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## OSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the oscillatory behavior of first order nonlinear neutral delay differential equation

$$(x(t) - q(t)x(t - \sigma(t)))' + f(t, x(t - \tau(t))) = 0,$$

where  $\sigma, \tau \in C([t_0, \infty), (0, \infty))$ ,  $q \in C([t_0, \infty), [0, \infty))$  and  $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ . The obtained results extended and improve several of the well known previously results in the literature. Our results are illustrated with an example.

**1. Introduction.** In recent years the literature on the oscillation of neutral delay differential equations is growing very fast. It is relatively a new field with interesting applications in real world life problems. In fact, the neutral delay differential equations appear in modelling of the networks containing lossless

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transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role, see Hale [20], Driver [5], Brayton and Willoughby [4], Popove [21] and Boe and Chang [3], and reference cited therein. Also this evidenced by the extensive references in the books of Ladde et al. [17] and Gyori and ladas [19]. In a paper [18], Li and Kuang obtained a sufficient condition for the oscillation of a nonlinear delay differential equation of the form

$$(1.1) \quad x'(t) + p(t)g(x(t - \tau(t))) = 0,$$

where

$$(1.2) \quad p \in C([0, \infty), [0, \infty)), \quad \tau \in C([0, \infty), (0, \infty)), \quad \lim_{t \rightarrow \infty} (t - \tau(t)) = \infty,$$

$$(1.3) \quad g \in C(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad ug(u) > 0, \quad \text{for } u \neq 0.$$

The main results in [18] is the following:

**Theorem 1.1.** *Assume that (1.2) and (1.3) hold and that for some  $\varepsilon > 0$ ,  $M \geq 0$  and  $r > 0$ ,*

$$|g(u) - u| \leq M|u|^{1+r}, \quad \text{for } |u| < \varepsilon.$$

*Furthermore, suppose that*

$$\int_{\delta(t)}^t p(s) ds \geq e^{-1}, \quad t \geq t_0,$$

*and*

$$\int_{t_0}^{\infty} p(t) \left[ \exp \left( \int_{\delta(t)}^t p(s) ds - e^{-1} \right) - 1 \right] dt = \infty,$$

*where  $\delta(t) := \max_{s \in [0, t]} \{s - \tau(s)\}$ . Then every solution of (1.1) oscillates.*

The purpose of this paper is to extend the above mentioned oscillation criteria to the nonlinear neutral delay differential equation

$$(1.4) \quad (x(t) - q(t)x(t - \sigma(t)))' + f(t, x(t - \tau(t))) = 0, \quad \text{for } t \geq t_0 > 0,$$

where

$$(1.5) \quad \sigma, \tau \in C([t_0, \infty), (0, \infty)), \quad \lim_{t \rightarrow \infty} (t - \sigma(t)) = \lim_{t \rightarrow \infty} (t - \tau(t)) = \infty,$$

$$(1.6) \quad q \in C([t_0, \infty), [0, \infty)), \quad 0 \leq q_1 \leq q(t) \leq q_2 \leq 1,$$

$$(1.7) \quad f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R}), \quad uf(t, u) \geq 0,$$

and there exist

$$(1.8) \quad p \in C([t_0, \infty), (0, \infty)) \quad \text{and} \quad g \in C(\mathbb{R}, \mathbb{R}),$$

such that

$$(1.9) \quad g'(u) \geq 0, \quad f(t, u) \geq p(t)g(u) \quad \text{and} \quad |g(u) - u| \leq M|u|^{1+r},$$

for  $u \in (-\varepsilon, \varepsilon)$  and, for some  $\varepsilon > 0$ ,  $M \geq 0$ ,  $r > 0$  and  $ug(u) > 0$ , for  $u \neq 0$ .

Let  $\varrho = \max\{\sigma, \tau\}$ , and  $t_1 \geq t_0$ . A function  $x(t) \in C([t_1 - \varrho, \infty), \mathbb{R})$  is said to be a solution of equation (1.4), for some  $t_1 \geq t_0$  if  $x(t) - q(t)x(t - \sigma)$  is continuously differentiable on  $[t_1, \infty)$  and satisfies (1.4), for  $t > t_1$ . For further research on the oscillation of neutral delay differential equations, see the recent papers by Elabbasy and Saker [10], Tang and Shen [23] and Elabbasy, Hassan and Saker [8] and [9]. As usual a solution  $x(t)$  of equation (1.4) is said to be oscillatory if it has arbitrarily large zeros on  $[t_0, \infty)$ , otherwise it is nonoscillatory. Equation (1.4) is called oscillatory if every solution of this equation is oscillatory.

**2. Main results.** In this section, we will establish oscillation criteria for equation (1.4), which improves and extends known results.

**Theorem 2.1.** *Assume that (1.5)–(1.9) hold. Furthermore, suppose that*

$$(2.1) \quad \int_{\delta(t)}^t p(s) ds \geq \frac{1}{e}, \quad \text{for } t \geq t_0,$$

and

$$(2.2) \quad \int_{t_0}^{\infty} p(t) \left[ \exp \left( \int_{\delta(t)}^t p(s) ds - \frac{1}{e} \right) - 1 \right] dt = \infty,$$

where  $\delta(t) := \max_{s \in [t_0, t]} \{s - \tau(s)\}$ . Then every solution of equation (1.4) oscillates.

**Proof.** Assume (1.4) has a nonoscillatory solution on  $[t_0, \infty)$ . Then, without loss of generality, there is a  $t_1 \in [t_0, \infty)$ , sufficiently large, so that  $x(t)$ ,  $x(t - \sigma(t))$ , and  $x(t - \tau(t)) > 0$  on  $[t_1, \infty)$ . Set

$$(2.3) \quad z(t) := x(t) - q(t)x(t - \sigma(t)).$$

Then, by (1.4) and (1.7), we find

$$z'(t) = -f(t, x(t - \tau(t))) < 0,$$

which implies that  $z(t)$  is strictly decreasing. We claim that  $z(t) > 0$  on  $[t_1, \infty)$ . If not, then there exists  $t_2 \geq t_1$  such that  $z(t_2) =: c < 0$  and hence  $z(t) \leq c$ , for all  $t \geq t_2$ . Then, from (2.3), we get

$$(2.4) \quad x(t) \leq c + q(t)x(t - \sigma(t)), \quad \text{for all } t \geq t_2.$$

We consider the following two possible cases. Case 1:  $x(t)$  is unbounded, i.e.,  $\limsup_{t \rightarrow \infty} x(t) = \infty$ . Thus, there is an increasing sequence  $\{t_k\}$  such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ ,

$$x(t_k) = \sup_{t \leq t_k} x(t) \quad \text{and} \quad \lim_{k \rightarrow \infty} x(t_k) = \infty.$$

From (1.6) and (2.4), we find

$$x(t_k) \leq c + q(t_k)x(t_k - \sigma(t_k)) \leq c + x(t_k),$$

which is a contradiction. Case 2:  $x(t)$  is bounded, i.e.,  $\limsup_{t \rightarrow \infty} x(t) = l < \infty$ . Thus, there is a sequence  $\{t_k\}$  such that  $\lim_{k \rightarrow \infty} x(t_k) = l$  and then  $\limsup_{k \rightarrow \infty} x(t_k - \sigma(t_k)) \leq l$ . By (1.6) and (2.4), we have

$$x(t_k) \leq c + q(t_k)x(t_k - \sigma(t_k)) \leq c + x(t_k - \sigma(t_k)),$$

and so

$$l \leq c + \lim_{k \rightarrow \infty} x(t_k - \sigma(t_k)) \leq c + l,$$

which is also a contradiction. Then, we get  $z(t) > 0$  on  $[t_1, \infty)$ . By using that fact  $z(t)$  is strictly decreasing and positive function, we find that  $\lim_{t \rightarrow \infty} z(t) = \alpha \geq 0$ . If  $\alpha > 0$ , from (1.4), (1.9) and (2.3), we obtain

$$\begin{aligned} z(t) - z(t_1) &= - \int_{t_1}^t f(s, x(s - \tau(s))) ds \leq - \int_{t_1}^t p(s)g(x(s - \tau(s))) ds \\ &\leq - \int_{t_1}^t p(s)g(z(s - \tau(s))) ds \leq - \frac{g(\alpha)}{2} \int_{t_1}^t p(s) ds. \end{aligned}$$

Hence by (2.1), we have  $\lim_{t \rightarrow \infty} z(t) = -\infty$ , which contradicts that  $z(t)$  being positive function, then  $\alpha = 0$ . It follows from (1.9), and (2.3) that, for  $t \geq t_1$

$$\begin{aligned} f(t, x(t - \tau(t))) &\geq p(t) g(x(t - \tau(t))) \geq p(t) g(z(t - \tau(t))) \\ &\geq p(t) (z(t - \tau(t)) - Mz^{1+r}(t - \tau(t))) \\ &\geq p(t) z(t - \tau(t)) (1 - Mz^r(t - \tau(t))). \end{aligned}$$

From this and equation (1.4), we find

$$(2.5) \quad z'(t) + p(t) z(t - \tau(t)) (1 - Mz^r(t - \tau(t))) \leq 0, \quad \text{for } t \geq t_1.$$

The rest of the proof is similar to that of Theorem 1 in [18] and hence is omitted.  $\square$

**Theorem 2.2.** *Assume that (1.5)–(1.9) hold. Furthermore, suppose that*

$$(2.6) \quad \liminf_{t \rightarrow \infty} \int_{\delta(t)}^t P(s) ds > \frac{1}{e},$$

or

$$(2.7) \quad \limsup_{t \rightarrow \infty} \int_{\delta(t)}^t P(s) ds > 1,$$

where  $\delta(t)$  is defined as in Theorem 2.1 and  $P(t) := (1 - \epsilon)p(t)$ , for  $\epsilon > 0$ . Then every solution of (1.4) oscillates.

**Proof.** Assume (1.4) has a nonoscillatory solution on  $[t_0, \infty)$ . Then, without loss of generality, there is a  $t_1 \in [t_0, \infty)$ , sufficiently large, so that  $x(t)$ ,  $x(t - \sigma(t))$ , and  $x(t - \tau(t)) > 0$  on  $[t_1, \infty)$ . As in the proof of Theorem 2.1, from (2.5) there exists  $T \geq t_1$ , sufficiently large such that

$$(2.8) \quad z'(t) + P(t) z(\delta(t)) \leq 0, \quad \text{for all } t \geq T.$$

But, then by Corollary 3.2.2 [19] the delay differential equation

$$(2.9) \quad z'(t) + P(t) z(\delta(t)) = 0,$$

has an eventually positive solution as well. It is also well known that (2.6) or (2.7) implies (2.9) has no eventually positive solution (see, [19] Theorem 3.4.3). This contradiction completes the proof.  $\square$

It is clear that there is a gap between (2.6) and (2.7) for the oscillation of all solutions of (1.4). The problem how to fill this gap for the equation (1.4) when the limit

$$\lim_{t \rightarrow \infty} \int_{\delta(t)}^t P(s) ds,$$

does not exist needs to be considered. This problem has been cleared for the linear (2.9). Let the numbers  $k$  and  $l$  be defined by

$$k := \liminf_{t \rightarrow \infty} \int_{\delta(t)}^t P(s) ds, \quad l := \limsup_{t \rightarrow \infty} \int_{\delta(t)}^t P(s) ds,$$

$$0 < k \leq \frac{1}{e}, \quad l < 1,$$

and  $\lambda$  is the smallest root of the equation  $\lambda = e^{k\lambda}$ . Then (2.9) will be oscillatory if either of the following conditions is satisfied:

(A)

$$l > \frac{\ln \lambda + 1}{\lambda}, \quad [15].$$

(B)

$$l > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \quad [24].$$

(C)

$$l > \frac{1 + \ln \lambda}{\lambda} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \quad [12].$$

(D)

$$l > 2k + \frac{2}{\lambda} - 1, \quad [13].$$

(E)

$$l > \frac{\ln \lambda - 1 + \sqrt{5 - 2\lambda + 2k\lambda}}{\lambda}, \quad [16].$$

(F)

$$l > \frac{e-1}{e-2} \left( k + \frac{1}{\lambda_1} \right) - \frac{1}{e-2}, \quad [7].$$

Note that Theorem 2.2 implies that (1.4) will also be oscillatory if either of the conditions (A)–(F) is satisfied.

**Theorem 2.3.** *Assume that (1.5)–(1.9) hold with  $\tau(t) = \tau \in (0, \infty)$ . Furthermore, suppose that*

$$\int_{t-\tau}^t p(s) ds \geq \frac{1}{a} > 0,$$

and

$$(2.10) \quad \int_{t_0}^{\infty} p(t) \left[ \exp \left( a \int_{t-\tau}^t p(s) ds \right) \right] dt = \infty.$$

Then every solution of (1.4) oscillates.

*Proof.* Assume (1.4) has a nonoscillatory solution on  $[t_0, \infty)$ . Then, without loss of generality, there is a  $t_1 \in [t_0, \infty)$ , sufficiently large, so that  $x(t)$ ,  $x(t - \sigma(t))$ , and  $x(t - \tau) > 0$  on  $[t_1, \infty)$ . As in the proof of Theorem 2.1, from (2.5) there exists  $T \geq t_1$ , sufficiently large such that

$$z'(t) + \frac{1}{2}p(t)z(t - \tau) \leq 0, \quad \text{for all } t \geq T,$$

since  $\lim_{t \rightarrow \infty} z(t) = 0$ . In view [18, Lemmas 3, 4], we get  $\frac{z(t - \tau)}{z(t)}$  is bounded

and  $\int_{t-\tau}^t p(s)ds \leq 2$ . Set

$$\lambda(t) = -\frac{z'(t)}{z(t)},$$

we have the generalized characteristic equation

$$\lambda(t) \geq \frac{1}{2}p(t) \exp \left( \int_{t-\tau}^t \lambda(s) ds \right).$$

It is easy to see that

$$(2.11) \quad e^{\frac{s}{r}} \geq 1 + \frac{s}{r^2}, \quad \text{for } s > 0, r \geq 1.$$

Let

$$A(t) := \exp \left( a \int_{t-\tau}^t p(s) ds \right),$$

then

$$\lambda(t) \geq \frac{1}{2}p(t) \exp \left( \frac{1}{A(t)} A(t) \int_{t-\tau}^t \lambda(s) ds \right).$$



By using (2.11), we have

$$\lambda(t) \geq \frac{1}{2} p(t) \left( 1 + \frac{1}{A(t)} \int_{t-\tau}^t \lambda(s) ds \right).$$

Hence

$$A(t) \lambda(t) - \frac{1}{2} p(t) \int_{t-\tau}^t \lambda(s) ds \geq \frac{1}{2} p(t) A(t).$$

Integrating from  $T$  to  $N$ , where  $T \leq N$  such that  $0 < N - \tau < T$ ,

$$\int_T^N A(t) \lambda(t) dt - \frac{1}{2} \int_T^N p(t) \left( \int_{t-\tau}^t \lambda(s) ds \right) dt \geq \frac{1}{2} \int_T^N p(t) A(t) dt.$$

Interchanging the order of integration, we get

$$\begin{aligned} \int_T^N p(t) \left( \int_{t-\tau}^t \lambda(s) ds \right) dt &\geq \int_T^{N-\tau} \lambda(s) \left( \int_{s-\tau}^s p(t) dt \right) ds \\ &= \int_T^{N-\tau} \lambda(t) \left( \int_{t-\tau}^t p(s) ds \right) dt. \end{aligned}$$

Thus, we have

$$\int_T^N A(t) \lambda(t) dt - \frac{1}{2} \int_T^{N-\tau} \lambda(t) \left( \int_{t-\tau}^t p(s) ds \right) dt \geq \frac{1}{2} \int_T^N p(t) A(t) dt,$$

and hence

$$\int_T^N A(t) \lambda(t) dt + \int_{N-\tau}^T \lambda(t) \left( \int_{t-\tau}^t p(s) ds \right) dt \geq \frac{1}{2} \int_T^N p(t) A(t) dt,$$

then,

$$\int_T^N A(t) \lambda(t) dt + \int_{N-\tau}^T \lambda(t) A(t) dt \geq \frac{1}{2} \int_T^N p(t) A(t) dt,$$

since

$$A(t) \geq \int_{t-\tau}^t p(s) ds.$$

Then

$$\int_{N-\tau}^N \lambda(t) A(t) dt \geq \frac{1}{2} \int_T^N p(t) A(t) dt,$$

On the other hand, as  $A(t) < \beta$ ,  $\beta = e^{2a}$ , we find

$$\int_{N-\tau}^N \lambda(t) dt \geq \frac{1}{2\beta} \int_T^N p(t) A(t) dt,$$

or

$$\ln \frac{z(N-\tau)}{z(N)} \geq \frac{1}{2\beta} \int_T^N p(t) A(t) dt.$$

In view of (2.10), we have

$$\lim_{t \rightarrow \infty} \frac{z(t-\tau)}{z(t)} = \infty,$$

which contradicts that  $\frac{z(t-\tau)}{z(t)}$  is bounded for  $t \geq T$ . The proof of Theorem 2.3 is complete.  $\square$

**Example 2.1.** Consider the neutral delay differential equation

$$(2.12) \quad \left( x(t) - \frac{1}{2}(1 + \sin t)x(\sqrt{t}) \right)' + f\left(t, x\left(\frac{t}{\lambda} - \lambda + 1\right)\right) = 0, \quad t \geq \lambda(\lambda - 1) + 1,$$

where  $\lambda > 1$  and

$$f\left(t, x\left(\frac{t}{\lambda} - \lambda + 1\right)\right) = \left[ \frac{1}{e \ln \lambda (t + \lambda)} + \frac{1}{(t + \lambda) \ln(t + \lambda)} \right] |x(t - \tau)|^r, \quad r > 1.$$

Let

$$p(t) = \frac{1}{e \ln \lambda (t + \lambda)} + \frac{1}{(t + \lambda) \ln(t + \lambda)}, \quad \lambda > 1$$

and

$$g(x(t - \tau)) = |x(t - \tau)|^r, \quad r > 1.$$

It is easy to see that assumptions (1.5)–(1.9) hold. Clearly, for  $t \geq \lambda(\lambda - 1) + 1$ ,

$$\int_{\frac{t}{\lambda} - \lambda + 1}^t p(s) ds = \frac{1}{e} + \ln \frac{\ln(t + \lambda)}{\ln\left(\frac{t}{\lambda} + 1\right)} = \frac{1}{e} - \ln\left(1 - \frac{\ln \lambda}{\ln(t + \lambda)}\right) > \frac{1}{e},$$

and

$$\lim_{t \rightarrow \infty} \int_{\frac{t}{\lambda} - \lambda + 1}^t p(s) ds = \frac{1}{e}.$$

On the other hand

$$\begin{aligned} & \int_{\lambda(\lambda-1)+1}^{\infty} p(t) \left[ \exp \left[ \int_{\frac{t}{\lambda}-\lambda+1}^t p(s) ds - \frac{1}{e} \right] - 1 \right] dt \\ & \geq \int_{\lambda(\lambda-1)+1}^{\infty} p(t) \left[ \int_{\frac{t}{\lambda}-\lambda+1}^t p(s) ds - \frac{1}{e} \right] dt \\ & \geq \frac{-1}{e \ln \lambda} \int_{\lambda(\lambda-1)+1}^{\infty} \frac{1}{t+\lambda} \ln \left( 1 - \frac{\ln \lambda}{\ln(t+\lambda)} \right) dt = \infty, \end{aligned}$$

since

$$\int_{\lambda(\lambda-1)+1}^{\infty} \frac{dt}{(t+\lambda) \ln(t+\lambda)} = \infty$$

and

$$\lim_{t \rightarrow \infty} (\ln(t+\lambda)) \ln \left( 1 - \frac{\ln \lambda}{\ln(t+\lambda)} \right) = -\ln \lambda.$$

Then Theorem 2.1 implies that every solution of (2.12) oscillates.

Additional examples may be readily given. We leave this to interested reader.

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