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On Generalized Models and Singular Products of Distributions in Colombeau Algebra $\mathfrak{G}(\mathbb{R})$

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Modelling of singularities given by discontinuous functions or distributions by means of generalized functions has proved useful in many problems posed by physical phenomena. We introduce in a systematic way generalized functions of Colombeau that model such singularities. Moreover, we evaluate some products of singularity-modelling generalized functions whenever the result admits an associated distribution.

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1. Introduction

The generalized functions of Colombeau [1] have become a powerful tool for treating differential equations with singular coefficients and data as well as singular products of Schwartz distributions. The flexibility of the Colombeau theory allows to model such singularities by means of appropriately chosen generalized functions, treat them in this framework and — via the association process — bring down the obtained results to distributional level.

In particular, generalized models of the Heaviside step-function θ have proved useful in solving problems that arise in Mathematical Physics [2]. Other examples involving θ- and δ-type singularities that describe realistic physical phenomena are jump conditions in hyperbolic systems leading to travelling δ-waves solutions [3], the so-called controlled hybrid systems [6], geodesics for impulsive gravitational waves [8]. A detailed presentation of results and citations on this topic can be found in [7] and [5].

Motivated by such works, we have introduced in a *unified way* generalized functions of Colombeau that model singularities of certain type and have additional properties [4]. Following the same idea, we model in this paper singularities given by functions that have discontinuities of first order in a point on the real line $\mathbb{R}$. Moreover, we evaluate products of singularity-modelling generalized functions whenever the result admits an associated distribution.

2. Notations and definitions

We recall first the basic definitions of the Colombeau algebra $\mathfrak{G}(\mathbb{R})$.

Notation 1. Let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\delta_{ij} = \{\, 1$ if $i = j, = 0$ if $i \neq j\,\}$, for $i, j \in \mathbb{N}_0$. Then we put for arbitrary $q \in \mathbb{N}_0$:

$$A_q(\mathbb{R}) = \{\varphi(x) \in \mathfrak{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j\, \varphi(x)\, dx = \delta_{0j},\ j = 0, 1, ..., q\},$$

where $\mathfrak{D}(\mathbb{R})$ is the space of infinitely-differentiable functions with compact support. For $\varphi \in A_0(\mathbb{R})$ and $\varepsilon > 0$, we will use the following notation throughout the paper: $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $s \equiv s(\varphi) := sup\ \{|x| : \varphi(x) \neq 0)\}$. Then clearly $s(\varphi_\varepsilon) = \varepsilon s(\varphi)$, and denoting $\sigma \equiv \sigma(\varphi, \varepsilon) := s(\varphi_\varepsilon) > 0$, we have $\sigma := \varepsilon s = O(\varepsilon)$, as $\varepsilon \to 0$, for each $\varphi \in A_0(\mathbb{R})$. Finally, the shorthand notation $\partial_x = d/dx$ will be used in the one-dimensional case too.

Definition 1. Let $\mathfrak{E}\,[\mathbb{R}]$ be the algebra of functions $F(\varphi, x) : \mathbb{A} \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable for fixed 'parameter' φ. Then the generalized functions of Colombeau are elements of the quotient algebra $\mathfrak{G} \equiv \mathfrak{G}(\mathbb{R}) = \mathcal{E}_{\mathrm{M}}[\mathbb{R}]\, /\, \mathcal{I}\,[\mathbb{R}]$. Here $\mathcal{E}_{\mathrm{M}}[\mathbb{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset K of $\mathbb{R}$ and $p \in \mathbb{N}$, there is a $q \in \mathbb{N}$, such that for each $\varphi \in A_q(\mathbb{R})$, $\sup_{x \in K} |\partial^p F(\varphi_\varepsilon, x)\,| = O(\varepsilon^{-q})$, as $\varepsilon \to 0_+$. The ideal $\mathcal{I}\,[\mathbb{R}]$ of $\mathcal{E}_{\mathrm{M}}[\mathbb{R}]$ consists of all functions such that for each compact $K \subset \mathbb{R}$ and any $p \in \mathbb{N}$, there is a $q \in \mathbb{N}$ such that, for every $r \geq q$ and $\varphi \in A_r(\mathbb{R})$, $\sup_{x \in K} |\partial^p F(\varphi_\varepsilon, x)\,| = O(\varepsilon^{r-q})$, as $\varepsilon \to 0_+$.

The algebra $\mathfrak{E}\,[\mathbb{R}]$ contains the distributions on $\mathbb{R}$, canonically embedded as a $\mathbb{C}$-vector subspace, by the map

$$i : \mathfrak{D}'(\mathbb{R}) \to \mathfrak{G} : u \mapsto \widetilde{u} = \{\, \widetilde{u}(\varphi, x) := (u * \check{\varphi})(x) |\, \varphi \in A_q(\mathbb{R})\,\},$$
$$\text{where } \check{\varphi}(x) = \varphi(-x). \quad (1)$$

The equality of generalized functions in $\mathfrak{G}$ is very strict and a weaker form of equality in the sense of *association* is introduced, which plays a fundamental role in the Colombeau theory.

Definition 2. **(a)** Two generalized functions $F, G \in \mathfrak{G}(\mathbb{R})$ are said to be 'associated', denoted $F \approx G$, if for some representatives $F(\varphi_\varepsilon, x), G(\varphi_\varepsilon, x)$

and arbitrary $\psi(x) \in \mathfrak{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$,

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} [F(\varphi_\varepsilon, x) - G(\varphi_\varepsilon, x)]\psi(x)\, dx = 0.$$

(b) A generalized function $F \in \mathfrak{G}(\mathbb{R})$ is said to be 'associated' with a distribution $u \in \mathfrak{D}'(\mathbb{R})$, denoted $F \approx u$, if for some representative $F(\varphi_\varepsilon, x)$, and arbitrary $\psi(x) \in \mathfrak{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$,

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} F(\varphi_\varepsilon, x)\psi(x)\, dx = \langle u, \psi \rangle.$$

These definitions are independent of the representatives chosen, and the association is a faithful generalization of the equality of distributions. The following relations hold in $\mathfrak{G}$:

$$F \approx u \quad \& \quad F_1 \approx u_1 \implies F + F_1 \approx u + u_1, \quad \partial F \approx \partial u.$$

Remark. Note that the relation $F \approx u$ is asymmetric in the sense that the terms cannot be moved over the $\approx$-sign: on the r.h.s. of it there stands a distribution. Of course, the equivalent relation $F \approx \widetilde{u}$ in $\mathfrak{G}$, is symmetric (and can be written as $F - \widetilde{u} \approx 0$ as well). We prefer however the first, simpler and suggesting, notation for the associated distribution.

Below we shall need also the following.

Notation 2. If $a \in \mathbb{R}$, $a > -1$, introduce the locally-integrable functions that are 'normed' powers of the variable $x \in \mathbb{R}$ supported in one of the real semiaxes:

$$\nu_+^a \equiv \nu_+^a(x) = \{\frac{x^{\,a}}{\Gamma(a+1)} \ \text{ if } x > 0, \ \ = 0 \ \text{ if } x < 0\}.$$

$$\nu_-^a \equiv \nu_-^a(x) = \{\frac{(-x)^{\,a}}{\Gamma(a+1)} \ \text{ if } x < 0, \ \ = 0 \ \text{ if } x > 0\}.$$

In particular, for $a = p \in \mathbb{N}_0$, we have

$$\nu_+^p = \{\ \frac{x^p}{p!},\ x > 0; \ = 0,\ x < 0\}, \qquad \nu_-^p = \{\ \frac{(-x)^p}{p!},\ x < 0; \ = 0,\ x > 0\}.$$

We can also define their 'even' and 'odd' compositions: $|\nu|^{\,p} = \nu_+^p + \nu_-^p$, $|\nu|^{\,p} \operatorname{sgn} x = \nu_+^p - \nu_-^p$. If $p = 0$, we come to the Heaviside step-function $\theta = \{1,\ x < 0; \ = 0,\ x > 0\}$.

All these functions have singularities at $x = 0$ and only considered as distributions, their derivatives exist and satisfy the relations

$$\partial_x \nu_\pm^{p+1} = \pm\, \nu_\pm^p, \quad \partial_x |\nu|^p = |\nu|^{p-1} \operatorname{sgn} x, \quad \partial_x |\nu|^p \operatorname{sgn} x = |\nu|^{p-1}.$$

Note that, due to the norming, no number coefficients are present here, which makes the calculations with these distributions easier.

The definition of $\nu_\pm^a$ can be extended for any $a \in \Omega := \mathbb{R}\backslash\{-\mathbb{N}\}$, by setting

$$\nu_+^a = \partial^r \nu_+^{a+r}(x),\ \nu_-^a = (-1)^r \partial^r \nu_-^{a+r}(x), \quad \text{where } r \in \mathbb{N} \text{ is such that } a+r > -1.$$

Notation 3. Let $C_d^k(\mathbb{R}\backslash\{0\})$ be the class of k-times differentiable functions on $\mathbb{R}\backslash\{0\}$ for some $k \in \mathbb{N}$, such that each function $f(x)$ and its derivatives $f^{(i)}, i = 1, ..., k$, have discontinuities of first order at the point $x = 0$, i.e. for each $i \le k$, the values $f^{(i)}(0_+)$ and $f^{(i)}(0_-)$ exist but in general differ from each other. Then we denote the mean values and the jumps at $x = 0$ of $f^{(i)}(x)$ by:

$$m_i \equiv m_i(f) = \frac{1}{2}\left[f^{(i)}(0_+) + f^{(i)}(0_-)\right], h_i \equiv h_i(f) = f^{(i)}(0_+) - f^{(i)}(0_-), i = 0, .., k.$$

3. Modelling of singularities in $\mathcal{G}(\mathbb{R})$

Consider first generalized functions that model the δ-type singularity in the sense of association, i.e. being associated with the δ-function.Since there is an abundant variety of such functions (together with the canonical imbedding $\widetilde{\delta}$ in $\mathfrak{G}$ of the distribution δ), we can put on the generalized functions in question an additional requirement. So define, following [7, §10], a generalized function $D \in \mathfrak{G}$ with the properties:

$$D \approx \delta, \qquad D^2 \approx \delta. \tag{2}$$

Let $\varphi \in A_0(\mathbb{R})$, $s \equiv s(\varphi)$, and $\sigma = s(\varphi_\varepsilon) = \varepsilon s$ be as in Notation 1, and $D \in \mathfrak{G}$ be the class $[\varphi \mapsto D(s(\varphi), x)]$. We then specify that $D(s, x) = f(x) + \lambda_s\, g(x)$, where $f, g \in \mathfrak{D}(\mathbb{R})$ are real-valued, symmetric, with disjoint support, and satisfying:

$$\int_{\mathbb{R}} f(x)dx = 1, \quad \int_{\mathbb{R}} g(x)dx = 0, \quad \text{and} \quad \lambda_s^2 = \frac{s - \int f^2(x)dx}{\int g^2(x)dx}.$$

Now, it is not difficult to check that, for each $\varphi \in A_0(\mathbb{R})$, the representative $D(s, x)$ of the generalized function D satisfies the conditions:

$$D(.,x) \in \mathfrak{D}(\mathbb{R}), \quad D(.,-x) = D(.,x), \quad \frac{1}{s}\int_{\mathbb{R}} D^2(s,x)\,dx = \int_{\mathbb{R}} D(s,x)dx = 1, \tag{3}$$

for each real positive value of the parameter s. Moreover, the generalized function D so defined satisfies the association relations (2). To show this, denote by

$$D_\sigma(x) \ := \ \frac{1}{\sigma} D\Big(\sigma, \frac{x}{\sigma}\Big), \quad \text{where} \ \ \sigma = s(\varphi_\varepsilon). \tag{4}$$

Now, for an arbitrary test-function $\psi \in \mathfrak{D}(\mathbb{R})$, evaluate the values

$$I_1(\sigma) = \langle D_\sigma(x), \psi(x)\rangle, \qquad I_2(\sigma) = \langle D^2_\sigma(x), \psi(x)\rangle,$$

as $\varepsilon \to 0_+$, or equivalently, as $\sigma \to 0_+$. But in view of (3), it is immediate to see that $\lim_{\sigma\to 0_+} I_1(\sigma) = \lim_{\sigma\to 0_+} I_2(\sigma) = \langle \delta, \psi\rangle$; which according to Definition 2 (b) implies (2).

Remark. The first equation in (2) is in consistency with the observation that $D_\sigma(x)$ is a strict δ-net as defined in distribution theory [7, §7]. But note that D is not the canonical embedding $\widetilde{\delta}$ of the δ-function since $\widetilde{\delta}^{\,2}$ does not admit associated distribution.

The flexible approach to modelling singularities allowed by the generalized functions so that the models satisfy auxiliary conditions can be systematically applied to defining generalized models of particular singularities. We recall that such approach has proved useful in studying Euler-Lagrange equations for classical particle in δ-type potential as well as the geodesic equation for impulsive gravitational waves; see [5, §1.5, §5.3].

We will consider models of singularities given by distributions with singular point support. For their definition, we intend to take advantage of the properties of δ-modelling function D. Observe that it holds

$$(\delta * D(s,.))(x) = \langle \delta_y,\, D(s, x-y)\rangle = D(s,x),$$

$$\begin{aligned}(\delta' * D(s,.))(x) = \langle \delta'_y,\, D(s,x-y)\rangle = -\langle \delta_y\ \partial_y D(s,x-y)\rangle = \\ = \langle \delta_y,\, D'(s,x-y)\rangle = D'(s,x).\end{aligned}$$

This can be continued by induction for any derivative to define a generalized function $D^{(p)}(x)$ that models the distribution $\delta^{(p)}(x)$ and has a representative

$$D^{(p)}(s,x) = (\delta^{(p)} * D(s,.))(x). \tag{5}$$

Clearly, this is in consistency with the differentiation: $\partial_x D^{(p)}(x) = D^{(p+1)}(x)$. Moreover,

$$D^{(p)}(-x) = \partial^p D(-x) = (-1)^p D^{(p)}(x). \tag{6}$$

In [4] we have employed such procedure for a unified modelling of singularities given by distributions with singular point support, i.e. (besides $\delta^{(p)}$)

the distributions $\nu_\pm^a$, $a \in \Omega$. Namely, choosing an arbitrary generalized function D with representative $D(s,x)$ that satisfies (3) for each $\varphi \in A_0(\mathbb{R})$, we have introduced generalized functions $X_\pm^a(x)$, modelling the above singularities, with representatives

$$X_\pm^a(s,x) := (\nu_\pm^a * D(s,.))(x), \quad a \in \Omega.$$

A consistency with the differentiation holds: $\partial_x X_\pm^a(x) = X_\pm^{a-1}(x)$, in particular, $H' = D$, where $H \in \mathfrak{G}$ is model of the step-function θ, with representative $H(s,x) = \theta * D(s,.)(x)$.

Now we will define and study models in $\mathfrak{G}$ of functions on the real line that have discontinuities in a point. Their generalized models will be obtained by the next definition that follows the idea suggested in [4].

Definition 3. For any function $f \in C_d^k(\mathbb{R} \setminus \{0\})$, choosing an arbitrary function $D \in \mathfrak{G}$ with representative $D(s,x)$ that satisfies (3) for each $\varphi \in A_0(\mathbb{R})$, define its generalized model as the function $F(x) \in \mathfrak{G}$ with representatives given by

$$F(s,x) := f * D(s,.)(x). \tag{7}$$

Note that each generalized functions F so introduced is really model of the corresponding discontinuous function f. Indeed, let $F_\sigma(x) = f * D_\sigma(x)$ be the representative of F depending on $\sigma = \varepsilon s$ and suppose (without loss of generality) that $\operatorname{supp} D(\sigma,x) \subseteq [-l,l]$ for some $l \in \mathbb{R}_+$. Then for an arbitrary test-function $\psi \in \mathfrak{D}(\mathbb{R})$, evaluate $I(\sigma) := \int_{\mathbb{R}} \psi(x) F_\sigma(x)\,dx$. Transformation of the variable $y = \sigma v + x$ yields

$$\begin{aligned} I(\sigma) &= \frac{1}{\sigma}\int_{\mathbb{R}} dx\,\psi(x)\left[\int_{-\sigma l+x}^{0} f(y) D\left(\sigma, \frac{x-y}{\sigma}\right) dy + \int_0^{\sigma l+x} f(y) D\left(\sigma, \frac{x-y}{\sigma}\right) dy\right] = \\ &= \int_{\mathbb{R}} dx\,\psi(x)\left[\int_{-l}^{-x/\sigma} f(\sigma v + x) D(\sigma, v)\,dv + \int_{-x/\sigma}^{l} f(\sigma v + x) D(\sigma, v)\,dv\right] \end{aligned}$$

Now taking the limit as $\varepsilon \to 0_+$, or else $\sigma = \varepsilon s \to 0_+$, and applying equation (3), we get

$$\lim_{\sigma \to 0_+} I(\sigma) \;=\; \int_{\mathbb{R}} dx\,\psi(x)\,f(x) \int_{-l}^{l} D(\sigma, v)\,dv = \langle\, f, \psi\,\rangle,$$

which according to Definition 2(b) implies the association $F(x) \approx f(x)$.

4. Products of some singularities modelled in $\mathcal{G}(\mathbb{R})$

The models of singularities we consider all have products in the Colombeau algebra as generalized functions, but we are seeking results that can be evaluated back in terms of distributions, i.e. products that admit associated distributions.

So, it was proved in [4] that, for an arbitrary p in $\mathbb{N}_0$, the generalized models $X_\pm^p, D^{(p+1)}$, and $D^{(p+2)}$ satisfy:

$$(\mp 1)^{p+1} X_\pm^p \,.\, D^{(p+1)} \;\approx\; \delta \mp \frac{(p+1)}{2}\delta'. \tag{8}$$

$$(\mp 1)^p \, X_\pm^p \,.\, D^{(p+2)} \;\approx \mp \frac{2p+3}{2}\delta' + \frac{1}{2}\binom{p+2}{2}\delta'', \tag{9}$$

Then the generalized functions $|X|^p = X_+^p + X_-^p$, $|X|^p \operatorname{sgn} x = X_+^p - X_-^p$, and D satisfy:

$$|X|^p \,.\, D^{(p+1)} \approx 2\,\delta, \quad |X|^p \,.\, D^{(p+2)} \;\approx\; (2p+3)\,\delta', \;\text{ for }\; p = 1,3,5,... \tag{10}$$

$$|X|^p \operatorname{sgn} x \,.\, D^{(p+1)} \approx -2\,\delta, |X|^p \operatorname{sgn} x \,.\, D^{(p+2)} \approx -(2p+3)\,\delta', \;\text{ for }\; p = 2,4,6... \tag{11}$$

Now we proceed to studying singular products, obtained by Definition 3, of generalized models of functions from the class $C_d^k(\mathbb{R} \setminus \{0\})$ with derivatives of the δ-modelling generalized function $D(x)$. Products of discontinuous functions with the derivatives of δ exist neither in the classical Distribution theory nor as so-called Colombeau products - their canonical embeddings in $\mathfrak{G}$ do not admit associated distributions. Nevertheless, their generalized models obey the following.

Theorem 1. *For each function $f(x) \in C_d^2(\mathbb{R} \setminus \{0\})$, its model $F(x)$ in $\mathfrak{G}(\mathbb{R})$ satisfies*

$$F(x) \,.\, D'(x) \;\approx\; -(h_0 + m_1)\;\delta \;+\; m_0\,\delta', \tag{12}$$

where $h_0 = f(0_+) - f(0_-)$, $m_0 = (f(0_-) + f(0_+))/2$, $m_1 = (f'(0_-) + f'(0_+))/2$.

P r o o f. For $\psi(x) \in \mathfrak{D}(\mathbb{R})$, we denote $I(\sigma) := \langle\, F_\sigma(x) \,.\, D'_\sigma(x),\, \psi(x)\,\rangle$. From equations (4), (5), and (7), we get on transforming the variables $y = \sigma v + x$, $x = -\sigma u$ and taking into account equation (6)

$$I(\sigma) = \frac{-1}{\sigma}\int_{-l}^{l} du\, \psi(-\sigma u) D'(\sigma,u)\left(\int_{-l}^{u} f(\sigma v - \sigma u) D(\sigma,v)\,dv \,+\right.$$

$$\left. + \int_{u}^{l} f(\sigma v - \sigma u) D(\sigma,v)\,dv\right.$$

Now applying Taylor theorem to the test-function ψ and changing the order of integration, we obtain

$$I(\sigma) = \frac{-\psi(0)}{\sigma} \int_{-l}^{l} dv\, D(\sigma, v) \left(\int_{v}^{l} f(\sigma v - \sigma u) D'(\sigma, u) du + \int_{-l}^{v} f(\sigma v - \sigma u) D'(\sigma, u) du \right)$$

$$+ \psi'(0) \int_{-l}^{l} dv\, D(\sigma, v) \left(\int_{v}^{l} f(\sigma v - \sigma u) u D'(\sigma, u) du + \int_{-l}^{v} f(\sigma v - \sigma u) u D'(\sigma, u)\, du \right) +$$

$$+ \; O(\sigma) \;\; =: \;\; \psi(0)\; I_1 \; + \; \psi'(0)\; I_2 + \; O(\sigma). \quad (13)$$

To obtain the above asymptotic evaluation, we have taken into account that the third term in the Taylor expansion is multiplied by definite integrals majorizable by constants. Integrating further by parts in the variable u, applying Lebesque theorem on bounded convergence, taking into account equation (3), and making use of Notation 3, we obtain

$$I_1 = \frac{f(0_-)}{\sigma} \int_{-l}^{l} D^2(\sigma, v)\, dv - \int_{-l}^{l} dv\, D(\sigma, v) \int_{v}^{l} f'(\sigma v - \sigma u) D(\sigma, u)\, du -$$

$$- \frac{f(0_+)}{\sigma} \int_{-l}^{l} D^2(\sigma, v)\, dv - \int_{-l}^{l} dv\, D(\sigma, v) \int_{-l}^{v} f'(\sigma v - \sigma u) D(\sigma, u)\, du =$$

$$= \; f(0_-) - f(0_+) - \frac{1}{2} \, \left[\, f'(0_-) + f'(0_+) \right] + \; O(\sigma) = - \left(h_0 + m_1 \right) \; + \; O(\sigma).$$

Here the Taylor theorem is applied to the function $f' \in C^1(\mathbb{R}_\pm)$ up to second order to get its expansion about the point $\sigma(v - u)$ which is respectively > 0 or < 0. This gives

$$f'(\sigma v - \sigma u) = f'(0_\pm) + \sigma(v - u)\; f''(0_\pm) + \; O(\sigma^2) = f'(0_\pm) + O(\sigma), \text{ for } v > u, \text{ resp. } v < u.$$

Proceeding similarly as above, we get for the second term in (13)

$$I_2 = - \, f(0_-) \int_{-l}^{l} v D^2(\sigma, v)\, dv - f(0_-) \; \int_{-l}^{l} dv D(\sigma, v) \int_{v}^{l} \; D(\sigma, u)\, du +$$

$$+ \; f(0_+) \int_{-l}^{l} v D^2(\sigma, v)\, dv - f(0_+) \int_{-l}^{l} dv\, D(\sigma, v) \int_{-l}^{v} \; D(\sigma, u)\, du + \; O(\sigma) =$$

$$= \frac{-1}{2} \, \left[\, f(0_-) + f'(0_+) \right] + \; O(\sigma) \;\; = \; - \, m_0 \; + \; O(\sigma).$$

We have used that $D(\sigma, v)$ satisfies

$$\int_{-l}^{l} v D^2(\sigma, v)\, dv = 0 \;\; \text{and} \;\; \int_{-l}^{l} dv\, D(\sigma, v) \int_{-l}^{v} D(\sigma, u)\, du = \frac{1}{2}.$$

Therefore

$$I(\sigma) = \langle\, F_\sigma(x)\,.\,D'_\sigma(x),\ \psi(x)\,\rangle = -\,(h_0+m_1)\,\psi(0)\ -\ m_0\,\psi'(0)\ +\ O(\sigma) = \\ = -\,(h_0+m_1)\,\langle\,\delta,\psi\,\rangle\ +\ m_0\,\langle\,\delta',\psi\,\rangle\ +\ O(\sigma).$$

Passing then to the limit as $\sigma\to 0_+$ and applying Definition 2 (b), we obtain equation (12). ■

Next we consider the product of models of discontinuous functions with the second derivative of the δ-modelling function $D(x)$. In view of Notation 3, one has the following.

Theorem 2. *For each function $f(x)\in C^3_d(\mathbb{R}\setminus\{0\})$, its generalized model $F(x)$ in $\mathfrak{G}(\mathbb{R})$ satisfies:*

$$F(x)\,.\,D''(x)\ \approx\ (h_1+m_2)\,\delta - \left(\frac{3}{2}\,h_0+2m_1\right)\,\delta' + m_0\,\delta''. \tag{14}$$

P r o o f. For $\psi(x)\in\mathfrak{D}(\mathbb{R})$, we denote $J(\sigma) := \langle\, F_\sigma(x)\,.\,D''_\sigma(x),\ \psi(x)\,\rangle$. From equations (4), (5), and (7), we get on transforming the variables $y=\sigma v+x$, $x=-\sigma u$ and taking into account equation (6)

$$J(\sigma) = \frac{1}{\sigma^2}\int_{-l}^{l} du\,\psi(-\sigma u)\,D''(\sigma,u)\Big(\int_{-l}^{u} f(\sigma v-\sigma u)D(\sigma,v)\,dv + \\ + \int_{u}^{l} f(\sigma v-\sigma u)D(\sigma,v)\,dv$$

Now applying the Taylor theorem to the test-function ψ and changing the order of integration, we obtain

$$J(\sigma)=\frac{\psi(0)}{\sigma^2}\int_{-l}^{l}dv\,D(\sigma,v)\left(\int_{v}^{l}f(\sigma v-\sigma u)D''(\sigma,u)\,du+\int_{-l}^{v}f(\sigma v-\sigma u)D''(\sigma,u)\,du\right) \\ -\frac{\psi'(0)}{\sigma}\int_{-l}^{l}dv\,D(\sigma,v)\Big(\int_{v}^{l}f(\sigma v-\sigma u)uD''(\sigma,u)\,du+\int_{-l}^{v}f(\sigma v-\sigma u)uD''(\sigma,u)\,du\Big)+ \\ +\frac{\psi''(0)}{2}\int_{-l}^{l}dv\,D(\sigma,v)\Big(\int_{v}^{l}f(\sigma v-\sigma u)u^2D''(\sigma,u)\,du+\int_{-l}^{v}f(\sigma v-\sigma u)u^2D''(\sigma,u)\,du\Big)+ \\ +\ O(\sigma)\ =:\psi(0)\ J_1\ -\psi'(0)\ J_2+\frac{\psi''(0)}{2}+J_3+O(\sigma). \tag{15}$$

Integrating twice by parts in the variable u, applying Lebesque theorem on bounded convergence, and taking account of equation (3), we get further

$$J_1 = \frac{-f(0_-)}{\sigma^2}\int_{-l}^{l} D(\sigma,v)D'(\sigma,v)\,dv + \frac{1}{\sigma}\int_{-l}^{l} dv D(\sigma,v)\int_{v}^{l} f'(\sigma v-\sigma u)D'(\sigma,u)\,du$$

$$+\frac{f(0_+)}{\sigma^2}\int_{-l}^{l} D(\sigma,v)D'(\sigma,v)\,dv + \frac{1}{\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} f'(\sigma v-\sigma u)D'(\sigma,u)\,du =$$

$$= -f'(0_-)\frac{1}{\sigma}\int_{-l}^{l} D^2(\sigma,v)\,dv + \int_{-l}^{l} dv\, D(\sigma,v)\int_{v}^{l} f''(\sigma v-\sigma u)D(\sigma,u)\,du +$$

$$+ f'(0_+)\frac{1}{\sigma}\int_{-l}^{l} D^2(\sigma,v)\,dv + \int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} f''(\sigma v-\sigma u)D(\sigma,u)\,du + \ O(\sigma) =$$

$$= -\, f'(0_-) + f'(0_+) + \frac{1}{2}\,\left[\, f''(0_-) + f''(0_+)\right] + \ O(\sigma) \ = \ (h_1 + m_2) \ + \ O(\sigma).$$

Here the Taylor theorem is applied to the function $f'' \in C^1(\mathbb{R}_\pm)$ up to second order to get its expansion about the point $\sigma(v-u)$, obtaining thus

$$f''(\sigma v - \sigma u) = f''(0_\pm) + \ O(\sigma).$$

Throughout the calculations, we have used that, due to $D(.,-x) = D(.,x)$, it holds:

$$\int_{-l}^{l} D(\sigma,v)D'(\sigma,v)\,dv = 0 \quad \text{and} \quad \int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} D(\sigma,u)\,du = \frac{1}{2}.$$

Proceeding similarly as above, we get for the second and third terms in (15):

$$J_2 = \frac{-\,f(0_-)}{\sigma}\int_{-l}^{l} vD(\sigma,v)D'(\sigma,v)\,dv + \frac{f(0_+)}{\sigma}\int_{-l}^{l} vD(\sigma,v)D'(\sigma,v)\,dv -$$

$$-\frac{1}{\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\left(\int_{v}^{l} f(\sigma v-\sigma u)D'(\sigma,u)\,du + \int_{-l}^{v} f(\sigma v-\sigma u)D'(\sigma,u)\,du\right) +$$

$$+\int_{-l}^{l} dv\, D(\sigma,v)\left(\int_{v}^{l} f'(\sigma v-\sigma u)uD'(\sigma,u)\,du + \int_{-l}^{v} f'(\sigma v-\sigma u)uD'(\sigma,u)\,du\right) =$$

$$= \frac{1}{2}\,f(0_-) - \frac{1}{2}\,f(0_+) + f(0_-)\frac{1}{\sigma}\int_{-l}^{l} D^2(\sigma,v)\,dv - f(0_+)\frac{1}{\sigma}\int_{-l}^{l} D^2(\sigma,v)\,dv -$$

$$-\,2\,f'(0_-)\int_{-l}^{l} dv\, D(\sigma,v)\int_{v}^{l} D(\sigma,u)\,du - 2\,f'(0_+)\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} D(\sigma,u)\,du +$$

$$+\,2\int_{-l}^{l} dv\, D(\sigma,v)\left(\int_{v}^{l} f(\sigma v-\sigma u)uD'(\sigma,u)\,du + 2\int_{-l}^{v} f(\sigma v-\sigma u)uD'(\sigma,u)\,du\right) =$$

$$= \frac{3}{2}f(0_-) - \frac{3}{2}f(0_+) - \,f'(0_-) - \,f'(0_+) + \ O(\sigma) = -\left(\frac{3}{2}h_0 + 2\,m_1\right) \ + \ O(\sigma).$$

$$J_3 = -2\int_{-l}^{l} dv\, D(\sigma,v)\left(\int_{v}^{l} f(\sigma v-\sigma u)uD'(\sigma,u)\,du + \int_{-l}^{v} f(\sigma v-\sigma u)uD'(\sigma,u)\,du\right) -$$
$$- f(0_-)\int_{-l}^{l} v^2 D(\sigma,v)D'(\sigma,v)\,dv + f(0_+)\int_{-l}^{l} v^2 D(\sigma,v)D'(\sigma,v)\,dv + O(\sigma) =$$
$$= 2\, f(0_-)\int_{-l}^{l} dv\, D(\sigma,v)\int_{v}^{l} D(\sigma,u)\,du + f(0_+)\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} D(\sigma,u)\,du + O(\sigma) =$$
$$= f(0_-) + f(0_+) + O(\sigma) = 2\,m_0 + O(\sigma).$$

During the calculations of J_2 and J_3, we have used that

$$\int_{-l}^{l} vD^2(\sigma,v)\,dv = \int_{-l}^{l} v^2 D(\sigma,v)D'(\sigma,v)\,dv = 0$$
$$\text{and}\quad \frac{1}{\sigma}\int_{-l}^{l} vD(\sigma,v)D'(\sigma,v)\,dv = -\frac{1}{2}.$$

Thus we finally get

$$\langle F_\sigma(x)\,.\,D''_\sigma(x),\,\psi(x)\rangle = (h_1+m_2)\,\psi(0) + \left(\frac{3}{2}\,h_0 + 2m_1\right)\psi'(0) + m_0\,\psi''(0) + O(\sigma)$$
$$= (h_1+m_2)\,\langle\,\delta,\psi\,\rangle - \left(\frac{3}{2}\,h_0 + 2m_1\right)\langle\,\delta',\psi\,\rangle + m_0\,\langle\,\delta'',\psi\,\rangle + O(\sigma).$$

Then passing to the limit as $\sigma \to 0_+$ and applying Definition 2 (a), we obtain (14). ∎

Remarks. **(1)** In the particular case of infinitely-differentiable functions, equations (12) and (14) are in consistence with the corresponding products obtained in classical distribution theory. Indeed, for $f \in C^\infty(\mathbb{R})$ and $\psi \in \mathfrak{D}(\mathbb{R})$, we have

$$\langle\, f\,.\,\delta',\,\psi\,\rangle = \langle\,\delta',\,f\,\psi\,\rangle = -\partial_x\,(\,f\,\psi)\,|_{x=0}\ , \quad \text{or else,} \quad f\,.\,\delta' = -f'(0)\,\delta + f(0)\,\delta'.$$

This clearly coincides with equation (12) since in this case $m_0 = f(0), m_1 = f'(0), h_0 = 0$. Similar argument applies to equation (14).

(2) In the results obtained so far, we have restricted ourselves to studying singularities at $x = 0$, but the considerations are clearly valid for any other singular point.

Recall next that any function can be canonically represented as a sum of its even and odd parts:

$$f(x) = \sum_{\sigma=0,1} f_\sigma(x), \quad \text{where } f_0(x) := \frac{1}{2}\,[f(x)+\check{f}(x)] \ \text{ and } f_1(x) := \frac{1}{2}\,[f(x)-\check{f}(x)]$$

are indeed even and odd functions: $f_\sigma(-x) = (-1)^\sigma f_\sigma(x)$, $\sigma = (0,1)$. Then, as consequences from the results of Theorems 1–2, we obtain this.

Corollary 1. *The even and odd parts $F_0(x), F_1(x)$ of the generalized model $F(x)$ of each function $f(x) \in C^3_d(\mathbb{R}\setminus\{0\})$ satisfy the equations:*

$$F_0(x)\,.\,D'(x) \approx m_0\,\delta'(x), \qquad F_1(x)\,.\,D'(x) \approx -(h_0+m_1)\,\delta(x). \tag{16}$$

$$F_0(x)\,.\,D''(x) \approx (h_1+m_2)\,\delta + m_0\,\delta'',\ F_1(x)\,.\,D''(x) \approx -\left(\frac{3}{2}h_0 + 2m_1\right)\delta'. \tag{17}$$

P r o o f. Rewrite equations (12) and (14) for the even and odd parts of the model $F(x)$. Then the above equations follow on the observation that the derivative of an even/odd function is, respectively, odd/even function, and also that $h(f_0) = m(f_1) = 0$. ∎

Examples. The following equations can be obtained on replacing the generalized model $F(x)$ of the function $f(x) \in C^3_d(\mathbb{R}\setminus\{0\})$ successively with:

(a) H in the first equation of (12): $H\,.\,D' \approx -\delta + \frac{1}{2}\,\delta'$.
(b) H in the first equation of (14): $H\,.\,D'' \approx -\frac{3}{2}\,\delta' + \frac{1}{2}\,\delta''$.
(c) X_+ in the first equation of (14): $X_+\,.\,D'' \approx \delta - \delta'$.
(d) $|X|$ in the first equation of (17): $|X|\,.\,D'' \approx 2\,\delta$.
(e) $|X|\,\mathrm{sgn}\,x$ in the second equation of (17): $|X|\mathrm{sgn}\,x\,.\,D'' \approx -2\,\delta$.

Note that these equations coincide correspondingly with: **(a)** equation (8) for $p=0$, **(b)** equation (9) for $p=0$, **(c)** equation (8) for $p=1$, **(d)** the first equation in (10) for $p=1$, **(e)** the first equation in (11) for $p=1$.

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