\widehat{U}$ the *core* of U, i.e. the set of those $x \in U$ such that $\mathcal{P}^m(x) \in \mathrm{Int}_\Lambda(R)$ for all $m \in \mathbb{Z}$. It is well-known (see [6]) that $\widehat{U}$ is a residual subset of U and has full measure with respect to any Gibbs measure on U. Clearly in general τ is not continuous on U, however τ is Lipschitz on $\widehat{U}$. The same applies to $\sigma : U \longrightarrow U$. Throughout we will mainly work with the restrictions of τ and σ to $\widehat{U}$.

Let $B(\widehat{U})$ be the *space of bounded functions* $g : \widehat{U} \longrightarrow \mathbf{C}$ with its standard norm $\|g\|_0 = \sup_{x \in \widehat{U}} |g(x)|$. Given a function $g \in B(\widehat{U})$, the *Ruelle operator* $L_g : B(\widehat{U}) \longrightarrow B(\widehat{U})$ is defined by

$$(L_g h)(u) = \sum_{\sigma(v)=u} e^{g(v)} h(v).$$

If $g \in B(\widehat{U})$ is Lipschitz on $\widehat{U}$, then L_g preserves the space $C^{\mathrm{Lip}}(\widehat{U})$ of *Lipschitz functions* $g : \widehat{U} \longrightarrow \mathbf{C}$.

In Theorem 1.2 above and its analogues in Sect. 4 below we assume that f is a **fixed function in** $C^1(U)$ (or $C^{\mathrm{Lip}}(\widehat{U})$, respectively) and $g = f - P\,\tau$, where P is the unique real number with $\mathrm{Pr}_\sigma(f - P\,\tau) = 0$, where $\mathrm{Pr}_\sigma(h)$ is the *topological pressure* of h with respect to the shift map σ (see e.g. [37]).

The set U described above and the shift map σ provide some kind of symbolic dynamics for the flow ϕ_t on Λ which is naturally related to the classic symbolic coding provided by the Markov family $\mathcal{R}$. Below we describe this relationship and define Gibbs measures as well.

Let $A = (A_{ij})_{i,j=1}^k$ be the matrix given by $A_{ij} = 1$ if $\mathcal{P}(\mathrm{Int}_\Lambda(R_i)) \cap$

$\text{Int}_\Lambda(R_j) \neq \emptyset$ and $A_{ij} = 0$ otherwise. Consider the symbol space

$$\Sigma_A = \{(i_j)_{j=-\infty}^{\infty} : 1 \leq i_j \leq k, A_{i_j\, i_{j+1}} = 1 \quad \text{for all} \quad j\},$$

with the product topology and the *shift map* $\sigma_A : \Sigma_A \longrightarrow \Sigma_A$ given by $\sigma_A((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all j. As in [6] one defines a natural surjection $\pi : \Sigma_A \longrightarrow R$ such that $\pi \circ \sigma_A = \mathcal{P} \circ \pi$ on a residual subset of R. Moreover π is Lispchitz if Σ_A is considered with the *metric* d_θ for some $\theta \in (0, 1)$, where $d_\theta(\xi, \eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi, \eta) = \theta^m$ if $\xi_i = \eta_i$ for $|i| \leq m$ and m is maximal with this property. Notice that $\hat{\tau} = \tau \circ \pi$ defines a Lipschitz function on a residual subset of Σ_A, so it has a Lipschitz extension $\hat{\tau} : \Sigma_A \longrightarrow \mathbb{R}_+$ ([6]).

Let $\Lambda(A, \tau)$ be the quotient space of

$$Y = \{(\xi, s) : \xi \in \Sigma_A, 0 \leq s \leq \hat{\tau}(\xi)\} \subset \Sigma_A \times \mathbb{R}$$

with respect to the equivalence relation identifying $(\xi, \hat{\tau}(\xi))$ and $(\sigma_A(\xi), 0)$ for all $\xi \in \Sigma_A$. The *suspension flow* $\hat{\phi}_t : \Lambda(A, \tau) \longrightarrow \Lambda(A, \tau)$ is defined by $\hat{\phi}_t(\xi, s) = (\xi, s + t)$, $t \in \mathbb{R}$, and it follows from [6] that π has a continuous extention $\pi : \Lambda(A, \tau) \longrightarrow \Lambda$ such that $\phi_t \circ \pi = \pi \circ \hat{\phi}_t$ for all $t \in \mathbb{R}$.

In a similar way one deals with the one-sided subshift of finite type

$$\Sigma_A^+ = \{(i_j)_{j=0}^{\infty} : 1 \leq i_j \leq k, A_{i_j\, i_{j+1}} = 1 \quad \text{for all} \quad j \geq 0\},$$

where the *shift map* $\sigma_A : \Sigma_A^+ \longrightarrow \Sigma_A^+$ is defined in a similar way: $\sigma_A((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all $j \geq 0$. Notice that $\hat{\tau}(\xi) = \tau(\pi(\xi))$ depends only on the forward coordinates of $\xi \in \Sigma_A$. Indeed, if $\xi_+ = \eta_+$, where $\xi_+ = (\xi_j)_{j=0}^{\infty}$, then for $x = \pi(\xi)$ and $y = \pi(\eta)$ we have $x, y \in R_i$ for $i = \xi_0 = \eta_0$ and $\mathcal{P}^j(x)$ and $\mathcal{P}^j(y)$ belong to the same R_{i_j} for all $j \geq 0$. This implies that x and y belong to the same local stable fibre in R_i and therefore $\tau(x) = \tau(y)$. Thus, $\hat{\tau}(\xi) = \hat{\tau}(\eta)$. In particular we can also consider $\hat{\tau}$ as a function on Σ_A^+ such that $\hat{\tau} = \tau \circ \pi$ on a residual subset of Σ_A^+.

The *space of Lipschitz functions* on Σ_A with respect to the metric d_θ will be denoted by $C_\theta(\Sigma_A)$. In a similar way one defines a metric d_θ on Σ_A^+ and the space of Lipschitz functions $C_\theta(\Sigma_A^+)$.

If $\hat{\pi} : \Sigma_A \longrightarrow \Sigma_A^+$ is the *natural projection*, one shows easily that there exists a continuous surjection $\pi^+ : \Sigma_A^+ \longrightarrow U$ such that then $\pi^+ \circ \hat{\pi} = \pi^{(U)} \circ \pi$. Moreover, $\sigma \circ \pi^+ = \pi^+ \circ \sigma_A^+$.

Given $F \in \mathcal{F}_\alpha(\Lambda)$ for some $\alpha > 0$, consider the function $\hat{F} = F \circ \pi$ on $\Lambda(A, \tau)$. Let $\hat{\mu}$ be the *Gibbs measure* related to $\hat{F}$ with respect to the suspended flow $\hat{\phi}_t$, and let $P = \Pr_{\hat{\phi}_t}(\hat{F})$ be the *topological pressure* of $\hat{F}$ with respect to $\hat{\phi}_t$ (see e.g. [37]). The function

$$\hat{f}_1(\xi) = \int_0^{\hat{\tau}(\xi)} \hat{F}(\xi, s)\, ds$$

is in $C_\theta(\Sigma_A)$ for some $\theta \in (0,1)$ and $\hat{f}_1 = f_1 \circ \pi$ on a residual subset of Σ_A, where $f_1(x) = \int_0^{\tau(x)} F(\phi_s(x))\, ds$, $x \in R$. It follows from Sinai's Lemma (see e.g. Proposition 1.2 in [37]) that there exist $\hat{f}, \hat{f}_2 \in C_{\theta^{1/2}}(\Sigma_A)$ such that

$$\hat{f}_1(\xi) = \hat{f}(\xi) + \hat{f}_2(\sigma_A(\xi)) - \hat{f}_2(\xi) \quad , \quad \xi \in \Sigma_A,$$

and $\hat{f}(\xi) = \hat{f}(\eta)$ whenever $\xi_+ = \eta_+$ (i.e. $\xi_j = \eta_j$ for all $j \geq 0$). That is, $\hat{f}$ can be regarded as a function on Σ_A^+. Notice that $\xi_+ = \eta_+$ is equivalent to $\pi^{(U)}(\pi(\xi)) = \pi^{(U)}(\pi(\eta))$. Thus, there exists a function f on U such that $f(\pi^{(U)}(\pi(\xi))) = \hat{f}(\xi)$ for all $\xi \in \Sigma_A$.

Let $\hat{\nu}$ be the *Gibbs measure* on Σ_A^+ determined by the function $\hat{f} - P\,\hat{\tau}$. Then (cf. [37])

$$d\hat{\mu}(\xi, s) = \frac{1}{\hat{\nu}(\hat{\tau})} d\hat{\nu}(\xi)\, ds, \text{ where } \quad \hat{\nu}(\hat{g}) = \int_{\Sigma_A^+} \hat{g}(\xi)\, d\hat{\nu}(\xi).$$

Moreover, we have $\Pr_{\sigma_A}(\hat{f} - P\,\hat{\tau}) = 0$. The Gibbs measures $\hat{\mu}$ and $\hat{\nu}$ give rise to measures μ and ν on Λ and U, respectively, via the surjections π and π^+. The measures μ and ν are called the *Gibbs measures* related to F and $f - P\tau$, respectively. It follows from above that $\Pr_{\phi_t}(F) = P$ and $\Pr_\sigma(f - P\tau) = 0$.

3. Some applications of Dolgopyat's result. Let M be a smooth (i.e. at least C^2) compact Riemann manifold and $\phi_t : M \longrightarrow M$ be an Anosov flow on M. We say that the flow is *topologically weak-mixing* if there do not exist a non-zero $a \in \mathbf{C}$ and a non-trivial continuous function $F : M \longrightarrow \mathbf{C}$ such that $F \circ \phi_t = e^{\mathbf{i}\,a\,t} F$ on M. Let h_T denote the *topological entropy* of ϕ_t (see e.g. [37]).

The so called *Ruelle* (or *dynamical*) *zeta function* is defined by

$$\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}, \tag{3.1}$$

where γ runs over the set of primitive closed orbits of $\phi_t : \Lambda \longrightarrow \Lambda$ and $\ell(\gamma)$ is the least period of γ.

Remark. If Λ is a basic set of an Axiom A flow ϕ_t, one defines the Ruelle zeta function again by (3.1), where γ runs over the set of primitive closed orbits of $\phi_t : \Lambda \longrightarrow \Lambda$. Topological weak-mixing is defined as in the case of Anosov flows.

For topologically weak-mixing Anosov flows Margulis ([32], [33]) proved that $\zeta(s)$ is analytic in $\mathrm{Re}(s) > h_T$ and has a meromorphic extension to a neighbourhood of the line $\mathrm{Re}(s) = h_T$ with a single simple pole at $s = h_T$. Moreover, Margulis showed that

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} \sim \frac{e^{h_T \lambda}}{h_T\, \lambda}$$

as $\lambda \to \infty$. Parry and Pollicott [36] extended this result to topologically weak-mixing Axiom A flows on basic sets.

In the case of a geodesic flow on a compact surface, using Dolgopyat's Theorem 1.2, Pollicott and Sharp proved a substantially stronger result.

Theorem 3.1 ([43])**.** *Let X be a compact surface of negative curvature and let ϕ_t be the geodesic flow on its sphere bundle. Then the related zeta function $\zeta(s)$ has an analytic and non-vanishing continuation in a half-plane* $\mathrm{Re}(s) > c_0$ *for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, there exists $c \in (0, h_T)$ such that*

$$\pi(\lambda) = \#\{\gamma \in \mathcal{P} : \ell(\gamma) \leq \lambda\} = \mathrm{li}(e^{h_T \lambda}) + O(e^{c\lambda})$$

as $\lambda \to \infty$.

Here $\mathcal{P}$ is the *set of all primitive closed geodesics* on X, and

$$\mathrm{li}(x) = \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}, \quad \text{as } x \to \infty.$$

Much more precise asymptotics have been obtained with respect to homology classes. Given a compact Riemann manifold X of negative curvature and a fixed homology class $\alpha \in H_1(X, \mathbb{Z})$, consider the counting function

$$\pi(\lambda, \alpha) = \#\{\gamma \in \mathcal{P} : \ell(\gamma) \leq \lambda,\ [\gamma] = \alpha\},$$

where $[\gamma]$ denotes the homology class determined by γ. The following was established independently by Anantharaman [2] and Pollicott and Sharp [44] using different methods however both relying heavily on Dolgopyat's Theorem 1.1.

Theorem 3.2 ([2], [44])**.** *Let X be a compact Riemann manifold of negative sectional curvature and first Betti number $b > 0$. Assume that the geodesic* *flow ϕ_t on $M = S(X)$ has C^1 stable and unstable foliations. Then there exist constants $C_0 > 0, C_1, C_2, \dots$ such that*

$$\pi(\lambda, \alpha) = \frac{e^{h_T \lambda}}{\lambda^{b/2}} \left(\sum_{j=0}^{n} \frac{C_j}{\lambda^{j/2}} + o(1/\lambda^{n/2}) \right), \quad \lambda \to \infty,$$

for any integer $n \geq 1$.

Let again X be a compact Riemann surface of negative curvature, and let $\pi_1(X)$ be the fundamental group of X. Fix an arbitrary symmetric set Γ of generators of $\pi_1(X)$. For $g \in \pi_1(X) \setminus \{1\}$, denote by $|g|$ the *length of the minimal (shortest) word* representing g with elements in Γ. For any real numbers $a < b$ set

$$\pi(n, [a,b]) = \#\{(\gamma, \gamma') \in \mathcal{P} \times \mathcal{P} : |\gamma|, |\gamma'| \leq n, \;\; a \leq \ell(\gamma) - \ell(\gamma') \leq b\},$$

Given a sequence $\epsilon_n \to 0$ with $\epsilon_n > 0$ for all n, an interval $[a,b]$ and $z \in \mathbb{R}$, set

$$I_n(z) = [z + \epsilon_n\, a, \; z + \epsilon_n\, b].$$

The following very delicate result concerning asymptotics of word lengths for pairs of closed geodesics on surfaces of negative curvature was obtained by Pollicott and Sharp making a fundamental use of Dolgopayt's Theorem 1.2.

Theorem 3.3 ([45]). *Let ϕ_t be as in Theorem 3.1 above. Then there exist constants $h_0 > 0$ and $\sigma > 0$ such that for any $a < b$ and any sequence $\{\epsilon_n\}$ of positive numbers converging to 0 at a subexponential rate, we have*

$$\lim_{n \to \infty} \sup_{z \in \mathbb{R}} \left[\frac{\sigma\, n^{5/2}}{\epsilon_n\, e^{2h_0 n}}\, \pi(n, I_n(z)) - \frac{(b-a)\, e^{2h_0}}{\sqrt{2\pi}\, (e^{h_0} - 1)^2} \cdot e^{-z^2/2\sigma^2 n} \right] = 0$$

as $n \to \infty$. In particular,

$$\pi(n, I_n(z)) \sim \frac{(b-a)\, e^{2h_0}\, \epsilon_n}{\sqrt{2\pi}\, \sigma\, (e^{h_0} - 1)^2}\, \frac{e^{2h_0 n}}{n^{5/2}}$$

as $n \to \infty$.

As another consequence of Theorems 1.1 and 1.2 and the procedure described in [15] (see also [14]) one gets exponential decay of correlations for the flow $\phi_t : M \longrightarrow M$.

Given $\alpha > 0$, denote by $\mathcal{F}_\alpha(M)$ the set of *Hölder continuous functions* with Hölder exponent α and by $\|h\|_\alpha$ the *Hölder constant* of $h \in \mathcal{F}_\alpha(M)$. Given a Hölder continuous function F on M let ν_F be the Gibbs measure determined by F on M.

Theorem 3.4 ([15]). *Under the assumptions in Theorem* 1.1, *let* $F = 0$ *and under the assumptions of Theorem* 1.2 *let* F *be an arbitrary Hölder continuous function on* M. *Assume in addition that the manifold* M *and the flow* ϕ_t *are* C^5. *Then, in each of these cases, for every* $\alpha > 0$ *there exist constants* $C = C(\alpha) > 0$ *and* $c = c(\alpha) > 0$ *such that*

$$\left|\int_M A(x)B(\phi_t(x))\, d\nu_F(x) - \left(\int_M A(x)\, d\nu_F(x)\right)\left(\int_M B(x)\, d\nu_F(x)\right)\right| \\ \leq Ce^{-ct}\|A\|_\alpha\, \|B\|_\alpha$$

for any two functions $A, B \in \mathcal{F}_\alpha(M)$.

There has been a considerable activity in recent times to establish exponential and other types of decay of correlations for various kinds of systems. Apart from the works referred to in Sect. 1, one should also mention [16], [5], [19], [18], [34]; see also the references there and Sect. 4 below.

4. Some recent results.

4.1. Eventually contracting Ruelle transfer operators. In this section we describe some recent results similar to Dolgopyat's Theorem 1.1 and 1.2 stated in Sect. 1.

Let M be a C^2 complete (not necessarily compact) Riemann manifold, let $\phi_t : M \longrightarrow M$ be a C^2 flow on M, and let Λ be a basic set for ϕ_t.

As in Sect. 1, let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t near Λ consisting of rectangles $R_i = [U_i, S_i]$ of arbitrarily small size $\epsilon > 0$. Consider again $U = \cup_{i=1}^k U_i$, the *shift map* $\sigma : U \longrightarrow U$, the first return function τ, and the core $\widehat{U}$ of U. Given a Lipschitz real-valued function f on $\widehat{U}$, set $g = f - P\tau$, where $P \in \mathbb{R}$ is the unique number such that the topological pressure of g with respect to σ is zero, and consider the Ruelle transfer operators $L_{g-(a+\mathbf{i}b)\tau}$ $(a, b \in \mathbb{R})$ on $C^{\mathrm{Lip}}(\widehat{U})$.

Definition. *We will say that the Ruelle (transfer) operators related to* $\phi_t : \Lambda \longrightarrow \Lambda$ *and the potential* g *on* $\widehat{U}$ *are eventually contracting if for every* $\epsilon > 0$ *there exist constants* $\rho \in (0,1)$, $a_0 \in (0,1)$ *and* $C > 0$ *such that for every integer* $m > 0$ *and every* $h \in C^{\mathrm{Lip}}(\widehat{U})$, *if* $a, b \in \mathbb{R}$, *are such that* $|a| \leq a_0$, $|b| \geq 1/a_0$, *then*

$$\|L^m_{f-(P+a+\mathbf{i}b)\tau} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\mathrm{Lip},b}. \tag{4.1}$$

As before, (4.1) implies that the spectral radius of the operator $L_{f-(P+a+\mathbf{i}b)\tau}$ on $C^{\mathrm{Lip}}(\widehat{U})$ does not exceed $\rho < 1$. Moreover, it follows from the arguments in [43] and the procedure described in [15] (see also [14]) that flows on basic sets as in the above definition have the following general properties.

General Properties 4.1. *Let $\phi_t : \Lambda \longrightarrow \Lambda$ be the restriction of an Axiom A flow to a basic set Λ and U be defined by means of a Markov family $\mathcal{R}$ as above.*

(a) *Assume that the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and the potential $f = 0$ on $\widehat{U}$ (then $g = -P\,\tau$) are eventually contracting. Then the zeta function $\zeta(s)$ determined by $\phi_t : \Lambda \longrightarrow \Lambda$ has an analytic and non-vanishing continuation in a half-plane $\mathrm{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, there exists $c \in (0, h_T)$ such that*

$$\pi(\lambda) = \#\{\gamma \in \mathcal{P} : \ell(\gamma) \leq \lambda\} = \mathrm{li}(e^{h_T \lambda}) + O(e^{c\lambda}), \quad \lambda \to \infty.$$

(b) *Assume that the manifold M and the flow ϕ_t are C^5 and the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and any Lipschitz potential f on $\widehat{U}$ are eventually contracting. Then for any Hölder continuous function F on Λ and any $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$(4.2) \quad \left| \int_\Lambda A(x) B(\phi_t(x))\, d\nu_F(x) - \left(\int_\Lambda A(x)\, d\nu_F(x) \right) \left(\int_\Lambda B(x)\, d\nu_F(x) \right) \right| \leq C e^{-ct} \|A\|_\alpha \, \|B\|_\alpha$$

for any two functions $A, B \in \mathcal{F}_\alpha(\Lambda)$, where ν_F is the Gibbs measure determined by F on Λ.

4.2. Open billiard flows. Let K be a subset of $\mathbb{R}^n$ of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where K_i are compact strictly convex disjoint domains in $\mathbb{R}^n$ with C^p $(p \geq 2)$ *boundaries* ∂K_i and $k_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^n \setminus K}$.

We will assume that K satisfies the following *no-eclipse condition*: for every pair K_i, K_j of different connected components of K the convex hull of $K_i \cup K_j$ has no common points with any other connected component of K. With this condition, the billiard generated in the exterior of K is sometimes called an *open billiard*.

As usual, we will denote by $T(\Omega)$ the *tangent bundle* of Ω and by $S(\Omega)$ its *sphere bundle*. The *billiard flow* ϕ_t (also known as the *broken geodesic flow*) is defined on $S(\Omega)$ (and more generally on $T(\Omega) \setminus \{0\}$) in the standard way. Clearly this flow has singularities, however its restriction to the *non-wandering set* Λ only has simple discontinuities at reflection points. Moreover, Λ is compact, ϕ_t is hyperbolic and transitive on Λ, and it follows from [55] that ϕ_t is non-lattice and therefore by a result of Bowen [6], it is topologically weak-mixing on Λ. It is well-known (and not difficult to see) that the cross-sections of Λ transversal to the flow are Cantor sets.

The following was established using a modification of Dolgopyat's method in [15].

Theorem 4.2 ([56])**.** *Let $n = 2$, K be as above, and let $\phi_t : \Lambda \longrightarrow \Lambda$ be the restriction of the billiard flow to the non-wandering set Λ. If U is defined by means of a Markov family $\mathcal{R}$ as above, then the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and any potential f on $\widehat{U}$ are eventually contracting.*

A similar result for open billiard flows in $\mathbb{R}^n$ with $n > 2$ follows from Theorem 4.4 below, however under some additional assumptions.

It follows from Theorem 4.2 that the open billiard flows in the plane have the properties (a) and (b) in 4.1. Apart from this, using Theorem 4.2 and some techniques developed in [45], a result similar to Theorem 3.3 above was proved in [42] for open billiard flows in the plane.

The motivation for Theorem 4.2 comes from investigations on scattering resonances. In this area two particular types of chaotic systems have been studied extensively – geodesic flows on manifolds of constant negative curvature and open billiard flows. The latter arises in scattering by an obstacle which is a finite union of strictly convex bodies with smooth boundaries (cf. [24], [39], [40]), while the former relates to studies on the distribution of resonances for convex co-compact hyperbolic surfaces or higher dimensional Schottky manifolds (see e.g. [63]), [20], [21]).

Theorem 4.2 and its multidimensional analogue, which follows from Theorem 4.4. below, have been recently used in [41] to prove a link between the existence of analytic continuations (in certain related regions in the complex plane) of a zeta function (defined by means of the billiard flow on Λ) on one side and the cut-off resolvent of the Dirichlet Laplacian in $\mathbb{R}^n \setminus K$ on the other. See Sect. 5 below for more details.

What concerns various types of decay of correlations for billiards, it seems most of the results in this direction concern the corresponding discrete dynamical system (generated by the billiard ball map from boundary to boundary). The first of these appears to be that of Bunimovich, Sinai and Chernov [8], where they established subexponential decay of correlations for a very large class of dispersing billiards. More recently, exponential decay of correlations for the billiard ball map was derived for some classes of dispersing billiards in the plane and on the two-dimensional torus by Young [61] and Chernov [11] as consequences of their more general arguments. See also the survey article [13].

Theorem 4.2 above (together with 4.1 (b)) provides a non-trivial class of billiard flows with exponential decay of correlations for any Hölder continuous potential. To our best knowledge this is the only result of this kind obtained so far. Very recently Melbourne [34] established rapid decay of correlations for generic dispersing billiard flows, while Chernov [12] proved subexponential decay of correlations for open billiard flows with finite horizon on the two-dimensional torus. By *rapid decay* we mean a decay in (4.2) faster than $C\,t^{-m}\,\|A\|_\alpha\,\|B\|_\alpha$ for any integer $m \geq 1$, while a *subexponential decay* means a decay $\leq C\,e^{-c\sqrt{t}}\,\|A\|_\alpha\,\|B\|_\alpha$.

4.3. Geodesic flows on manifolds of constant negative curvature. Let X be a complete (not necessarily compact) connected Riemann manifold of constant curvature $K = -1$ and dimension $\dim(X) = n+1$, $n \geq 1$, and let $\phi_t : M = S(X) \longrightarrow M$ be the geodesic flow on the *unit sphere bundle* of X. According to a classical result of Killing and Hopf (cf. e.g. Corollary 2.4.10 in [60]), any such X is a *hyperbolic manifold*, i.e. X is isometric to $\mathbb{H}^{n+1}/\Gamma$, where $\mathbb{H}^{n+1} = \{(x_1, \ldots, x_1) \in \mathbb{R}^{n+1} : x_1 > 0\}$ is the upper half-space in $\mathbb{R}^{n+1}$ with the *Poincaré metric* $ds^2(x) = \frac{1}{x_1^2}(dx_1^2 + \ldots + dx_1^2)$ and Γ is a discrete group of isometries (Möbius transformations) of $\mathbb{H}^{n+1}$ acting freely and discontinuously on $\mathbb{H}^{n+1}$. Moreover, $\mathbb{H}^{n+1}$ is isometric to the universal covering of X. See e.g. [47] for basic information on hyperbolic manifolds. Given a hyperbolic manifold $X = \mathbb{H}^{n+1}/\Gamma$, the *limit set* $L(\Gamma)$ is defined as the set of accumulation points of all Γ orbits in $\overline{\partial\mathbb{H}^{n+1}}$, the topological closure of $\partial\mathbb{H}^{n+1} = \{0\} \times \mathbb{R}^n$ including ∞. The *non-wandering set* Λ of $\phi_t : M \longrightarrow M$ is the image in M of the set of all points of $S(\mathbb{H}^{n+1})$ generating geodesics with end points in $L(\Gamma)$. In what follows we will assume that Λ is compact and *non-trivial*, i.e. $L(\Gamma)$ has more than two points and $L(\Gamma) \neq \overline{\partial\mathbb{H}^{n+1}}$; then $L(\Gamma)$ is a closed non-empty nowhere dense subset of $\overline{\partial\mathbb{H}^{n+1}}$ without isolated points (see e.g. Sect. 12.1 in [47]). The compactness of Λ is present when Γ is *convex cocompact* (see e.g. [59]).

Using the method of Dolgopyat [15] and some ideas from [56] to deal with Cantor sets, Naud [35] showed that on convex cocompact surfaces $\mathbb{H}^2/\Gamma$ the Ruelle (transfer) operators related to the geodesic flow on the non-wandering set Λ for the special potential $f = 0$ (this is the case when $g = -P\,\tau$, where again the topological pressure of $-P\,\tau$ is zero) are eventually contracting. (In fact, Naud considered Ruelle operators on a space of the form $C^1(I)$, where I is a finite disjoint union of intervals.)

For convex cocompact hyperbolic manifolds $X = \mathbb{H}^{n+1}/\Gamma$ of arbitrary dimension a more general fact was proved in [57] as a consequence of a general procedure.

Theorem 4.3 ([57]). *Let $X = \mathbb{H}^{n+1}/\Gamma$ be a convex cocompact hyperbolic manifold, and let $\phi_t : \Lambda \longrightarrow \Lambda$ be the restriction of the geodesic flow of X to the non-wandering set $\Lambda \subset S(X)$. If U is defined by means of a Markov family $\mathcal{R}$, then the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and any Lipschitz potential f on $\widehat{U}$ are eventually contracting.*

The general result proved in [57] is rather technical and we are not going to discuss it here.

4.4. Contact flows on basic sets. Let again M be a C^2 complete (not necessarily compact) Riemann manifold and let $\phi_t : M \longrightarrow M$ be a C^2 flow on M. In this subsection we assume in addition that ϕ_t is a *contact flow* on M, i.e. there exists a smooth (C^2) flow-invariant one-form ω on M such that $\omega \wedge (d\omega)^n \neq 0$ on M, where $\dim(M) = 2n+1$.

Consider the following *pinching condition* for ϕ_t on Λ:

(P): *There exist constants $C > 0$ and $0 < \alpha \leq \beta$ such that for every $x \in \Lambda$ we have*

$$\frac{1}{C}\, e^{\alpha_x\, t}\, \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C\, e^{\beta_x\, t}\, \|u\| \quad , \quad u \in E^u(x) \ , t > 0,$$

for some constants $\alpha_x, \beta_x > 0$ depending on x but independent of u with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in \Lambda$.

Notice that when $n = 1$ this condition is always satisfied. We should also remark that, as shown in [22], the 1/4-pinching condition for the sectional curvature over a basic set for the geodesic flow on a Riemann manifold of negative curvature implies (P).

A vector $b \in E^u(z) \setminus \{0\}$ will be called *tangent to* Λ at z if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp^u_z(t_m\, v^{(m)}) \in \Lambda \cap W^u_\epsilon(z)$ for all m, $v^{(m)} \to b$ and $t_m \to 0$ as $m \to \infty$. It is easy to see that a non-zero vector $b \in E^u(z)\setminus$ is tangent to Λ at z iff there exists a C^1 curve $z(s)$, $0 \leq s \leq \delta$, on $W^u_{\epsilon_0}(z)$ for some $\delta > 0$ such that $z(s) \in \Lambda$ for arbitrarily small $s > 0$ and $\dot{z}(0) = b$.

In the main results below we impose two additional conditions ϕ_t and Λ. The first of these is the following *non-flatness condition*:

(NF): *For every $x \in \Lambda$ there exists $\epsilon_x > 0$ such that there is no C^1 submanifold X of $W^u_{\epsilon_x}(x)$ of positive codimension with $\Lambda \cap W^u_{\epsilon_x}(x) \subset X$.*

The following condition says that $d\omega$ is in some sense non-degenerate on the 'tangent space' of Λ near some its points:
(ND): *There exist $z_0 \in \Lambda$, $\epsilon > 0$ and $\mu_0 > 0$ such that for any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any unit vector $b \in E^u(\hat{z})$ tangent to Λ at $\hat{z}$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(z_0)$ arbitrarily close to $\hat{z}$ and a unit vector $a \in E^s(\tilde{z})$ tangent to Λ at $\tilde{z}$ with $|d\omega_{\tilde{z}}(a, b_{\tilde{z}})| \geq \mu_0$, where $b_{\tilde{z}}$ is the parallel translate of b along the geodesic in $W^u_\epsilon(z_0)$ from $\hat{z}$ to $\tilde{z}$.*

Clearly both conditions (NF) and (ND) are always satisfied if $n = 1$ or $\Lambda = M$. The latter means that M is compact and ϕ_t is an Anosov flow on M.

The following result was proved very recently by further developing Dolgopyat's method in [15].

Theorem 4.4 ([58]). *Let $\phi_t : M \longrightarrow M$ be a contact flow on a C^2 complete Riemann manifold and let Λ be a basic set for ϕ_t. Assume that Λ satisfies the conditions (P), (NF) and (ND). If U is defined by means of a Markov family $\mathcal{R}$, then the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and any potential f on $\widehat{U}$ are eventually contracting.*

In fact the main result in [58] concerns more general Axiom A flows on basic sets, however for the sake of simplicity here we state it only in the case of contact flows.

As a consequence one obtains again that the properties (a) and (b) in 4.1 hold for $\phi_t : \Lambda \longrightarrow \Lambda$ as in Theorem 4.4.

Notice that, as mentioned above, when $n = 1$ the conditions (P), (NF) and (ND) are always satisfied, so one gets the following immediate consequence.

Corollary 4.5. *Let $\phi_t : M \longrightarrow M$ be a contact flow on a C^2 complete Riemann manifold with $\dim(M) = 3$, and let Λ be a basic set for ϕ_t. If U is defined by means of a Markov family $\mathcal{R}$, then the Ruelle transfer operators related to $\phi_t : \Lambda \longrightarrow \Lambda$ and any potential f on $\widehat{U}$ are eventually contracting.*

5. Analytic continuation of the resolvent of the Laplacian in the exterior of several convex bodies. Let K be a subset of $\mathbb{R}^N (N \geq 2)$ of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where K_i are compact strictly convex disjoint domains in $\mathbb{R}^N$ with smooth (C^∞) *boundaries* ∂K_i and $k_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^N \setminus K}$. We assume again that K satisfies the no-eclipse condition stated in Subsection 4.2. Moreover we assume that the billiard flow ϕ_t and the non-wandering set Λ satisfies the conditions (P), (NF) and (ND). Thus, the conclusions of Theorem 4.4 above and therefore the properties (a) and (b) in 4.1 hold for $\phi_t : \Lambda \longrightarrow \Lambda$.

Given a periodic reflecting ray $\gamma \subset \Omega$ with m_γ reflections, denote by d_γ the period (return time) of γ, by T_γ the primitive period (length) of γ, by P_γ the linear Poincaré map associated to γ, and by $\lambda_{\gamma,i}$ $(i = 1, \dots, N-1)$ the eigenvalues of P_γ with $|\lambda_{\gamma,i}| > 1$. Let $\mathcal{P}$ be the set of all primitive periodic rays. For $\gamma \in \mathcal{P}$ set

$$\delta_\gamma = -\frac{1}{2}\log(\lambda_{1,\gamma} \dots \lambda_{N-1,\gamma}),$$

where $r_\gamma = 0$ if m_γ is even and $r_\gamma = 1$ if m_γ is odd. Consider the *(scattering) zeta function*

$$Z(s) = \sum_{m}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}.$$

It is easy to show that there exists an abscissa of absolute convergence for $Z(s)$, i.e. a number $s_0 \in \mathbb{R}$ such that for $\mathrm{Re}(s) > s_0$ the series $Z(s)$ is absolutely convergent and s_0 is minimal with this property. On the other hand, using symbolic dynamics and the results of [37], we deduce that $Z(s)$ is meromorphic for $\mathrm{Re}(s) > s_0 - a$, $a > 0$. Moreover, property (a) in 4.1 implies there exists $0 < \epsilon < a$ so that the zeta function $Z(s)$ admits an analytic continuation for $\mathrm{Re}(s) \geq s_0 - \epsilon$. We refer the reader to [39] and [40] for various other interesting results about the zeta function $Z(s)$.

For $\Im(z) < 0$ consider the cut-off resolvent

$$R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi : \; L^2(\Omega) \longrightarrow L^2(\Omega),$$

where $\chi \in C_0^\infty(\mathbb{R}^N)$, $\chi = 1$ on K and Δ_D is the Dirichlet Laplacian in $\Omega = \mathbb{R}^N \backslash K$. The cut-off resolvent $R_\chi(z)$ has a meromorphic continuation in $\mathbf{C}$ for N odd with poles z_j such that $\Im(z_j) > 0$ and in $\mathbf{C} \backslash \{\mathbf{i}\,\mathbb{R}^+\}$ for n even. The analytic properties and the estimates of $R_\chi(\lambda)$ play a crucial role in many problems related to the local energy decay, distribution of the resonances etc. In the physical literature and in many works concerning numerical calculation of the resonances it is often conjectured that the poles μ_j (with $\mathrm{Re}(\mu_j) < 0$) of $Z(s)$ and the poles $(-\mathbf{i}\,\mu_j)$ of $R_\chi(\lambda)$ are closely related.

The above conjecture is true for convex co-compact hyperbolic manifolds $X = \Gamma \setminus \mathbb{H}^{n+1}$, where Γ is a discrete group of isometries with only hyperbolic elements admitting a finite fundamental domain. More precisely, the zeros of the corresponding Selberg's zeta function coincide with the poles (resonances) of the Laplacian Δ_g on X ([38]).

In the case of two strictly convex disjoint domains it was proved by M. Ikawa (1982) and C. Gérard (1988) that the poles of $R_\chi(\lambda)$ are in small neighborhoods of the pseudo-poles

$$m\frac{\pi}{d} + \mathbf{i}\,\alpha_k, \; m \in \mathbb{Z}, k \in N.$$

Here $d > 0$ is the distance between the obstacles and $\alpha_k > 0$ are determined by the eigenvalues λ_j of the Poincaré map related to the unique primitive periodic ray.

The case of several (more than two) convex obstacles is generally speaking much more complicated, however the case $s_0 > 0$ is much easier, since it is known that for $-\mathbf{i}\, s_0 \leq \Im(z) \leq 0$ the cut-off resolvent $R_\chi(z)$ is analytic.

Next, **assume that** $s_0 < 0$. In this case Ikawa established that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ so that the cut-off resolvent $R_\chi(z)$ is analytic for

$$\Im(z) < -\mathbf{i}\,(s_0 + \epsilon) \quad , \quad |\mathrm{Re}(s)| \geq C_\epsilon .$$

The proof of this is based on a construction of an asymptotic solution $U_M(x, s; k)$ given by a finite sum of series having the form

$$\sum_{n=0}^{\infty} \sum_{|i|=n+2, i_{n+2}=j} \sum_{q=0}^{M} e^{-s\varphi_i(x)} \sum_{\nu=0}^{2q} \Big(a_{i,q,\nu}(x, s; k)(s + \mathbf{i}\, k)^\nu \Big) (\mathbf{i}\, k)^{-q}, \tag{5.1}$$

where $i = (i_0, \ldots, i_m)$ are configurations (itineraries, corresponding to a given sequence of reflections at connected components of K), $\varphi_i(x)$ are phase functions and the amplitudes $a_{i,q,\nu}(x, s; k)$ depend on $s \in \mathbf{C}$ and $k \in \mathbb{R}$.

It is an interesting problem to examine the link between the existence of an analytic continuation of $R_\chi(z)$ for $\Im(z) \geq -\mathbf{i}\, s_0$ and an analytic continuation of the dynamical zeta function $Z(s)$ beyond its line of absolute convergence. In [26] Ikawa announced a result in this direction under some rather stringent conditions about K. It appears no proof of this result has ever been published.

The following rather more general result was established very recently.

Theorem 5.1 ([41]). *Let $s_0 < 0$. There exists $\sigma_2 < s_0$ such that $Z(s)$ is analytic for* $\mathrm{Re}(s) > \sigma_2$ *and the cut-off resolvent $R_\chi(z)$ is analytic in the domain*

$$\Im(z) < -\mathbf{i}\,\sigma_2, \quad |\mathrm{Re}(s)| \geq C.$$

In the proof of the above, as in [24], the idea is to construct an asymptotic solution $U_M(x, s; k)$ which is analytic for $\sigma_2 \leq \mathrm{Re}(s) \leq s_0$, $|s + \mathbf{i}\, k| \leq 1$, $k \geq 1$. To achieve this one constructs approximations of the terms after the first sum in (5.1). These terms are compared with others involving powers of Ruelle transfer operators L^n_{-sf+g}, where f and g are fixed functions determined by K, and here Theorem 4.4 above plays a crucial role. The whole procedure is rather long and technical and we refer the reader to [41] for details.

6. Sketch of the proof of Theorem 4.4. In this section we sketch the proof of Theorem 4.4 above. For details the reader is referred to [58].

Let $\phi_t : M \longrightarrow M$ be a contact flow on a complete (not necessarily compact) Riemann manifold of dimension $\dim(M) = 2n+1$ and let Λ be a basic set of ϕ_t. Throughout we assume that ϕ_t and Λ satisfy the pinching condition (P) from Sect. 4.

6.1. Linearization and ball size comparison. With the condition (P) it turns out that the flow ϕ_t on unstable (or stable) manifolds is conjugate to its linearization $d\phi_t$ near the set Λ.

Take $\epsilon_0 > 0$ such that for any $x \in \Lambda$, the *unstable exponential map*

$$\exp^u_x : E^u(x;\epsilon_0) \longrightarrow \exp^u_x(E^u(x;\epsilon_0)) \subset W^u_{\epsilon_0}(x)$$

is a diffeomorphism. For any $x \in \Lambda$ and any $t \in \mathbb{R}$ the map

$$\hat{\phi}^t_x = (\exp^u_{\phi_t(x)})^{-1} \circ \phi_t \circ \exp^u_x : E^u(x) \longrightarrow E^u(\phi_t(x))$$

is well-defined and smooth locally near 0.

The following proposition shows how to define a family of local C^1 diffeomorphisms $F_x : E^u(x) \longrightarrow E^u(x)$ which conjugate $\hat{\phi}^t_x$ and $d\hat{\phi}^t_x(0)$ near 0. In a similar way one can construct local conjugacies on stable manifolds.

Proposition 6.1. *There exist constants $\rho \in (0,1)$, $C > 0$ and $\hat{\epsilon} \in (0, \epsilon_0/2)$ such that:*

(a) *For every $x \in \Lambda$ and every $u \in E^u(x;\hat{\epsilon})$ there exists*

$$F_x(u) = \lim_{t\to\infty} d\hat{\phi}^t_{\phi_{-t}(x)}(0) \cdot \hat{\phi}^{-t}_x(u) \in E^u(x;2\hat{\epsilon}).$$

Moreover, $\|F_x(u) - u\| \le C\,\|u\|^2$ and $\|F_x(u) - d\hat{\phi}^t_{\phi_{-t}(x)}(0)\cdot\hat{\phi}^{-t}_x(u)\| \le C\,\rho^t\,\|u\|^2$ for any $u \in E^u(x,\hat{\epsilon})$ and any $t \ge 0$.

(b) *The maps $F_x : E^u(x;\hat{\epsilon}) \longrightarrow F_x(E^u(x;\hat{\epsilon})) \subset E^u(x;2\hat{\epsilon})$ ($x \in \Lambda$) are C^1 diffeomorphisms with uniformly bounded derivatives.*

(c) *For any $x \in \Lambda$ and any integer $t \ge 0$, setting $x_t = \phi_{-t}(x)$, we have*

$$d\hat{\phi}^t_{x_t}(0) \circ F_{x_t}(v) = F_x \circ \hat{\phi}^t_{x_t}(v)$$

for any $v \in E^u(x_t;\hat{\epsilon})$ with $\|\hat{\phi}^t_{x_t}(v)\| \le \hat{\epsilon}$.

The above proposition will now be used to compare diameters of sets of the form $\Lambda \cap B^u_T(x,\epsilon)$, where

$$B^u_T(x,\epsilon) = \{y \in W^u_{\epsilon_0}(x) : d(\phi_t(x),\phi_t(y)) \le \epsilon,\ 0 \le t \le T\},$$

$x \in \Lambda$, $T > 0$ and $\epsilon \in (0, \epsilon_0]$. We should stress that comparing diameters of sets of the form $B^u_T(x, \epsilon)$ is generally speaking much easier. When intersections with Λ are involved the problem becomes much more difficult, since in general near a point $x \in \Lambda$, the set Λ 'might look in a rather different way in different magnifications'.

We denote by $\mathrm{diam}(A)$ the *diameter* of a subset A of M with respect to the distance d on M induced by the Riemann metric.

Definition 6.2. *We will say that the basic set Λ of ϕ_t has a regular distortion along unstable manifolds if there exists a constant $\hat{\epsilon}_0 > 0$ with the following properties:*

(a) *For any $0 < \delta, \epsilon \leq \hat{\epsilon}_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that*

$$\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R\, \mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) *For any $\epsilon \in (0, \hat{\epsilon}_0]$ and any $\rho \in (0, 1)$ there exists $\delta_0 = \delta_0(\epsilon, \rho) \in (0, \epsilon]$ such that for any $0 < \delta \leq \delta_0$, any $z \in \Lambda$ and any $T > 0$ we have*

$$\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho\, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)).$$

One would notice that the property (a) in the above definition has some analogy with one of the volume lemmas in [7].

Using the linearization described in Proposition 6.1 above, one derives the following.

Proposition 6.3. *Assume that ϕ_t and Λ satisfy the conditions* (P) *and* (NF). *Then Λ has a regular distortion along unstable manifolds.*

6.2. Symbolic coding and diameters of cylinders. Form now on we will assume that $\mathcal{R} = \{R_j\}_{j=1}^k$ is a fixed Markov family for $\phi_t : \Lambda \longrightarrow \Lambda$ of a sufficiently small size. Define $U = \cup_{j=1}^k U_i$, τ and $\sigma : U \longrightarrow U$ as in Sect. 2, and let again $\widehat{U}$ be the *core* of U.

Assume that $f \in C^{\mathrm{Lip}}(\widehat{U})$ is a **fixed function** and set $g = f - P\,\tau$, where $P \in \mathbb{R}$ is such that $\Pr_\sigma(g) = 0$. Denote by ν the *Gibbs measure* on U determined by g.

Let $A = (A_{ij})_{i,j=1}^k$ be the matrix defined in Sect. 2. Given a finite string $\imath = (i_0, i_1, \ldots, i_m)$ of integers $i_j \in \{1, \ldots, k\}$, we will say that $\imath$ is *admissible* if for any $j = 0, 1, \ldots, m-1$ we have $A_{i_j i_{j+1}} = 1$. Given an admissible string $\imath$, denote by $\overset{\circ}{C}[\imath]$ the set of those $x \in U$ so that $\sigma^j(x) \in \mathrm{Int}_\Lambda(U_{i_j})$ for all $j = 0, 1, \ldots, m$. The set

$$C[\imath] = \overline{\overset{\circ}{C}[\imath]} \subset \Lambda$$

will be called a *cylinder* of length m in U, while $\overset{\circ}{C}[\imath]$ will be called an *open cylinder* of length m. It follows from the properties of the Markov family that $\overset{\circ}{C}[\imath]$ is an open dense subset of $C[\imath]$ and $\nu(C[\imath] \setminus \overset{\circ}{C}[\imath]) = 0$ ([6]). Any cylinder of the form $C[i_0, i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+q}]$ will be called a *subcylinder* of $C[\imath]$ of *co-length* q.

It follows from the properties of Gibbs measures (cf. [54], [49] or [37]) and $\mathrm{Pr}_\sigma(g) = 0$ that there exist constants $c_2 > c_1 > 0$ such that

$$c_1 \leq \frac{\nu(C[\imath])}{e^{g_m(y)}} \leq c_2$$

for any cylinder $C[\imath]$ of length m in U and any $y \in C[\imath]$.

Given $x \in U_i$ for some i and $r > 0$ we will denote by $B_U(x, r)$ the set of all $y \in U_i$ with $d(x, y) < r$.

The following lemma describes the main consequences of Proposition 6.3 that will be needed later on.

Lemma 6.4. *Assume that the basic set Λ of ϕ_t has a regular distortion along unstable manifolds and that the local stable and unstable laminations over Λ are Lipschitz. Then there exist a global constants $0 < \rho < 1$ and a positive integer $p_0 \geq 1$ such that:*

(a) *For any cylinder $C[\imath] = C[i_0, \ldots, i_m]$ and any subcylinder $C[\imath'] = C[i_0, i_1, \ldots, i_{m+1}]$ of $C[\imath]$ of co-length 1 we have*

$$\rho\, \mathrm{diam}(C[\imath]) \leq \mathrm{diam}(C[\imath']).$$

(b) *For any cylinder $C[\imath] = C[i_0, \ldots, i_m]$ and any subcylinder $C[\imath'] = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+p_0}]$ of $C[\imath]$ of co-length p_0 we have*

$$\mathrm{diam}(C[\imath']) \leq \rho\, \mathrm{diam}(C[\imath]).$$

6.3. Consequences of the non-degeneracy of $d\omega$ over Λ. Katok and Burns [27] showed that for contact hyperbolic flows there is a certain relationship between the temporal distance function $\Delta(x, y)$ (see Sect. 2 above) and the two-form $d\omega$. Later Liverani (see Lemma B.7 in [31]) proved a stronger form of the lemma of Katok and Burns. We state it here under the general assumption that the local holonomy maps are Lispchitz, which is a consequence of the pinching condition (P) (see [46]).

Lemma 6.5 ([31]). *Assuming the constant $\epsilon_0 > 0$ is sufficiently small, there exists a constant $C \geq 1$ such that for any $z \in \Lambda$ and any $x \in W^u_{\epsilon_0}(z) \cap \Lambda$ and $y \in W^s_{\epsilon_0}(z) \cap \Lambda$ we have*

$$|\Delta(x, y) - d\omega_z(u, v)| \leq C\, \|u\|^2\, \|v\|^2,$$

where $u \in E^u(z)$ and $v \in E^s(z)$ are such that $\exp^u_z(u) = x$ and $\exp^s_z(v) = y$.

Next, **fix** an arbitrary $z_0 \in \Lambda$, an arbitrary orthonormal basis $u_1, \ldots, u_n$ in $E^u(z_0)$ and a C^1 parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V_0'$, of a small neighbourhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V_0' is a convex compact neighbourhood of 0 in $\mathbb{R}^n \approx \mathrm{span}(u_1, \ldots, u_n) = E^u(z_0)$. Then $r(0) = z_0$ and $\left.\frac{\partial}{\partial s_i} r(s)\right|_{s=0} = u_i$ for all $i = 1, \ldots, n$. Set

$$U_0' = W_0 \cap \Lambda.$$

Fix any constants

$$0 < \theta_0 < \theta_1 < 1.$$

Definitions 6.6. (a) *For a cylinder $\mathcal{C} \subset U_0'$ and a non-zero vector $a = (a_1, \ldots, a_n)$ in the parameter space $\mathbb{R}^n = E^u(z_0)$ we will say that a separation by an a-plane occurs in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u, v) \geq \frac{1}{2}\,\mathrm{diam}(\mathcal{C})$ such that*

$$\left|\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|},\, a \right\rangle\right| \geq \theta_1.$$

Let $\mathcal{S}_a$ be the family of all cylinders $\mathcal{C}$ in U_0' such that a separation by an a-plane occurs in $\mathcal{C}$.

(b) *A subset V of U will be called regular if there exist finitely many cylinders $D_1, \ldots, D_p$ in U with $V \subset \cup_{j=1}^p D_j$ and $\nu(\cup_{j=1}^p D_j \setminus V) = 0$. (Then clearly $\overline{V} = \cup_{j=1}^p D_j$.)*

(c) *Given a regular subset V of U_0' and $\delta > 0$, let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ ($p = p(\delta) \geq 1$) be the family of maximal cylinders in $\overline{V}$ with $|\mathcal{C}_j| \leq \delta$ such that $\overline{V} = \cup_{j=1}^p \mathcal{C}_j$. Set*

$$M_a^{(\delta)}(V) = \cup\{\mathcal{C}_j : \mathcal{C}_j \in \mathcal{S}_a\,,\ 1 \leq j \leq p\}.$$

In what follows we will construct, amongst other things, a sequence of unit vectors $b_1, b_2, \ldots, b_{j_0} \in \mathbb{R}^n$. For each $\ell = 1, \ldots, j_0$ set

$$B_\ell = \{a \in \mathbb{S}^{n-1} : |\langle a, b_\ell \rangle| \geq \theta_0\}.$$

Below we use the notation

$$I_{v,t} g(s) = \frac{g(s + t\, v) - g(s)}{t}, \quad t \neq 0.$$

The following technical lemma provides the main geometric tools used in the next subsection.

Lemma 6.7. *Assume that the basic set Λ of ϕ_t has a regular distortion along unstable manifolds, the local stable and unstable laminations over Λ are Lipschitz, and Λ satisfies the condition (ND). Then there exist vectors $b_1, \ldots, b_{j_0} \in \mathbb{S}^{n-1}$, an open subset U_0 of U_0' which is a finite union of cyinders, a point $\zeta \in U_0 \cap \widehat{U}$ and integers $N_0 > n_1 \geq 1$ such that the following hold:*

(i) *$\mathcal{U} = \sigma^{n_1}(U_0)$ is an open dense subset of U and $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is a homeomorphism.*

(ii) *For any regular open neighbourhood V of ζ in U_0 there exist a constant $\delta' = \delta'(V) \in (0, \delta_0)$ such that*

$$M_{b_1}^{(\delta)}(V) \cup M_{b_2}^{(\delta)}(V) \cup \ldots \cup M_{b_{j_0}}^{(\delta)}(V) \supset V, \quad \delta \in (0, \delta'].$$

(iii) *For any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(j)}, v_2^{(j)} : \mathcal{U} \longrightarrow U$ ($j = 1, \ldots, j_0$) such that $\sigma^N(v_i^{(j)}(x)) = x$ for all $x \in \mathcal{U}$, $v_i^{(j)}(\mathcal{U})$ is a finite union of open cylinders of length N, and*

$$I_{a,h}[\tau_N(v_2^{(j)}(\tilde{r}(s))) - \tau_N(v_1^{(j)}(\tilde{r}(s)))] \geq \frac{\hat{\delta}}{2}$$

for all $j = 1, \ldots, j_0$, $s \in V_0$, $0 < |h| \leq \hat{\delta}$ and $a \in B_j$ such that s and $s + h\, a$ are in $r^{-1}(U_0 \cap \Lambda) \subset V_0$.

Fix vectors $b_1, \ldots, b_{j_0} \in \mathbb{S}^{n-1}$, an open subset U_0 of U_0' which is a finite union of cylinders, $\zeta \in U_0 \cap \widehat{U}$, and integers $N_0 > n_1 \geq 1$ with the properties described in the above lemma.

Consider the inverse homeomorphism

$$\psi : \mathcal{U} \longrightarrow U_0 \text{ such that } \sigma^{n_1}(\psi(x)) = x, \ x \in \mathcal{U}.$$

6.4. Contracting operators. Throughout we assume that the basic set Λ of ϕ_t has a regular distortion along unstable manifolds, the local stable and unstable laminations over Λ are Lipschitz, and Λ satisfies the condition (ND). (Notice that these assumptions follows from these in Theorem 4.4.)

Given a real number a (with $|a|$ small), denote by λ_a the *largest eigenvalue* of $L_{f-(P+a)\tau}$ and by $h_a \in C^{\mathrm{Lip}}(U)$ the corresponding (positive) eigenfunction such that $\sup_{u\in U} h_a(u) = 1$. Since $\Pr(f - P\tau) = 0$, it follows from the main properties of pressure (cf. e.g. Ch. 3 in [37]) that $|\Pr(f-(P+a)\tau)| \leq |\tau|_\infty |a|$. Moreover, for small $|a|$ the maximal eigenvalues λ_a and the eigenfunctions h_a depend analytically on a. In particular, there exist constants $a_0 > 0$ and $C_0 > 0$ such that $h_a \geq 1 - C_0|a|$ on U for $|a| \leq a_0$.

For $|a| \leq a_0$, as in [15], consider the function

$$f^{(a)}(u) = f(u) - (P+a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a$$

and the operators

$$L_{ab} = L_{f^{(a)} - \mathbf{i}\, b\tau} : C^{\mathrm{Lip}}(U) \longrightarrow C^{\mathrm{Lip}}(U), \quad \mathcal{M}_a = L_{f^{(a)}} : C^{\mathrm{Lip}}(U) \longrightarrow C^{\mathrm{Lip}}(U).$$

One checks that $\mathcal{M}_a\ 1 = 1$ and $|(L_{ab}^m h)(u)| \leq (\mathcal{M}_a^m |h|)(u)$ for all $u \in U$, $h \in C^{\mathrm{Lip}}(U)$ and $m \geq 0$.

As in [15] (see also Corollary 3.3. in [56]), the main ingredient needed to prove Theorem 4.4 is the following integral estimate.

Theorem 6.8. *There exist a positive integer N and constants $\hat{\rho}_1 \in (0,1)$ and $a_0 > 0$ such that for any $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $|b| \geq 1/a_0$ and every $h \in C^{\mathrm{Lip}}(U)$ with $\|h\|_{\mathrm{Lip},b} \leq 1$ we have*

$$\int_U |L_{ab}^{Nm} h|^2 \, d\nu \leq \hat{\rho}_1^m$$

for every positive integer m, where ν is the Gibbs measure determined by $f - P\tau$ on U.

Set $\widehat{U}_i = U_i \cap \widehat{U}$ for any $i = 1, \ldots, k$. Define a *new metric* D on $\widehat{U}$ by

$$D(x,y) = \min\{\mathrm{diam}(\mathcal{C}) : x, y \in \mathcal{C},\ \mathcal{C} \text{ a cylinder in } U_i\}$$

if $x, y \in U_i$ for some $i = 1, \ldots, k$, and $D(x,y) = 1$ otherwise. We assume that the Markov family is chosen so that $\mathrm{diam}(U_i) < 1$ for all i. Denote by $C_D^{\mathrm{Lip}}(\widehat{U})$ the space of all functions $h : \widehat{U} \longrightarrow \mathbf{C}$ which a Lipschitz with respect to the metric D on $\widehat{U}$ and by $\mathrm{Lip}_D(h)$ the Lipschitz constant of h with respect to D.

Next, given $A > 0$, denote by $K_A(\widehat{U})$ *the set of all functions* $h \in C_D^{\mathrm{Lip}}(\widehat{U})$ such that $h > 0$ and

$$\frac{|h(u) - h(u')|}{h(u')} \leq A\, D(u, u')$$

for all $u, u' \in \widehat{U}$ that belong to the same $\widehat{U}_i$ for some $i = 1, \dots, k$. Notice that $h \in K_A(\widehat{U})$ implies $|\ln h(u) - \ln h(v)| \leq A\ D(u, v)$ and therefore

$$e^{-A\ D(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A\ D(u,v)}, \quad u, v \in \widehat{U}_i;\ i = 1, \dots, k.$$

As in [15], Theorem 6.8 is easily derived from the following lemma which is similar to Lemma $10''$ in [15].

Lemma 6.9. *There exist a positive integer* N *and constants* $\hat{\rho} = \hat{\rho}(N) \in (0, 1)$, $a_0 = a_0(N) > 0$ *and* $E \geq 1$ *such that for every* $a, b \in \mathbb{R}$ *with* $|a| \leq a_0$, $1/|b| \leq a_0$, *there exists a finite family* $\{\mathcal{N}_J\}_{J \in \mathsf{J}}$ *of operators* $\mathcal{N}_J = \mathcal{N}_J(a, b) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$, *where* $\mathsf{J} = \mathsf{J}(\mathsf{a}, \mathsf{b})$ *is a finite set depending on* a *and* b, *with the following properties:*

(a) *The operators* $\mathcal{N}_J$ *preserve the cone* $K_{E|b|}(\widehat{U})$*;*

(b) *For all* $H \in K_{E|b|}(\widehat{U})$ *and* $J \in \mathsf{J}$ *we have* $\displaystyle\int_{\widehat{U}} (\mathcal{N}_J H)^2\, d\nu \leq \hat{\rho} \int_{\widehat{U}} H^2\, d\nu.$

(c) *If* $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ *are such that* $H \in K_{E|b|}(\widehat{U})$, $|h(u)| \leq H(u)$ *for all* $u \in \widehat{U}$ *and* $|h(u) - h(u')| \leq E|b| H(u')\, D(u, u')$ *whenever* $u, u' \in \widehat{U}_i$ *for some* $i = 1, \dots, k$, *then there exists* $J \in \mathsf{J}$ *such that* $|L_{ab}^N h(u)| \leq (\mathcal{N}_J H)(u)$ *for all* $u \in \widehat{U}$ *and*

$$|(L_{ab}^N h)(u) - (L_{ab}^N h)(u')| \leq E|b| (\mathcal{N}_J H)(u')\, D(u, u')$$

whenever $u, u' \in \widehat{U}_i$ *for some* $i = 1, \dots, k$.

In the remaining part of this sub-section we sketch the proof of Lemma 6.9.

We will use the objects constructed in the previous subsection, notably $\hat{\delta} > 0$, the integers $N_0 > n_1 \geq 1$, the sets $U_0 \subset U_1$ and $\mathcal{U} = \sigma^{n_1}(U_0) = \mathrm{Int}_\Lambda(U)$.

Choose a sufficiently large constant $E > 0$ and **fix an integer** $N > N_0$. Then choose the maps $v_i^{(j)}$ so that the conclusion of Lemma 6.7(iii) holds and a small constant $\epsilon^{(0)} > 0$.

It follows from Lemma 6.7(ii) with $V = U_0$ that there exist a constant $\kappa_0 = \kappa_0(U_0) \in (0, \delta_0)$ such that

$$M_{b_1}^{(\delta)}(U_0) \cup \ldots \cup M_{b_{j_0}}^{(\delta)}(U_0) \supset U_0 \quad , \quad \delta \in (0, \kappa_0],$$

Fix κ_0 with the above properties. Let $\epsilon_1 > 0$ and $b \in \mathbb{R}$ be such that $|b| \geq 1$,

$$\frac{\epsilon^{(0)}}{2} \leq \epsilon_1 < \epsilon^{(0)}, \tag{6.1}$$

and

$$\frac{\epsilon_1}{|b|} \in (0, \kappa_0]. \tag{6.2}$$

In what follows most of the time ϵ_1 and b will stay fixed, however at the end of the section we will vary them so that (6.1) and (6.2) hold.

Let $\mathcal{C}_j = \mathcal{C}_j^{(\epsilon_1/|b|)}$ $(1 \leq j \leq p)$ be the fixed family of *maximal cylinders* in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_j) \leq \frac{\epsilon_1}{|b|}$ such that $U_0 \subset \cup_{j=1}^p \mathcal{C}_j$ and $\overline{U_0} = \cup_{j=1}^p \mathcal{C}_j$ (see Definitions 6.2). Then $\nu(\cup_{j=1}^p \mathcal{C}_j \setminus U_0) = 0$. Moreover, Proposition 6.4(a) implies that $\mathrm{diam}(\mathcal{C}_j) \geq \rho \frac{\epsilon_1}{|b|}$ for all j, so

$$\rho \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{C}_j) \leq \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq p.$$

Fix an integer $q_0 \geq 0$ so large that

$$\theta_0 < \theta_1 - 32\,\rho^{q_0 - 1},$$

and let $p_0 \geq 1$ be the constant from Lemma 6.4(b).

Next, let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all cylinders in $\overline{U_0}$ that are subcylinders of *co-length* $p_0\, q_0$ of some $\mathcal{C}_j$ $(1 \leq j \leq p)$. That is, if k_j is the length of $\mathcal{C}_j$, we consider the subcylinders of length $k_j + p_0\, q_0$ of $\mathcal{C}_j$, and we do this for any $j = 1, \ldots, p$. Then

$$\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q.$$

Moreover, it follows from the properties of $\mathcal{C}_j$ and Lemma 6.4 that

$$\rho^{p_0\, q_0 + 1} \cdot \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq j \leq q. \tag{6.3}$$

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, j_0$ and $i = 1, 2$, set

$$Z_j = \sigma^{n_1}(\mathcal{D}_j) \quad , \quad X_{i,j}^{(\ell)} = \{v_i^{(\ell)}(u) : u \in Z_j\},$$

$\hat{Z}_j = Z_j \cap \widehat{U}$, $\widehat{\mathcal{D}}_j = \mathcal{D}_j \cap \widehat{U}$ and $\widehat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \widehat{U}$. It then follows that $\mathcal{D}_j = \psi(Z_j)$, and $U = \cup_{j=1}^{q} Z_j$.

$$\begin{array}{ccccc}
\mathcal{U} & \overset{\psi}{\underset{\sigma^{n_1}}{\rightleftarrows}} & U_0 & \overset{v_i^{(\ell)}}{\underset{\sigma^{N-n_1}}{\rightleftarrows}} & U_1 \\
\bigcup & & \bigcup & & \bigcup \\
Z_j = \sigma^{n_1}(\mathcal{D}_j) & \xrightarrow{\ \psi\ } & \mathcal{D}_j & \xrightarrow{\ v_i^{(\ell)}\ } & X_{i,j}^{(\ell)}
\end{array}$$

Let J be a *subset of the set*

$$\Xi = \{\ (i, j, \ell) : 1 \le i \le 2,\ 1 \le j \le q\ ,\ 1 \le \ell \le j_0\ \}.$$

Fix for a moment a sufficiently small $\mu = \mu(N, \epsilon_1) > 0$ and define $\beta = \beta_{\mu,b,J} : \widehat{U} \longrightarrow [0, 1]$ by

$$\beta = 1 - \mu \sum_{(i,j,\ell) \in J} \eta_{i,j}^{(\ell)}.$$

Then $\beta \in C_D^{\mathrm{Lip}}(\widehat{U})$ and one derives using (6.3) that

$$\mathrm{Lip}_D(\beta) \le \Gamma_1(\mu, N)\, \frac{|b|}{\epsilon_1}$$

for some constant $\Gamma_1(\mu, N)$.

Next, given J, define *Dolgopyat's operator* $\mathcal{N} = \mathcal{N}_J : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$ by

$$(\mathcal{N}h)(u) = \left(\mathcal{M}_a^N(\beta \cdot h)\right)(u),\ \ u \in \widehat{U}.$$

Parts (a) and (c) in the definitions below are analogues of corresponding notions in [15]; part (b) is related to the fact that we consider the flow over a basic set Λ which in general is a proper subset of the manifold M.

Definition 6.10. (a) *Given $t > 0$ and $m > 0$, a subset W of $\widehat{U}$ will be called (t, m)-dense in $\widehat{U}$ if for every $u \in \widehat{U}$ there exist a cylinder $\mathcal{C}$ containing u with* $\mathrm{diam}(\mathcal{C}) \le mt$ *and a cylinder $\mathcal{C}'$ with* $\mathrm{diam}(\mathcal{C}') \ge t$ *such that $\hat{\mathcal{C}}' \subset W \cap \mathcal{C}$.*

(b) *We will say that the cylinders $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are* adjacent *if they are subcylinders of the same $\mathcal{C}_i$. If $\mathcal{D}_j, \mathcal{D}_{j'} \subset \mathcal{C}_i$ and for some $\ell = 1, \ldots, j_0$ there exist $u \in \mathcal{D}_j$ and $v \in \mathcal{D}_{j'}$ such that $d(u, v) \ge \frac{1}{2}$* $\mathrm{diam}(\mathcal{C}_i)$ *and*

$$\left| \left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|},\ b_\ell \right\rangle \right| \ge \theta_1$$

we will say that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are b_ℓ-separable in $\mathcal{C}_i$.

(c) *A subset J of Ξ will be called dense if for any $j = 1, \ldots, q$ there exists $(i, j', \ell) \in J$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent. Denote by J the set of all dense subsets J of Ξ.*

In what follows we assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are such that

$$H \in K_{E|b|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u), \quad u \in \widehat{U}, \tag{6.4}$$

and

$$|h(u) - h(u')| \leq E|b| H(u')\, D(u, u') \quad \text{whenever } u, u' \in \widehat{U}_i,\ i = 1, \ldots, k. \tag{6.5}$$

Given $\mu \in (0, 1/2)$, define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbf{C}$ ($\ell = 1, \ldots, j_0$, $i = 1, 2$) by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1 - \mu) e^{f_N^{(a)}(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_N^{(a)}(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{f_N^{(a)}(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1 - \mu) e^{f_N^{(a)}(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))}.$$

For any $\ell = 1, \ldots, j_0$ consider the function

$$\gamma_\ell(u) = b\, [\tau_N(v_1^{(\ell)}(u)) - \tau_N(v_2^{(\ell)}(u))], \quad u \in \widehat{U}.$$

The proof of the following lemma incorporates the main differences between the case of a flow over a basic set Λ and an Anosov flow (over a smooth compact manifold).

Lemma 6.11. *There exists a constant $c_2 > 0$ such that if $j, j' \in \{1, 2, \ldots, q\}$ are such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are b_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \ldots, p$ and $\ell = 1, \ldots, j_0$, then*

$$|\gamma_\ell(u) - \gamma_\ell(u')| \geq c_2 \epsilon_1 \quad , \quad u \in \hat{Z}_j,\ u \in \hat{Z}_{j'}.$$

From this, by and large as in [15], one derives the following.

Lemma 6.12. *Assume ϵ_1 and b are chosen in such a way that* (6.4) *and* (6.5) *hold. Then for any $j = 1, \ldots, q$ there exist $i = 1, 2$, $j' = 1, \ldots, q$ and $\ell = 1, \ldots, j_0$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \hat{Z}_{j'}$.*

Sketch of Proof of Lemma 6.9. Fix ϵ_1 with (6.1), and assume that $N > N_0$ and $\mu = \mu(N, \epsilon_1) > 0$ is sufficiently small. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ be such that $|a| \leq \epsilon^{(0)}$ and $|b| \geq \epsilon^{(0)}/\kappa_0$, and assume that b satisfies (6.2). Let $\mathsf{J} = \mathsf{J}(\mathsf{a}, \mathsf{b})$ be the set of all dense subsets of $\Xi = \Xi(a, b)$.

Define the subset $J = J(b)$ of Ξ in the following way. First, include in J all $(1, j, \ell) \in \Xi$ such that $\chi_\ell^{(1)}(u) \leq 1$ for all $u \in \hat{Z}_j$. Then for any $j = 1, \dots, q$ and $\ell = 1, \dots, j_0$ include $(2, j, \ell)$ in J if and only if $(1, j, \ell)$ has not been included in J (that is, $\chi_\ell^{(1)}(u) > 1$ for some $u \in \hat{Z}_j$) and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \hat{Z}_j$. It follows from Lemma 6.12 that J is dense, i.e. $J \in \mathsf{J}$. Consider the operator $\mathcal{N} = \mathcal{N}_J : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$. Properties (a) and (b) in Lemma 6.9 are easier to derive, so we are not going to discuss these here.

To check (c) in Lemma 6.9, assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ satisfy (6.4) and (6.5). Then one derives that $|(L_{ab}^N h)(u) - (L_{ab}^N h)(u')| \leq E|b|(\mathcal{N} H)(u')\, D(u, u')$ whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \dots, k$.

So, it remains to show that

$$\left|(L_{ab}^N h)(u)\right| \leq (\mathcal{N}H)(u), \;\; u \in \widehat{U}. \tag{6.6}$$

Let $u \in \widehat{U}$. If $u \notin \hat{Z}_j$ for any $(i, j, \ell) \in J$, then $\beta(v) = 1$ whenever $\sigma^N v = u$ (since $v \in X_{i,j}^{(\ell)}$ implies $u = \sigma^N v \in Z_j$). Hence

$$\left|(L_{ab}^N h)(u)\right| = \left|\sum_{\sigma^N v = u} e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v)} h(v)\right| \leq (\mathcal{M}_a^N(\beta H))(u) = (\mathcal{N}H)(u).$$

Assume that $u \in \hat{Z}_j$ for some $(i, j, \ell) \in J$. We will consider the case $i = 1$; the case $i = 2$ is similar. (Notice that by the definition of J, we cannot have both $(1, j, \ell)$ and $(2, j, \ell)$ in J.) Then $\chi_\ell^{(1)}(u) \leq 1$, and therefore

$$\begin{aligned}
\left|(L_{ab}^N h)(u)\right| &\leq \left|\sum_{\sigma^N v = u,\; v \neq v_1^{(\ell)}(u), v_2^{(\ell)}(u)} e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v)} h(v)\right| \\
&\quad + \left|e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_N^{(a)} + \mathbf{i} b \tau_N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))\right| \\
&\leq \sum_{\sigma^N v = u,\; v \neq v_1^{(\ell)}(u), v_2^{(\ell)}(u)} e^{f_N^{(a)}(v)} |h(v)| \\
&\quad + \left[(1 - \mu) e^{f_N^{(a)}(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_N^{(a)}(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))\right] .
\end{aligned}$$

Since $(1, j, \ell) \in J$ and $(2, j, \ell) \notin J$, the definitions of the functions $\eta_{i,j}^{(\ell)}$ and β give $\beta(v_1^{(\ell)}(u)) \geq 1 - \mu$ and $\beta(v_2^{(\ell)}(u)) = 1$. This and (6.4) imply

$$\begin{aligned} \left|(L_{ab}^N h)(u)\right| \leq & \sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{f_N^{(a)}(v)} \beta(v) H(v) \\ & + \left[e^{f_N^{(a)}(v_1(u))} \beta(v_1(u)) H(v_1(u)) + e^{f_N^{(a)}(v_2(u))} \beta(v_2(u)) H(v_2(u)) \right] = (\mathcal{N} H)(u), \end{aligned}$$

which proves (6.6). □

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