Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **34** (2008), 467–482

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ZEROS OF SEQUENCES OF PARTIAL SUMS AND OVERCONVERGENCE

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Communicated by P. Pflug

Abstract. We are concerned with overconvergent power series. The main idea is to relate the distribution of the zeros of subsequences of partial sums and the phenomenon of overconvergence. Sufficient conditions for a power series to be overconvergent in terms of the distribution of the zeros of a subsequence are provided, and results of Jentzsch-Szegö type about the asymptotic distribution of the zeros of overconvergent subsequences are stated.

Introduction and statement of the main results. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

be a power series of final radius of convergence $R(f)$. Throughout the paper will be assumed that $R(f) = 1$. We set

$$S_k(z) = \sum_{n=0}^{k} a_n z^n.$$

2000 *Mathematics Subject Classification:* 30B40, 30B10, 30C15, 31A15.
Key words: Overconvergence, Hadamard-Ostrowski gaps, equilibrium measure.

It is known that the sequence $\{S_n\}_{n=0}^{\infty}$ of all partial sums diverges at every point z_0 outside the unit disk. The classical result of Ostrowski shows that this conclusion may be wrong if we consider sequences $\{S_{n_k}\}$ instead of $\{S_n\}$. If a sequence $\{S_{n_k}\}$ converges inside some domain $\mathcal{U}$ (e.g. uniformly with respect to the Chebyshev norm on compact subsets) which contains the unit disk $\mathcal{D} := \{z, |z| < 1\}$ then $\{S_{n_k}\}$, resp. f is called *overconvergent.* In what follows, we will say that $\{S_{n_k}\}$ is overconvergent in $\mathcal{U}$. In the case of overconvergence, the function f is necessarily analytically continuable beyond $\partial\mathcal{D}$, and moreover, f is analytic in $\mathcal{U}$ (we write $f \in \mathcal{A}(\mathcal{U})$.)

We say further that the function f possesses *Hadamard-Ostrowski gaps* (H.-O. gaps) if there exist sequences $\{p_k\}$ and $\{q_k\}$ such that

a) $q_{k-1} \le p_k < q_k, k = 1, 2, \dots,$

aa) $\liminf_{k\to\infty} q_k/p_k > 1$ and

aaa) $\limsup_l |a_l|^{1/l} < 1,$ if $l \in \bigcup_{k=1}^{\infty}[p_k, q_k]$.

The relation between the overconvergence of some subsequence $\{S_{n_k}\}$ and the presence of H.-O. gaps in (1) was first revealed by A. Ostrowski:

Theorem of Ostrowski [1, 2]**.** *Let* (1) *be a power series with radius of convergence $R(f) = 1$, which is analytically continuable beyond $\mathcal{D}$. Then f is overconvergent iff it possesses H.-O. gaps. In this case, the sequence $\{S_{p_k}\}$ converges inside a domain that contains all regular points of f on the unit circle.*

We draw the attention to the fact that merely the existence of H.-O. gaps does not imply overconvergence. The classical result by Hadamard and the next example convince us of this fact.

Theorem of Hadamard [3]. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be a power series with radius* 1*. Then f is not analytically continuable beyond $\partial\mathcal{D}$, if $k/n_k \to 0$ as $k \to \infty$.*

Consider as an illustration the function $f(z) = \sum\limits_{n=0}^{\infty} z^{2^n}$; it possesses H.-O. gaps. But according to Hadamard's theorem, the unit circle is the natural boundary of analycity.

Set $\mathcal{D}(f)$ for the complete domain where (1) admits a continuation as a single-valued analytic function. Given a sequence of partial sums $\{S_{n_k}\}$, we denote by $\mathcal{D}(\{S_{n_k}\})$ the greatest domain containing the point $z = 0$ where the sequence $\{S_{n_k}\}$ converges (uniformly on compact subsets). Apparently, $\mathcal{D}(f) \supseteq \mathcal{D}(\{S_{n_k}\})$. In 1993 J. Mueller [5] showed that in general $\mathcal{D}(f)$ could be "essentially" larger than $\mathcal{D}(\{S_{n_k}\})$. He constructed a power series f with radius of convergence 1 such that no sequence $\{S_{n_k}\}$ converges locally uniformly inside the domain $\mathcal{D}(f)$.

Our interest is devoted to the behavior of the zeros of overconvergent series. This subject was first posed by A. Ostrowski in 1922 by considering maximally convergent polynomial sequences [4]. In [9], sequences of polynomials of best Chebyshev approximation were investigated. Under the basic assumption that the function f is analytically continuable beyond the unit circle, in [6] sufficient conditions for overconvergence were provided. A detailed study on the distribution of zeros of overconvergent sequences of partial sums in Forier series was done in [7]. The purpose of the present paper is to gain a further insight into this subject by establishing new sufficient conditions for overconvergence of a given sequence $\{S_{n_k}\}$ of partial sums in terms of its zero distribution, as well as to obtain results of Jentzsch-Szegö type for the asymptotical distribution of its zeros. In our coming results no previous information about the analytical continuability of the function f will be needed.

Before continuing, we set $\nu_n(K)$, $K-$ a compact set in $\mathbf{C}$, for the number of the zeros of S_n on K.

Now, let $\{S_{n_k}\}$ be a sequence of partial sums which is overconvergent in some domain $\mathcal{U}$. Then by the classical theorem of Hurwitz

$$\limsup \nu_{n_k}(K) < \infty$$

on compact subsets K of $\mathcal{U}$.

We raise the question whether the "asymptotically small" number of the zeros of some sequence $\{S_{n_k}\}$ inside (e.g. on compact subsets) some domain ensures existence of H.-O. gaps and an overconvergence?

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius* 1 *and suppose that there exist a domain $\mathcal{U} \supset \mathcal{D}$ with $\partial\mathcal{U} \bigcap \partial\mathcal{D} \neq \emptyset$ and a sequence $\{S_{n_k}\}$ such that inside $\mathcal{U}$ the condition*

$$\nu_{n_k}(K) = o(n_k),\ k \to \infty \tag{2}$$

holds.

Then the function f possesses H.-O. gaps.

The next result provides a sufficient condition for a sequence $\{S_{n_k}\}$ to be overconvergent in a domain.

Theorem 2. *Let f be a power series as in Theorem* 1 *and suppose that the sequence $\{S_{n_k}\}$ is overconvergent in a domain $\mathcal{U} \supset \mathcal{D}$, $\partial\mathcal{U} \bigcap \partial\mathcal{D} \neq \emptyset$.*

Moreover, suppose that there are a point $z_0 \in \partial\mathcal{U}$ *and a neighborhood* V *of* z_0 *such that*

a) $f \in \mathcal{A}(V)$

and

b) the sequence $\{S_{n_k}\}$ *satisfies condition* (2) *inside* V.

Then the sequence $\{S_{n_k}\}$ *is overconvergent in* $\mathcal{U} \bigcup V$.

The fact that the function f is analytical in an *essentially* larger domain than the domain of overconvergence of the sequence $\{S_{n_k}\}$ plays a crucial role in the proof of Theorem 2. It turns out that if the sequence $\{S_{n_k}\}$ is not *too rare,* then merely condition (2) yields an overconvergence. So, the above assertion about the analytic continuability of f can be omitted. We prove

Theorem 3. *Let* f *be a power series with radius* 1. *Suppose that there exist an infinite sequence* $\{S_{n_k}\}$ *and a domain* $\mathcal{U} \supset \mathcal{D}$ *with* $\partial\mathcal{U} \bigcap \partial\mathcal{D} \neq \emptyset$ *such that condition* (2) *holds inside* $\mathcal{U}$. *Assume further that*

$$\limsup_{n_k \to \infty} \frac{n_{k+1}}{n_k} < \infty. \tag{3}$$

Then the sequence $\{S_{n_k}\}$ *is overconvergent in the domain* $\mathcal{U}$.

Given a sequence $\{S_{n_k}\}$ of partial sums, we denote by $\mathcal{O}(\{S_{n_k}\})$ the largest domain containing the point $z = 0$ such that condition (2) holds inside. By the classical Hurwitz's theorem, $\mathcal{O}(\{S_{n_k}\}) \supseteq \mathcal{D}(\{S_{n_k}\})$. Apparently, $\mathcal{O}(\{S_{n_k}\}) \equiv \mathcal{D}(\{S_{n_k}\})$, provided $\{S_{n_k}\}$ is overconvergent in $\mathcal{O}(\{S_{n_k}\})$. We deduce the following

Corollary 4. *Under the conditions of Theorem* 3, *the domains* $\mathcal{O}(\{S_{n_k}\})$ *and* $\mathcal{D}(\{S_{n_k}\})$ *coincide.*

From Theorem 3 we easily deduce

Corollary 5. *Under the conditions of Theorem* 3, *each point* $z_0 \in \partial\mathcal{D}(\{S_{n_k}\})$ *is a concentration point of zeros of* $\{S_{n_k}\}$ *and the inequality*

$$\lim_{\delta \to 0} \left(\limsup_{n_k} \frac{\nu_{n_k}(\overline{\mathcal{D}}(z_0, \delta))}{n_k} \right) > 0,$$

with $\mathcal{D}(z_0, \delta)$ *being a disk of radius* δ *centered at* z_0, *holds.*

A further consequence of Theorem 3 is

Theorem 6. *Let f be a power series with radius* 1 *and let $\mathcal{U} \supset \mathcal{D}$ be a domain, $\partial\mathcal{U} \bigcap \partial\mathcal{D} \neq \emptyset$. Suppose that there is a sequence $\{S_{n_k}\}$ of partial sums such that condition* (3) *holds, and, in addition,*

$$\limsup \|S_{n_k}\|_K^{1/n_k} \leq 1$$

on compact subsets K of $\mathcal{U}$.

Then $\{S_{n_k}\}$ is overconvergent in $\mathcal{U}$.

We now recall some known facts and definitions. For further particulars and results we refer to [11] and [12]. We say that a compact set E in $\mathbf{C}$ is regular, if its complement $E^c := \overline{\mathbf{C}} \setminus E$ is connected and possesses a Green's function $G_E(z, \infty)$ with singularity at infinity such that $\lim G_E(z, \infty) \to 0$ whenever z approaches the boundary ∂E of E. Set $\Gamma_\rho := \{z, G_E(z, \infty) = \ln \rho\}$, $\rho > 1$, $\text{cap}(E)$ for its logarithmic capacity and μ_E for the equilibrium measure. The regularity of E ensures the positiveness of the capacity as well as the equality

$$\log \ \text{cap}\ (E) = \lim_{z \to \infty} (\log |z| - g_E(z, \infty)).$$

The equilibrium measure μ_E is the unique unit measure supported on ∂E, which minimizes the energy integral

$$\iint \log \frac{1}{|z-t|} d\mu(z) d\mu(t)$$

over all unit measures supported on E. In the case being considered $\text{supp}\ \mu_E \equiv \partial E$.

Further, if τ_n is an infinite sequence of Borel measures, supported on some set F, we say that τ_n converges in the weak topology to the measure τ (we write $\tau_n \longrightarrow \tau$), if

$$\int \phi d\tau_n \to \int \phi d\tau$$

for any function ϕ continuous in $\mathbf{C}$ and having a compact support. Finally, we associate with each polynomial S_n the counting measure μ_n; that is

$$\mu_n(K) := \frac{\nu_n(K)}{\text{number of all zeros of } S_n \text{ in } \mathbf{C}}.$$

Before we continue, we note the following fact: suppose that a domain $\mathcal{U}$ of overconvergence of $\{S_{n_k}\}$ is regular. Then the obvious estimate

$$\limsup_{n_k \to \infty} |a_{n_k}|^{1/n_k} \leq 1/\text{cap}\,(\overline{\mathcal{U}})$$

is true.

We now introduce the set $\mathcal{G}(\{S_{n_k}\}$ as the largest closed set in $\overline{\mathbf{C}}$ where on each compact subset K the inequality

$$\limsup_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} \le 1$$

holds. By the maximum principle for polynomials the complement of $\mathcal{G}(\{S_{n_k}\})$ is connected; further, the set $\overline{\mathcal{O}}(\{S_{n_k}\})$ coincides with that component of $\mathcal{G}(\{S_{n_k}\})$ which contains the origin (see the proofs below).

An inequality stronger than the former is valid, namely, if $\tilde{\mathcal{G}}$ is a closed regular subset of $\mathcal{G}(\{S_{n_k}\})$, then

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} \le 1/\mathrm{cap}\,(\tilde{\mathcal{G}}). \tag{4}$$

Indeed, let T_n be a Chebyshev polynomial on $\tilde{\mathcal{G}}$ of degree exactly n with all zeros on $\tilde{\mathcal{G}}$; as it is known [10]

$$|T_n(z)|^{1/n} \to \mathrm{cap}(\tilde{\mathcal{G}})e^{G(z,\infty)}$$

uniformly inside the complement of $\tilde{\mathcal{G}}$; $G(z,\infty)$ is its Green's function. The functions $\frac{S_{n_k}}{T_{n_k}}$ are subharmonic in $\tilde{\mathcal{G}}^c$. Fix now arbitrary numbers $\rho > 1$ and $\Theta > 0$. On writing $S_{n_k} = a_{n_k}P_{n_k}$ with P_{n_k} being a monic polynomial of degree n_k and using the Lemma of Bernstein-Walsh (see Lemma 3 below), we get for all n_k great enough

$$\left\|\frac{a_{n_k}P_{n_k}}{T_{n_k}}\right\|_{\Gamma_\rho} \le C_1\frac{e^{n_k\Theta}}{\mathrm{cap}\,(\tilde{\mathcal{G}})^{n_k}}.$$

(In what follows we will denote by C_i, $i = 1, 2, \ldots$ positive constants independent on i and different on different occasions.) By the maximum principle for subharmonic functions, the inequality remains valid at the point of infinity. Using finally the arbitrariness of Θ, we arrive at (4).

The next result provides an exact estimate for $\limsup_{n_k\to\infty}|a_{n_k}|^{1/n_k}$ as well as a result of Jentzsch-Szegö type for the distribution of the zeros.

Theorem 7. *Suppose that there is a closed compact set S and an infinite sequence $\{S_{n_k}\}$ such that for each compact subset K of S*

$$\limsup_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} \le 1.$$

Assume, further that S is regular and there is a point $z_0 \notin S$ with

$$\limsup_{n_k\to\infty} \frac{1}{n_k} \log |S_{n_k}(z_0)| = G_S(z_0, \infty). \tag{5}$$

Then

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} = 1/cap\ (S). \tag{6}$$

If

$$\limsup_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} = 1$$

compactly in S, then there is a subsequence $\{n_{k_l}\}$ such that the counting measures $\mu_{n_{k_l}}$ converge weakly to the equilibrium measure μ_S of S.

The second part of Theorem 7 is in fact a result of Jentzsch-Szegö type about the distribution of the zeros of $\{S_{n_{k_l}}\}$. Theorem 7 is an analogue of the main theorem of [8].

We note that the sequence $\{S_{n_{k_l}}\}$ from Theorem 7 is determined by the condition

$$\lim_{n_{k_l}\to\infty} |a_{n_{k_l}}|^{1/n_{k_l}} = 1/\text{cap}\ (S). \tag{7}$$

A direct consequence of Theorem 7 and of (4) is

Corollary 8. *Suppose that S is a regular set such that the sequence $\{S_{n_k}\}$ converges as $n_k \to \infty$ locally uniformly inside S^0 and suppose that condition* (5) *holds.*

Then

$$cap(S) = cap\ (\mathcal{G}(S_{n_k}))$$

and S is the largest set where $\{S_{n_k}\}$ converges locally uniformly inside.

If $f \not\equiv 0$ on each component of S, then there is a subsequence $\{n_{k_l}\}$ such that

$$\mu_{n_{k_l}} \longrightarrow \mu_S.$$

Remark. We note that if $\mathcal{G}(S_{n_k})$ is regular, then $S \equiv \mathcal{G}(S_{n_k})$. Otherwise S coincides with the regular component of $\mathcal{G}(S_{n_k})$.

We relate Corollary 8 to Theorem 5.2.2. in [7] where under the same conditions on f and $\{S_{n_k}\}$ as above and condition (5) replaced by (6) a weak convergence of the counting measures μ_{n_k} through an appropriate subsequence to the equilibrium measure of the set S is established.

We note also the corollaries

Corollary 9. *Let* (1) *be a power series with radius* 1 *and let the sequence* $\{S_{n_k}\}$ *be given. Suppose that* $\overline{\mathcal{D}}(\{S_{n_k}\})$ *is regular and that condition* (5) *holds for some point* $z_0 \in \mathcal{D}^c(\{S_{n_k}\})$.

Then

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} = 1/\ cap\ (\overline{\mathcal{D}}(\{S_{n_k}\}))$$

and for any sequence $\{n_{k_l}\}$ *with* (7) *the convergence*

$$\mu_{n_{k_l}} \to \mu_{\overline{\mathcal{D}}(\{S_{n_k}\})},\ n_{k_l} \to \infty$$

takes place.

For the case, when condition (3) holds, one may omit involving $\mathcal{D}(\{S_{n_k}\})$ into considerations. Namely, one can prove

Corollary 10. *Let* (1) *be a power series with radius* 1. *Let the sequence* $\{S_{n_k}\}$ *be such that condition* (3) *holds. Suppose, in addition, that* $\mathcal{O}(\{S_{n_k}\})$ *is regular and that condition* (5) *holds for some point* $z_0 \notin \overline{\mathcal{O}}(\{S_{n_k}\})$. *Then*

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} = 1/cap\ (\overline{\mathcal{D}}(\{S_{n_k}\})).$$

In both cases for any sequence $\{n_{k_l}\}$ *with* (7) *the measures* $\mu_{n_{k_l}}$ *converges weakly to the equilibrium measures. (Compare the last result with Corollary* 5.*)*

Condition (3) plays an essential role in the proofs of Theorem 3 and of preceding corollaries. It is an open problem whether these statements are true when (3) is not necessarily satisfied. Also, it is an open question whether the presented results are extendable to Fourier (see [7]) and Faber series.

Preliminaries.

Lemma 1. *Given a regular compact set* E *in* **C** *with connected complement and* $\{P_n\}_{n\in\Lambda}$ *be a sequence of polynomials, such that on each compact subset* K *of* E *condition* (2) *holds. Let* a_n *be the leading coefficients of* P_n, *i.e.,* $P_n(z) := a_n z^n + \cdots, a_n \neq 0,\ n \in \Lambda$. *Suppose that*

$$\lim_{n\in\Lambda} |a_n|^{1/n} = 1/cap(E)$$

and

$$\limsup_{n\in\Lambda} \|P_n\|_E{}^{1/n} \le 1.$$

Then the counting measures μ_n associated with P_n converge weakly to the equilibrium measure μ_E of E, $n \in \Lambda$.

The proof of this result was first presented in [17] (see also in [13]). In an independent way it was completed in [18].

Let $F := \{F_n\}_{n\in\Lambda}$ be a family of functions locally single-valued and analytic in some domain B except perhaps for branch points, and let $|F_n|$, $n \in \Lambda$ be single valued. We say that a function H harmonic in B is *a harmonic majorant of F in B,* if for any compact subset $K \subset B$ the inequality

$$\limsup_{n\in\Lambda} \|F_n\|_K \le e^{\|h\|_K}$$

holds. If for each compact subset of B and each subsequence a strict equality takes place then h is *an exact harmonic majorant.* The next lemma is due to J. L. Walsh ([14] and [15]).

Lemma 2. *Let B be a domain in $\mathbf{C}$, $F-$ be a family as above and h be a harmonic majorant for F in B. If there is a continuum $M \subset B$ where a strict inequality holds, then a strict inequality holds on every compact subset of B.*

Further, given a sequence of polynomials $\{P_n\}$, deg $P_n \le n$, let h be an exact harmonic majorant for $\{P_n\}^{1/n}$ in B for $\{P_n\}$ and for any subsequence. Then on each compact subset of B condition (2) *holds.*

For the sake of completeness, we provide the next two lemmas

Lemma 3 (The Lemma of Bernstein-Walsh), [16]**.** *Given a regular compact set E in $\mathbf{C}$ with connected complement, let P_n be a polynomial of degree n. Then, for every compact set $F \subset E^c$ the inequality*

$$\|P_n(z)\|_F \le \|P_n\|_E e^{n\|G_E(z,\infty)\|_F}$$

is true.

Lemma 4 (The two-constants-lemma), [10]**.** *Given a regular domain G, let $\partial G = \Gamma_1 \bigcup \Gamma_2, \Gamma_i-$ family of curves, $i = 1, 2$. Let $f \in \mathcal{A}(G) \bigcap C(\overline{G})$. Then for every compact subset K of G there exists a constant $\alpha(K) := \alpha, 0 < \alpha < 1$ such that*

$$\|f\|_K \le \|f\|_{\Gamma_1}^{\alpha} \|f\|_{\Gamma_2}^{1-\alpha}.$$

Proofs.

Proof of Theorem 1. We suppose without loss of generality that $f(0) = a_0 \neq 0$. Select in an arbitrary way simply connected and regular domains $0 \in W_1 \subset W_2 \subset \mathcal{U}$ with cap $(W_1) > 1$ and let $\zeta_{n,l}, l = 1, \ldots, l_{n_k}$ be all zeros of S_{n_k} on $\overline{W}_2$. By (2),

$$l_{n_k} = o(n_k),\ n_k \to \infty. \tag{8}$$

Set

$$t_{n_k}(z) := \begin{cases} \prod_{l=1}^{l_{n_k}}(z - \zeta_{n_k,l}), & l_{n_k} > 0 \\ 1, & \text{otherwise} \end{cases}$$

Let $0 < r < 1$ be a positive number such that $|f(z)| > 0$ for $z \in D_r := \{z, |z| \le r\} \subset W_1$ and fix a positive number $\rho < r$.

Introduce into considerations the functions $\phi_{n_k}(z) := \left\{\dfrac{S_{n_k}}{t_{n_k}}\right\}^{1/(n_k - l_{n_k})}$ with ϕ_{n_k} being that holomorphic branch for which $|\arg \phi_{n_k}(0)| \le \pi/n_k$. The functions ϕ_{n_k} are analytic on $\overline{W}_2$. Thanks the choice of r and of ρ, the inequalities

$$(r-\rho)^{l_{n_k}} \le |t_{n_k}(z)| \le (\operatorname{diam} W_2)^{l_{n_k}}$$

hold for every $z \in \overline{D}_\rho$ and for n_k large enough. Using the last inequalities and keeping in mind the choice of the holomorphic branch imply that

$$\phi_{n_k} \to 1, n_k \to \infty \tag{9}$$

uniformly on the disk $\overline{D_\rho}$. On the other hand, the sequence ϕ_{n_k} is uniformly bounded inside W_1. Indeed, let K be a compact subset of W_1 and $\Theta > 0$ be arbitrary. Making use of (8) and applying Bernstein-Walsh's lemma leads to

$$\|\phi_{n_k}\|_K \le e^{\Theta} \|z/\rho\|_K^{1 - l_{n_k}/n_k},\ n_k \ge n_{k_0}.$$

Hence, viewing (9) and applying the uniqueness theorem for analytic functions and by Montel's theorem, we get that

$$\phi_{n_k} \to 1,\ n_k \to \infty$$

uniformly inside the entire domain W_1.

Condition (2) yields

$$\limsup_{n_k \to \infty} \|S_{n_k}\|_K^{1/n_k} \le 1,\ n_k \to \infty \tag{10}$$

on each compact subset K of W_1 and, thus,

$$\limsup_{n_k \to \infty} \|S_{n_k}\|_{\partial W_1}^{1/n_k} \le 1,\ n_k \in \{n_k\},\ k \to \infty.$$

Now we apply a result by Gehlen [6] saying that if condition (10) holds for some closed regular set F then for each $\epsilon > 0$ there exists a number $q(\epsilon) := q \in (0, 1)$ such that

$$\limsup_{\nu \in \bigcup [qn_k, n_k]} |a_\nu|^{1/n_k} \le \frac{1+\epsilon}{\text{cap }(F)}.$$

Our statement becomes true, if we take $F := \overline{W}_1$ and select the number ϵ small enough. □

P r o o f o f T h e o r e m 2. First we remark that under the conditions of the theorem the domain $\mathcal{U}$ is necessarily simply connected, as well as that $\mathcal{U} \supset \mathcal{D}$ and $f \in \mathcal{A}(\mathcal{U} \bigcup V)$. On the other hand, the polynomials S_{n_k}, $n_k \to \infty$ satisfy inside V condition (10). Hence, on any compact subset K of the domain $\mathcal{U} \bigcup V$ the inequality

$$\limsup \|S_{n_k}\|_K^{1/n_k} \le 1$$

takes place. Since $f \in \mathcal{A}(\mathcal{U} \bigcup V)$, we deduce that

$$\limsup \|f - S_{n_k}\|_K^{1/n_k} \le 1$$

on each compact subset $K \subset \mathcal{U} \bigcup V$.

Hence, the function $h \equiv 0$ is a harmonic majorant for the sequence $\{|f - S_{n_k}|^{1/n_k}\}$ in the domain $\mathcal{U} \bigcup V$. Moreover, the last inequality is strict on compact subsets of the unit disk $\mathcal{D}$. Then, by Lemma 2, a strong inequality holds on each compact set in $\mathcal{U} \bigcup V$; with other words

$$\limsup \|f - S_{n_k}\|_K^{1/n_k} < 1$$

on compact subsets K in $\mathcal{U} \bigcup V$.

The last inequality proves our statement. □

P r o o f o f T h e o r e m 3. The proof is based on the considerations of the proof of Theorem 1. In the same way as before we introduce the numbers $\rho < r < 1$ and the regular domains W_1, W_2.

Using (10) yields

$$\limsup_{n_k \to \infty} \|S_{n_{k+1}} - S_{n_k}\|_{\partial W_1}^{1/n_{k+1}} \le 1. \tag{11}$$

Next, we establish an uniform convergence of S_{n_k} inside W_1. If in (11) a strict inequality takes place, then we are done. That is why we suppose that

$$\limsup \|S_{n_{k+1}} - S_{n_k}\|_{\partial W_1}^{1/n_{k+1}} = 1. \tag{12}$$

Coming back to the circle D_ρ, (recall that $D_\rho \subset \mathcal{D}$), we note that

$$\|S_{n_{k+1}} - S_{n_k}\|_{\overline{D_\rho}} \leq 2\rho^{n_k}.$$

On keeping track of (3), we may write

$$\|S_{n_{k+1}} - S_{n_k}\|_{\overline{D_\rho}} \leq C_3 q^{n_{k+1}} \tag{13}$$

with $q < 1$ being a suitable positive constant, $q > \rho^{\liminf n_k/n_{k+1}}$.

Next, we apply the two-constants-theorem with respect to W_1 and D_ρ. Let $S \subset W_1 \setminus \overline{D}_\rho$ be an arbitrary closed regular set. There exists a positive constant $\alpha(S) := \alpha$, $0 < \alpha < 1$ such that for every n_k

$$\|S_{n_{k+1}} - S_{n_k}\|_S \leq \|S_{n_{k+1}} - S_{n_k}\|_{\overline{D_\rho}}{}^{\alpha} \|S_{n_{k+1}} - S_{n_k}\|_{\partial W_1}{}^{1-\alpha}. \tag{14}$$

Select now a positive number Θ such that $e^\theta < 1/q^{\alpha/2(1-\alpha)}$. Using (12), for n_k large enough we obtain

$$\|S_{n_{k+1}} - S_{n_k}\|_{\partial W_1} \leq e^{n_{k+1}\Theta}.$$

Estimating the term $\|S_{n_{k+1}} - S_{n_k}\|_S$ in (14) by making use of (13) and of the last inequality and keeping track of the choice of Θ yield

$$\limsup_{k\to\infty} \|S_{n_{k+1}} - S_{n_k}\|_S^{1/n_{k+1}} \leq q^{\alpha/2} < 1.$$

Hence, $\{S_{n_k}\}$ converges uniformly on S, and thus, by arguments of the arbitrariness of S, locally uniformly inside W_1. Letting ∂W_1 approach $\partial \mathcal{U}$, we arrive at the statement of the theorem. □

P r o o f o f o f C o r o l l a r y 4. By definition, $\mathcal{D}(\{S_{n_k}\}) \subseteq \mathcal{O}(\{S_{n_k}\})$. Suppose that $\mathcal{D}(\{S_{n_k}\}) \subset \mathcal{O}(\{S_{n_k}\})$. Let $z_0 \in \partial\mathcal{D}(\{S_{n_k}\}) \bigcap \mathcal{O}(\{S_{n_k}\})$ and $V \subset \mathcal{O}(\{S_{n_k}\})$ be a neighborhood of z_0. Condition (2) takes place on compact subsets of the domain $\tilde{V} := V \bigcup \mathcal{D}(\{S_{n_k}\})$. By the previous proof, the sequence $\{S_{n_k}\}$ converges inside the domain $\tilde{V}$ (recall that (3) is valid). Hence, $\tilde{V} \subset \mathcal{D}(\{S_{n_k}\})$ which due the definition of $\mathcal{D}(\{S_{n_k}\})$ is impossible. □

P r o o f o f o f C o r o l l a r y 5. Suppose that the first statement is not true. Then there is a point $z_0 \in \partial\mathcal{D}(\{S_{n_k}\})$ and a neighborhood V of z_0 such that $\nu_{n_k}(\overline{V}) = 0$ for all n_k large enough. But according to Theorem 3 the sequence $\{S_{n_k}\}$ would be then overconvergent in a larger domain than $\mathcal{D}(\{S_{n_k}\})$. Further, suppose that z_0 be a boundary point of $\mathcal{D}(\{S_{n_k}\})$ for which the second statement is false. This means that (2) holds for closed disks $\overline{\mathcal{D}}(z_0, \delta)$ for all δ sufficiently small, which in turns means that $z_0 \in \mathcal{O}(\{S_{n_k}\})$. This conclusion contradicts Corollary 4. □

P r o o f o f T h e o r e m 6. We first remark that

$$\limsup_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} = 1$$

on compact subsets K of U, since otherwise $f \equiv 0$. The proof of the theorem follows immediately from the second part of Lemma 2. Indeed, applying it with respect to the sequence $\{S_{n_k}\}$ and the domain $\mathcal{U}$, we obtain (2). Adding condition (3), we get the statement of the theorem. □

Before we continue, we set $\tilde{\mathcal{G}}(\{S_{n_k}\})$ for that component of $\mathcal{G}(\{S_{n_k}\})$ which contains the closed unit disk $\overline{\mathcal{D}}$. In what follows, we will show that $\overline{\mathcal{O}}(\{S_{n_k}\}) \equiv \tilde{\mathcal{G}}(\{S_{n_k}\})$.

Indeed, if K is a regular compact subset of $\tilde{\mathcal{G}}(\{S_{n_k}\})$, then the inequality

$$\limsup_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} < 1$$

is impossible (for $\{n_k\}$ and for any subsequence). Otherwise we may suppose thanks Lemma 2 that on a closed disk $\overline{D}_\rho$ with $\rho < 1$ a strict inequality takes place. Since

$$\limsup_{n_k\to\infty} \|f - S_{n_k}\|_{\overline{D}_\rho}^{1/n_k} \le \rho,$$

then $f \equiv 0$ on D_ρ, which in turn implies that $f \equiv 0$ in $\mathcal{D}$. Consequently,

$$\lim_{n_k\to\infty} \|S_{n_k}\|_K^{1/n_k} = 1 \tag{15}$$

on compact subsets of $\tilde{\mathcal{G}}(\{S_{n_k}\})$. Now, applying again Lemma 2 we see that condition (2) holds necessarily on compact subsets there. On the other hand, $\mathcal{O}(\{S_{n_k}\}) \subseteq \tilde{\mathcal{G}}(\{S_{n_k}\})$ (see the proof of Theorem 2). This proves our proposition. We note in particular that on compact subsets of $\mathcal{O}(\{S_{n_k}\})$ inequality (15) holds.

P r o o f o f T h e o r e m 7. We recall (see (4)) that under the conditions of the theorem

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} \le 1/\ \text{cap}\ (S).$$

Suppose that

$$\limsup_{n_k\to\infty} |a_{n_k}|^{1/n_k} < 1/\ \text{cap}\ (S). \tag{16}$$

and introduce the functions

$$v_{n_k}(z) := 1/n_k \log |S_{n_k}(z)| - G_S(z,\infty).$$

The functions v_{n_k} are subharmonic in S^c and

$$v_{n_k}(\infty) = (1/n_k)\log |a_{n_k}| + \log\ \text{cap}\ S. \tag{17}$$

Set

$$v(z) := \limsup_{n_k} v_{n_k}(z).$$

By the theorem of Arzela-Askoli the function v is either equivalent to $-\infty$, or is continuous and subharmonic in S^c. The conditions of the theorem ensure that $v \equiv -\infty$ is impossible.

Using now the Lemma of Bernstein-Walsh, we obtain

$$v(z) \le 0,\ z \in S^c.$$

According to condition (5),

$$v(z_0) = 0,$$

from what we deduce that

$$v(z) \equiv 0,\ z \in S^c. \tag{18}$$

But in view of (16) and of (17),

$$v(\infty) < 0.$$

So, the last inequality is in view of (18) impossible; hence, assumption (16) is false.

Further, Lemma 1 is applicable to the subsequence $\{S_{n_{k_l}}\}$, where n_{k_l} is determined by (7). Thus the sequence of the counting measures $\mu_{n_{k_l}}$ converges weakly to the equilibrium measure μ_S. $\square$

At the end of our paper, we give two examples. Let

$$f(z) := \sum_{n=0}^{\infty} (z(z+1)/2)^{3^n}.$$

The series is absolutely convergent inside the lemniscate $\mathcal{L} := \{z, \, |z(z+1)| = 2\}$ and diverges at every point outside. Hence, $\mathcal{D}(f) \equiv \mathcal{L}^o$. Since $1 \in \mathcal{L} \bigcap \partial\mathcal{D}$ and $\mathcal{L}^o \supset \mathcal{D}$, the radius of convergence equals 1. We note that Cap $\mathcal{L} = \sqrt{2}$ and $G_{\mathcal{L}}(z, \infty) = \ln|z(z+1)/2|$ (see [10].) Further, in the case being considered $\mathcal{L}^o$ is a regular domain. Consider now the partial sums $S_{2.3^k}(z) = \sum_{l=0}^{k} (z(z+1)/2)^{3^l}$, $k = 0, 1, \cdots$. The sequence $\{S_{2.3^k}\}$ is overconvergent in $\mathcal{L}$ and, as it is easy to check, condition (5) holds at each point outside. Hence, Theorem 7 is applicable with respect to $\{S_{2.3^k}\}_{k=1}^{\infty}$.

Consider as a second example the function

$$f(z) := \sum_{n=0}^{\infty} (z(z-10)/11)^{3^n}.$$

This series converges inside the lemniscate $\mathcal{M} := \{z, \, |z(z-10)| = 11\}$ and diverges outside. $\partial\mathcal{M}$ consists of two closed analytic disjoint curves, and $\mathcal{M} \bigcap \partial\mathcal{D} = \{-1\}$. As before, the conditions of Theorem 7 refer to the sequence of the partial sums $S_{2.3^k}(z) = \sum_{n=0}^{k} (z(z-10)/11)^{3^n}$.

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Received June 1, 2007
Revised September 07, 2007